Non-harmonious logics and ecumenical logics: what do we they teach us about the meaning of connectives?

Gilles Dowek

Meaning

Two expressions e and e' (propositions, terms, connectives, predicate symbols, function symbols...) have the same meaning if

If what is needed to judge $K[e]$ true is the same as what is needed to judge $K[e']$ true ($K[e]$ and $K[e']$ have the same proofs)

This implies that $K[e]$ has a proof if and only if $K[e']$ does

Introduction rules

The meaning of a logical constant $\triangle$ is defined by its introduction rules
The meaning of a logical constant $\triangle$ is defined by the way a proof of $A \triangle B$ is built
(the way it is used is a consequence of the way it is built)

(Dual view possible: elimination rules / used)

Well-established thesis for constructive natural deduction

Does it generalize beyond constructive natural deduction?
Or does it need to be adjusted when other logics are considered?

I. The $\odot$-logic

Dead and alive? Dead or alive?

$$\frac{\sqrt{2}}{2}|\phi\rangle + \frac{\sqrt{2}}{2}|\psi\rangle$$

Is it a conjunction? a disjunction?

When you build it, you need $|\phi\rangle$ and $|\psi\rangle$
When you use it (measurement), you get $|\phi\rangle$ or $|\psi\rangle$

A connective $\odot$ with the introduction rules of $\wedge$ and the elimination rules of $\vee$

A non-harmonious connective, but not of the kind of Prior's tonk

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \textit{ tonk } B}\ \textit{tonk}\text{-i} \qquad\qquad \frac{\Gamma \vdash A \textit{ tonk } B \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ \textit{tonk}\text{-e}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \vdash A\end{array}}{\Gamma \vdash A \textit{ tonk } B}\ \textit{tonk}\text{-i} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\ \textit{tonk}\text{-e}$$

B provided by the elimination rule, but not required by the introduction rule

No cut reduction

A non-harmonious connective, but not of the kind of Prior's tonk

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \odot B}\ \odot\text{-i} \qquad\qquad \frac{\Gamma \vdash A \odot B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ \odot\text{-e}$$

The elimination rule provide A or B, and the introduction rule requires A and B
The introduction rule requires more than what the elimination rule provides

A non-harmonious connective, but not of the kind of Prior's tonk

The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \vdash A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \odot B}\ \odot\text{-i} \quad \begin{array}{c}\pi_3\\ \vdots\\ \Gamma, A \vdash C\end{array} \quad \begin{array}{c}\pi_4\\ \vdots\\ \Gamma, B \vdash C\end{array}}{\Gamma \vdash C}\ \odot\text{-e}$$

can be reduced in several ways: $(\pi_1/A)\pi_3$ and $(\pi_2/B)\pi_4$
Keep the symmetry: in a non-deterministic way, either to $(\pi_1/A)\pi_3$ or to $(\pi_2/B)\pi_4$

This is fortunate: measurement also is non-deterministic

The proof language of the $\odot$-logic: a quantum programming language with measurement

Harmonious and non-harmonious connectives

Harmonious connectives: information-preservation, reversibility, and determinism

Non-harmonious ones: information-erasure, non-reversibility, and non-determinism

Is the meaning of $\odot$ defined by its introduction rule alone?

No: the same introduction rule as $\wedge$, but not same meaning

Not defined by its elimination rule either

Both rules (and their discrepancy) contribute to the definition of this meaning

II. Peirce’s law again

Independent definitions

- The meaning of $\odot$ defined by all the rules of $\odot$
- The meaning of $\wedge$ defined by all the rules of $\wedge$ (or its introduction rule only)
- The meaning of $\vee$ defined by all the rules of $\vee$ (or its introduction rules only)
- The meaning of $\bot$ defined by all the rules of $\bot$ (or its introduction rules only)
- The meaning of $\Rightarrow$ defined by all the rules of $\Rightarrow$ (or its introduction rule only)
- The meaning of $\neg$ defined by all the rules of $\neg$ (or its introduction rule only)
- ...

Independent definitions

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-i}$$

$\vee$ does not appear in $\wedge$-i

How could $\wedge$-i contribute to the meaning of $\vee$?

A theorem: the sub-formula property
Conservative extension: if a proposition not containing $\wedge$ has a proof, then it has a proof that does not use the rule $\wedge$-i

The sub-formula property holds for constructive natural deduction (commuting cuts...)

Independent definitions?

The rule

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

is another rule of $\neg$ (what else?)

But it changes the meaning of $\Rightarrow$ as

$$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$$

then has a proof

$$\frac{\dfrac{\ldots}{\vdash \neg\neg(((P \Rightarrow Q) \Rightarrow P) \Rightarrow P)}}{\vdash ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P}$$

Rules and rule schemes

What is used is not the rule

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

per se

But the instance for $\Gamma = \varnothing$ and $A = ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$

$$\frac{\vdash \neg\neg(((P \Rightarrow Q) \Rightarrow P) \Rightarrow P)}{\vdash ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P}$$

and $\Rightarrow$ does occur in this rule

Deduction rules are rule schemes: every connective occurs in (instances of) every rule

All the deductions rules holistically define the meaning of each connectives

A more dramatic example: Naive Ecumenical logic

Mix the languages and the rules of constructive and classical logic

$$
\dfrac{
\dfrac{}{\Gamma \vdash A \vee_c \neg_c A}\,\text{em}
\qquad
\dfrac{\dfrac{}{\Gamma, A \vdash A}\,\text{ax}}{\Gamma, A \vdash A \vee \neg A}\,\vee\text{-i}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg_c A, A \vdash \neg_c A}\,\text{ax} \quad \dfrac{}{\Gamma, \neg_c A, A \vdash A}\,\text{ax}}{\Gamma, \neg_c A, A \vdash \bot_c}\,\neg_c\text{-e}}{\Gamma, \neg_c A, A \vdash \bot}\,\bot_c\text{-e}}{\Gamma, \neg_c A \vdash \neg A}\,\neg\text{-i}}{\Gamma, \neg_c A \vdash A \vee \neg A}\,\vee\text{-i}}{}
}{\Gamma \vdash A \vee \neg A}\,\vee_c\text{-e}
$$

(more in Émilie's talk)

But no problem if we mix the rule instances rather that the rules

"Sub-formula" property: a proof of a purely constructive proposition only contains purely constructive propositions

Here $\vee_c$-elim with constructive connectives

III. $\wedge$ and $\wedge_c$ in Ecumenical logics

Are $\wedge$ and $\wedge_c$ the same connective?

Answer 1: Of course, their (introduction) rules are the same
Answer 2: Not quite sure...

For example $\Rightarrow$ and $\Rightarrow_c$ different

$$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$$

has no proof

$$((P \Rightarrow_c Q) \Rightarrow_c P) \Rightarrow_c P$$

has a proof

For $\wedge$ and $\wedge_c$ more difficult to answer
Requires a precise definition of Ecumenical logic(s?)

IV. Diversity in Ecumenism

Not possible to mix languages and rules

Try negative translation instead

- $|A \wedge B| = \neg\neg(|A| \wedge |B|)$
- $|A \Rightarrow B| = \neg\neg(|A| \Rightarrow |B|)$
- $|\top| = \neg\neg\top$
- ...
- $|P(t_1, ..., t_n)| = \neg\neg P(t_1, ..., t_n)$

Examples:

$$|\top \Rightarrow \top| = \neg\neg((\neg\neg\top) \Rightarrow (\neg\neg\top))$$

$$|(0 < 1) \Rightarrow (0 < 1)| = \neg\neg((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$$

First attempt: double negations on the top

$X \wedge_c Y = \neg\neg(X \wedge Y)$, ...

But too naive

$$\top_c \Rightarrow_c \top_c = \neg\neg((\neg\neg\top) \Rightarrow (\neg\neg\top))$$

$$(0 < 1) \Rightarrow_c (0 < 1) = \neg\neg((0 < 1) \Rightarrow (0 < 1))$$

and not $\neg\neg((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$

Second attempt: double negations on the bottom

$X \wedge_c Y = ((\neg\neg X) \wedge (\neg\neg Y))$, ...
Inspired by Hermant's and Allali's *light negative translation*

But too naive

$$\top_c \Rightarrow_c \top_c = (\neg\neg\top) \Rightarrow (\neg\neg\top)$$

and not $\neg\neg((\neg\neg\top) \Rightarrow (\neg\neg\top))$

$$(0 < 1) \Rightarrow_c (0 < 1) = ((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$$

and not $\neg\neg((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$

Third attempt: the best of both worlds

If each classical connective brings two negations only on one branch, no hope that

$$(0 < 1) \Rightarrow_c (0 < 1) = \neg\neg((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$$

$X \wedge_c Y = \neg\neg((\neg\neg X) \wedge (\neg\neg Y))$, ...

$$\top_c \Rightarrow_c \top_c = \neg\neg((\neg\neg\neg\neg\top) \Rightarrow (\neg\neg\neg\neg\top))$$

and not $\neg\neg((\neg\neg\top) \Rightarrow (\neg\neg\top))$

$$(0 < 1) \Rightarrow_c (0 < 1) = \neg\neg((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$$

Fixes: Third attempt (double negations on the top and on the bottom)

D. (2015)

$$\top_c \Rightarrow_c \top_c = \neg\neg((\neg\neg\neg\neg\top) \Rightarrow (\neg\neg\neg\neg\top))$$

and not $\neg\neg((\neg\neg\top) \Rightarrow (\neg\neg\top))$
Not really an issue three negations are equivalent to one

But another problem:
Taking all the connectives to be classical in $0 < 1$ yields $0 < 1$ and not $\neg\neg(0 < 1)$
Not really an issue for absolute provability $\varnothing \vdash A$ (atomic propositions have no proofs anyway)
But one for relative provability $\Gamma \vdash A$ (more on this later)

Fixes: First attempt (double negations on the top)

- $|A \wedge B| = \neg\neg(|A| \wedge |B|)$
- ...
- $|P(t_1, ..., t_n)| = \neg\neg P(t_1, ..., t_n)$

Prawitz (2015): introduce also classical predicate symbols
$x <_c y = \neg\neg(x < y)$
This way

$$(0 <_c 1) \Rightarrow_c (0 <_c 1) = \neg\neg((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$$

But an expensive solution

Douglas Bridges: constructive analysis not the theory of constructive reals but the constructive theory of reals
Same notion of real number as in classical analysis

But what does this mean precisely?

- The meaning of the connectives and quantifiers defined by the deduction rules
- The meaning of the function symbols and predicate symbols defined by the axioms

Reject extreme holism: the deduction rules and the axioms globally define the meaning of connectives, quantifiers, function symbols, and predicate symbols

Could work for fixed set of axioms, but different theories (analysis, arithmetic, geometry, set theory, type theory...) yield different meanings for connectives and quantifiers

Do not mess with predicate symbols

Tweak Predicate logic instead

$$\begin{array}{ll} t & = x \mid f(t, ..., t) \\ A & = P(t, ..., t) \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \Rightarrow A \mid ... \end{array}$$

$$\begin{array}{ll} t & = x \mid f(t, ..., t) \\ L & = P(t, ..., t) \\ A & = \ L \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \Rightarrow A \mid ... \end{array}$$

Do not mess with predicate symbols

Tweak Predicate logic instead

$$
\begin{aligned}
t &= x \mid f(t, ..., t) \\
A &= P(t, ..., t) \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \Rightarrow A \mid ...
\end{aligned}
$$

$$
\begin{aligned}
t &= x \mid f(t, ..., t) \\
L &= P(t, ..., t) \\
A &= \circ L \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \Rightarrow A \mid ...
\end{aligned}
$$

$0 < 1$ and $\circ(0 < 1)$

$0 < 1$ is the propositional content of the proposition $\circ(0 < 1)$
$0 < 1$: the proposition minus assertiveness, $\circ$: assertiveness

A long philosophical tradition: lekton, lexis(?)...

But also that-clauses ("that the Sun shines") $\circ$ = "it is true"
It is true that the Sun shines = The Sun shines

Also in expressions of second-order logic in first-order predicate logic

$$\forall X \; ((X \; 0) \Rightarrow (X \; 0)) \qquad \forall X \; ((0 \, \epsilon \, X) \Rightarrow (0 \, \epsilon \, X))$$

ϵ: copula

$$\forall Y \; (Y \Rightarrow Y) \qquad \forall Y \; (\varepsilon(Y) \Rightarrow \varepsilon(Y))$$

ε, $\circ$: unary copulas

Constructive and classical assertiveness

$$
\begin{array}{ll}
nt & = x \mid f(t, ..., t) \\
L & = P(t, ..., t) \\
A & = \circ L \mid \ \circ_c L \mid \top \mid \top_c \mid \bot \mid \bot_c \mid \neg A \mid \neg_c A \mid A \wedge A \mid A \wedge_c A \mid A \Rightarrow A \mid A \Rightarrow_c A \mid ...
\end{array}
$$

$$\circ_c L = \neg\neg \circ L$$

The $\circ_c$ connective provides the missing double negation (without messing with the predicate symbols)

Grienenberger (2019)

Also a fix for the second attempt (double negations on the bottom)

$$(0 < 1) \Rightarrow_c (0 < 1) = ((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$$

and not $\neg\neg((\neg\neg(0 < 1)) \Rightarrow (\neg\neg(0 < 1)))$

Add a copula $\triangleright$ at the root of the proposition

$$\begin{array}{ll} t & = x \mid f(t, ..., t) \\ L & = P(t, ..., t) \mid \dot{\top} \mid \dot{\bot} \mid \dot{\neg} L \mid L \mathbin{\dot{\wedge}} L \mid L \mathbin{\dot{\Rightarrow}} L \mid ... \\ A & = \triangleright L \end{array}$$

$\triangleright_c L = \neg\neg \triangleright L$

Gilbert (2018)

Universes *à la* Tarski

Proposition / judgement

Relative provability with double negations on the top and on the bottom

$P \wedge Q \vdash P$ has a proof in classical logic
But $P \wedge_c Q \vdash P$ (that is $\neg\neg(\neg\neg P \wedge \neg\neg Q) \vdash P$) has no proof in Ecumenical logic

Hence a theorem for absolute provability only

$\vdash (P \wedge Q) \Rightarrow P$ has a proof in classical logic
and $\vdash (P \wedge_c Q) \Rightarrow_c P$ (that is $\vdash \neg\neg((\neg\neg\neg\neg(\neg\neg P \wedge \neg\neg Q)) \Rightarrow (\neg\neg P)))$ also does

In the first case $\vdash (P \wedge_c Q) \Rightarrow P$ and in the second $\vdash (P \wedge_c Q) \Rightarrow_c P$
To get a relative provability theorem we should also introduce a $\vdash_c$ adding double negations to the hypotheses and conclusion

But the side discovery is that

$(P \wedge Q) \Rightarrow P$ has a proof
$(P \wedge_c Q) \Rightarrow_c P$ has a proof
but not the mixed proposition $(P \wedge_c Q) \Rightarrow P$

An Ecumenical logic should

- coincide with constructive logic for purely constructive propositions
- coincide with classical logic for purely classical propositions

But has some degree of freedom for mixed propositions
Yet...

The non provability of $(P \wedge_c Q) \Rightarrow P$ common to many Ecumenical logics

$$\neg\neg(P \wedge Q) \Rightarrow P$$

$$\neg\neg(\circ P \wedge \circ Q) \Rightarrow \circ P$$

$$(\neg\neg P \wedge \neg\neg Q) \Rightarrow P$$

$$\triangleright((\neg\neg P \wedge \neg\neg Q) \Rightarrow P)$$

has no proof

$\wedge$ and $\wedge_c$ have a different meaning in all these logics

$\wedge$ and $\wedge_c$ have a different meaning

Not surprising for $\neg$ and $\neg_c$, $\vee$ and $\vee_c$ (excluded middle)

Not completely surprising for $\exists$ and $\exists_c$ (witness)

Quite surprising for $\Rightarrow$ and $\Rightarrow_c$ (Peirce)

Surprising for $\wedge$ and $\wedge_c$ (requires mixed propositions in Ecumenical logic)

V. Lessons learnt about the meaning of connectives (and about Ecumenical logics)

The meaning of logical constants

The meaning of a logical constant is defined by its introduction rules
Works for constructive natural deduction

But needs to be adjusted when other logics are considered

- Introduction rules are not enough for non harmonious connectives ($\odot$...)
- (Due to their schematic nature) the rules of all connectives holistically define the meaning of each (Peirce, Naive Ecumenical logic...)

Even $\wedge$ and $\wedge_c$ have a different meaning

Ecumenical logics

An Ecumenical logic should coincide with constructive (resp. classical) logic for purely constructive (resp. classical) propositions
But has some degree of freedom for mixed propositions

The idea that we can determine the truth of

$$(P \wedge_c Q) \Rightarrow P$$

from the meaning of $\Rightarrow$ as defined in constructive logic and that of $\wedge_c$ as defined in classical logic contradicted by the fact that the meaning of these symbols are defined by the whole set of deduction rules, hence differently in different systems