From the universality of mathematical truth to the interoperability of proof systems

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I. Yet another crisis of the universality of mathematical truth

The universality of mathematical truth

The truth conditions of a mathematical statement must be the object of unanimous agreement

- Constitutive of the notion of mathematical truth itself
- Yet, constantly jeopardized
- When mathematicians disagree on the truth of some statements: a crisis of the universality of mathematical truth

In the past

- The incommensurability of the diagonal and side of a square

$$\exists x \, (x \textit{ is a number} \wedge x^2 = 2)$$

- The introduction of infinite series

$$\sum_n \frac{1}{2^n} = 2 \qquad\qquad \sum_n (-1)^n = 0$$

- The non-Euclidean geometries

The sum of the angles in a triangle equals the straight angle

- The independence of the axiom of choice

Every vector space has a basis

- Constructivity

If $A \cup B$ infinite, then A infinite or B infinite

All these crises have been resolved

The incommensurability of the diagonal and side of a square: rational numbers and real numbers

Infinite series: limit

Non-Euclidean geometries: several solutions

- Different spaces: truth of

  On a space of zero curvature, the sum of the angles in a triangle equals the straight angle

  but not of

  On a space of negative curvature, the sum of the angles in a triangle equals the straight angle

- Axiomatic theories: E and H, truth of

  $$E \vdash \textit{the sum of the angles in a triangle equals the straight angle}$$

  but not of

  $$H \vdash \textit{the sum of the angles in a triangle equals the straight angle}$$

Equivalent (soundness and completeness)

The second solution

- A true $\longrightarrow \Gamma \vdash A$ true
- Truth conditions: for the statements of geometry $\longrightarrow$ for arbitrary sequents
- Separation between the definition of the truth conditions of a sequent: the logical framework and the definition of the various geometries as theories
- A logical framework: Predicate logic
- The various geometries defined in this logical framework

The axiom of choice

First solution: truth of

In a model of ZFC, every vector space has a basis

but not of

In a model of ZF, every vector space has a basis

Second: *Every vector space has a basis* consequence of the axiom of choice

First solution does not work:
- Too far from the original formulation
- Problem of the “absolute” theory in which this should be proved

Thus, second chosen, paving the way to Reverse mathematics

Constructivity

First solution: truth of

In a model valued in a Boolean algebra, if $A \cup B$ *infinite, then* A *infinite or* B *infinite*

but not of

In a model valued in a Heyting algebra, if $A \cup B$ *infinite, then* A *infinite or* B *infinite*

Again, too far from the original formulation and question of the “absolute” theory

Second: *if* $A \cup B$ *infinite, then* A *infinite or* B *infinite* consequence of the excluded middle

A third solution: Ecumenism

Changing the axioms while keeping the same symbols?

Axioms express the meaning of the symbols:
different axioms $\longrightarrow$ different meanings $\longrightarrow$ different symbols (just like $\vee$ and $\oplus$)

The only "mistake" is not to accept or to reject the excluded middle, but to use the same symbol for $\vee$ and $\vee_c$

Nothing prevents from using them both

Truth of

$$Infinite(A \cup B) \Rightarrow_c Infinite(A) \vee_c Infinite(B)$$

but not of

$$Infinite(A \cup B) \Rightarrow Infinite(A) \vee Infinite(B)$$

$\sqrt{2}$: $\mathbb{Q}$ vs. $\mathbb{R}$ already Ecumenical (mass vs. weight...)

Past crises ($\sqrt{2}$, $\sum_n$, non-Euclidean geometries, AC, Constructivism) have been resolved

But... yet another crisis: computerized proof systems

Computerized proof systems

Coq, Isabelle/HOL, PVS, HOL Light, Lean...
A major step forward in the quest of mathematical rigor

But jeopardizes, once again, the universality of mathematical truth

A proof of Fermat's little theorem $\longrightarrow$ a Coq proof of Fermat's little theorem, a PVS proofs of Fermat's little theorem...

Each proof system: its own language and its own truth conditions

Yet another crisis to be resolved

II. Logical frameworks

A solution that (already) worked for several crises

Express the theories implemented in Coq, Isabelle/HOL, PVS, HOL Light, Lean... in Predicate logic

- (if we are lucky) many common axioms and few differentiating the theories
- (if we are lucky) mixing the axioms differentiating the symbols (Ecumenism)
- analyze which proof uses which axiom (just like for the axiom of choice)
- try to find better proofs using less axioms (just like constructivization, Reverse mathematics...)

A solution that (already) worked for several crises

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Beyond Predicate logic

In a century: some limitations of Predicate logic

Other logical frameworks: λ-Prolog, Isabelle, the Edinburgh logical framework, Pure type systems, Deduction modulo theory, Ecumenical logic, DEDUKTI

In DEDUKTI

- Function symbols can bind variables (like in λ-Prolog, Isabelle, the Edinburgh logical framework)
- Proofs are terms (like in the Edinburgh logical framework)
- Deduction and computation are mixed (like in Deduction modulo theory)
- Both constructive and classical proofs can be expressed (like in Ecumenical logic)

The two features of Dedukti

Dedukti is a typed λ-calculus with

- Dependent types
- Computation rules

Several implementations: Dkcheck, Lambdapi, Kocheck...

No typing rules today, but illustration of these features with examples

In a logical framework, you can

- Define your theory
- Check proofs expressed in this theory

A theory in Predicate logic: a language (sorts, function symbols, and predicate symbols) and a set of axioms

A theory in Dedukti: a set of symbols (replace sorts, function symbols, predicate symbols, and axioms) and a set of computation rules

III. Examples of axioms in Dedukti

Catching up with Predicate logic

Predicate logic is a sophisticated framework with notions of sort, function symbol, predicate symbol, arity, variable, term, proposition, proof...

A typed λ-calculus is much more primitive

These notions must be constructed

A good exercise to start with, but also an interest in itself: the first book of Euclid's elements (originally formalized in Coq) can be expressed in Predicate logic + the axioms of geometry and exported to many systems (Géran)

Terms and propositions: a first attempt

I : TYPE
$Prop$: TYPE

function symbols: $I \to \ldots \to I \to I$
predicate symbols: $I \to \ldots \to I \to Prop$
connectives: $Prop \to \ldots \to Prop \to Prop$
$\forall$: $(I \to Prop) \to Prop$

- $\forall$ binds (higher-order abstract syntax: $\forall x\ A$ expressed as $\forall\ \lambda x\ A$)
- Symbol declarations only (no computation rules yet)
- Simply typed λ-calculus (no dependent types yet)
- Types are terms of type TYPE

Works if we want one sort

But if we want several (like in geometry: points, lines, circles...)

I_1 : TYPE

I_2 : TYPE

I_3 : TYPE

Several (an infinite number of?) symbols and several (an infinite number of?) quantifiers

$\forall_1 : (I_1 \to Prop) \to Prop$

$\forall_2 : (I_2 \to Prop) \to Prop$

$\forall_3 : (I_3 \to Prop) \to Prop$

Making the universal quantifier generic

Something like
$\forall : \Pi X : \mathsf{TYPE}, ((X \to \mathit{Prop}) \to \mathit{Prop})$

But does not work for two reasons

- (a minor one) no dependent products on TYPE in DEDUKTI
- (a major one) many things in TYPE beyond I_1, I_2, and I_3 (for example *Prop*)

Making the universal quantifier generic

I : TYPE
I_1 : TYPE, I_2 : TYPE, I_3 : TYPE

Set : TYPE

ι : Set
ι_1 : Set, ι_2 : Set, ι_3 : Set

El : $Set \to$ TYPE

$El\ \iota \longrightarrow I$
$El\ \iota_1 \longrightarrow I_1$, $El\ \iota_2 \longrightarrow I_2$, $El\ \iota_3 \longrightarrow I_3$

$Prop$: TYPE

$\forall : \Pi x : Set, (El\ x \to Prop) \to Prop$

Uses dependent types and computation rules

Reminiscent of expression of Simple type theory in Predicate logic, universes *à la* Tarski...

Proofs

So far: terms and propositions. Now: proofs

Proofs are trees, they can be expressed in Dedukti

Curry-de Bruijn-Howard: $P \Rightarrow P$ should be the type of its proofs
But not possible here $P \Rightarrow P : \mathit{Prop} : \mathrm{TYPE}$ is not itself a type

$\mathit{Prf} : \mathit{Prop} \to \mathrm{TYPE}$
mapping each proposition to the type of its proofs: $\mathit{Prf}\,(P \Rightarrow P) : \mathrm{TYPE}$

Not all types are types of proofs (for example I, $\mathit{El}\ \iota$, Prop...)

Proofs

Brouwer-Heyting-Kolmogorov: $\lambda x : (\mathit{Prf}\ P), x$ should be a proof of $P \Rightarrow P$
But has type $(\mathit{Prf}\ P) \to (\mathit{Prf}\ P)$ and not $\mathit{Prf}\ (P \Rightarrow P)$
$\mathit{Prf}\ (P \Rightarrow P)$ and $(\mathit{Prf}\ P) \to (\mathit{Prf}\ P)$ must be identified

A computation rule

$$\mathit{Prf}\ (x \Rightarrow y) \longrightarrow (\mathit{Prf}\ x) \to (\mathit{Prf}\ y)$$

In the same way

$$\mathit{Prf}\ (\forall\ x\ p) \longrightarrow \Pi z : (\mathit{El}\ x), (\mathit{Prf}\ (p\ z))$$

The function Prf is an injective morphism from propositions to types: it is the Curry-de Bruijn-Howard isomorphism

If you want to express Predicate logic proofs, you know enough

Simple type theory (HOL4, HOL Light, Isabelle/HOL...): two features

- Propositions as objects

  $o : Set$

  $El\ o \longrightarrow Prop$

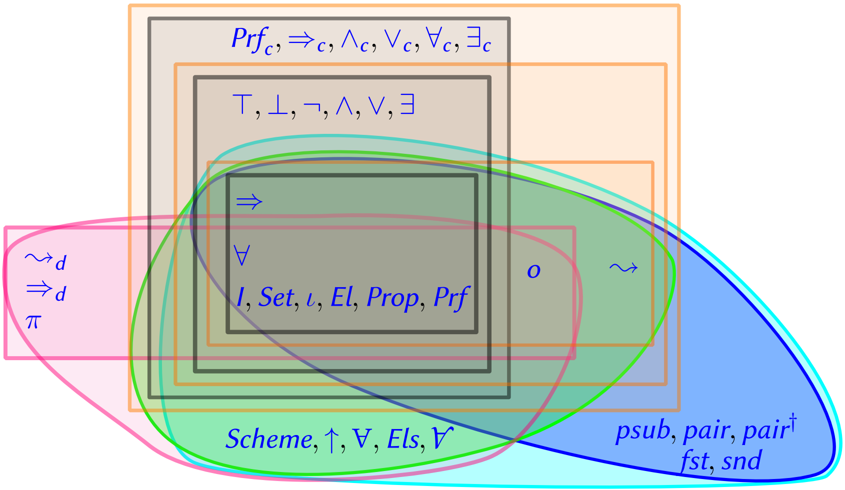
- Functions

  $\leadsto\ : Set \to Set \to Set$

  $El\ (x \leadsto y) \longrightarrow (El\ x) \to (El\ y)$

More symbols

Pick cherries according to your taste



Enough to express Predicate logic, Simple type theory, Simple type theory with predicate subtyping, The Calculus of constructions...

More symbols: universes, universe polymorphism, predicativity, inductive types, cubical type theory (Barras), set theory (Traversié)

IV. The benefits of universality

- Reverse engineering proofs

  First book of Euclide's Elements in Coq $\longrightarrow$ in Predicate logic

  Fermat's little theorem in Matita $\longrightarrow$ in constructive Simple type theory (Thiré)

  Bertrand's postulate in Matita $\longrightarrow$ in Predicative type theory (Felicíssimo)

- Interoperability

  The first book of Euclide's element in Isabelle/HOL, TSTP...

  Fermat's little theorem in Isabelle/HOL, HOL Light, Coq, Lean, PVS...

  Bertrand's postulate in Agda

- Cross-verification

A social motivation: mathematicians and industrials more likely to develop proofs in mathematics (possibly with some axioms they can debate) than in an exotic system

And a philosophical one: Universality has survived many crises: we ought not to give up on it (and we do not need to)

Mathematics is necessarily always in crisis, and always in the process of resolving it.

Michel Serres