

Linear lambda-calculus is linear

Alejandro Díaz-Caro and Gilles Dowek

Linear logic: something to do with algebraic linearity

A proof of a linear implication between A and B : any a linear function mapping proofs of A to proofs of B

Fruitfully exploited to build models of linear logic

Difficult to even formulate within the proof language: $f(u + v) = f(u) + f(v)$,
 $f(a.u) = a.f(u)$
What are $+$ and $.$?

Changed with quantum programming languages (our motivation): mix programming language constructions (abstraction, application) with algebraic operations ($+$, $.$)

This work

A extension of linear logic with $+$ and $.$

A linearity theorem: if f is a proof of an implication between two propositions (of some specific form), then

$$f(u + v) = f(u) + f(v)$$

$$f(a.u) = a.f(u)$$

A first step: interstitial rules

Deduction rule: **premises = conclusion**

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A}{\Gamma \vdash A} \text{ sum} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A} \text{ prod}$$

Constructors \oplus and \odot in the proof language

That express the addition and the multiplication by a scalar

Trivial = internal

Commuting cuts

$$\frac{\frac{\frac{\pi_1}{\Gamma \vdash A} \quad \frac{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B} \wedge\text{-i} \quad \frac{\frac{\pi_3}{\Gamma \vdash A} \quad \frac{\pi_4}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B} \wedge\text{-i}}{\Gamma \vdash A \wedge B} \text{sum} \quad \frac{\pi_5}{\Gamma, A \vdash C} \wedge\text{-e1}}{\Gamma \vdash C} \wedge\text{-e1}$$

Needs to commute sum either with $\wedge\text{-e1}$ or with $\wedge\text{-i}$

Here **commutation with introductions** when possible (all but \vee)

Stronger introduction property

Second step: scalars

A field \mathcal{S} of scalars

Replace

- ▶ the rule \top -i with a **family** of rules \top -i(a), one for each scalar
- ▶ the rule prod with a **family** of rules prod(a), one for each scalar

A choice of linear connectives

Intuitionistic linear logic: no multiplicative falsehood, no additive implication, no multiplicative disjunction

- ▶ multiplicative truth (additive truth not **useful**)
- ▶ additive falsehood
- ▶ multiplicative implication
- ▶ additive conjunction (multiplicative conjunction not **possible**)
- ▶ additive disjunction

Sum rule additive

Often written 1 , 0 , \multimap , $\&$, and \oplus

We use the usual \top , \perp , \Rightarrow , \wedge , and \vee (but notations are arbitrary)

$$\begin{array}{c}
\frac{}{x:A \vdash x:A} \text{ ax} \quad \frac{\Gamma \vdash t:A \quad \Gamma \vdash u:A}{\Gamma \vdash t+u:A} \text{ sum} \quad \frac{\Gamma \vdash t:A}{\Gamma \vdash a \bullet t:A} \text{ prod}(a) \\
\frac{}{\vdash a.\star : \top} \top\text{-i}(a) \quad \frac{\Gamma \vdash t:\top \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash \delta_{\top}(t, u):A} \top\text{-e} \quad \frac{\Gamma \vdash t:\perp}{\Gamma, \Delta \vdash \delta_{\perp}(t):C} \perp\text{-e}
\end{array}$$

$$\frac{\Gamma, x:A \vdash t:B}{\Gamma \vdash \lambda x.t : A \Rightarrow B} \Rightarrow\text{-i} \quad \frac{\Gamma \vdash t:A \Rightarrow B \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash t u : B} \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash t:A \quad \Gamma \vdash u:B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \wedge\text{-i}$$

$$\frac{\Gamma \vdash t:A \wedge B \quad \Delta, x:A \vdash u:C}{\Gamma, \Delta \vdash \delta_{\wedge}^1(t, x.u) : C} \wedge\text{-e1} \quad \frac{\Gamma \vdash t:A \wedge B \quad \Delta, x:B \vdash u:C}{\Gamma, \Delta \vdash \delta_{\wedge}^2(t, x.u) : C} \wedge\text{-e2}$$

$$\frac{\Gamma \vdash t:A}{\Gamma \vdash \text{inl}(t) : A \vee B} \vee\text{-i1} \quad \frac{\Gamma \vdash t:B}{\Gamma \vdash \text{inr}(t) : A \vee B} \vee\text{-i2}$$

$$\frac{\Gamma \vdash t:A \vee B \quad \Delta, x:A \vdash u:C \quad \Delta, y:B \vdash v:C}{\Gamma, \Delta \vdash \delta_{\vee}(t, x.u, y.v) : C} \vee\text{-e}$$

$$\frac{}{x:A \vdash x:A} \text{ ax} \quad \frac{\Gamma \vdash t:A \quad \Gamma \vdash u:A}{\Gamma \vdash t+u:A} \text{ sum} \quad \frac{\Gamma \vdash t:A}{\Gamma \vdash a \bullet t:A} \text{ prod}(a)$$

$$\frac{}{\vdash a.\star : \top} \top\text{-i}(a) \quad \frac{\Gamma \vdash t:\top \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash \delta_{\top}(t, u):A} \top\text{-e} \quad \frac{\Gamma \vdash t:\perp}{\Gamma, \Delta \vdash \delta_{\perp}(t):C} \perp\text{-e}$$

$$\frac{\Gamma, x:A \vdash t:B}{\Gamma \vdash \lambda x.t : A \Rightarrow B} \Rightarrow\text{-i} \quad \frac{\Gamma \vdash t:A \Rightarrow B \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash t u : B} \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash t:A \quad \Gamma \vdash u:B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \wedge\text{-i}$$

$$\frac{\Gamma \vdash t:A \wedge B \quad \Delta, x:A \vdash u:C}{\Gamma, \Delta \vdash \delta_{\wedge}^1(t, x.u):C} \wedge\text{-e1} \quad \frac{\Gamma \vdash t:A \wedge B \quad \Delta, x:B \vdash u:C}{\Gamma, \Delta \vdash \delta_{\wedge}^2(t, x.u):C} \wedge\text{-e2}$$

$$\frac{\Gamma \vdash t:A}{\Gamma \vdash \text{inl}(t) : A \vee B} \vee\text{-i1} \quad \frac{\Gamma \vdash t:B}{\Gamma \vdash \text{inr}(t) : A \vee B} \vee\text{-i2}$$

$$\frac{\Gamma \vdash t:A \vee B \quad \Delta, x:A \vdash u:C \quad \Delta, y:B \vdash v:C}{\Gamma, \Delta \vdash \delta_{\vee}(t, x.u, y.v):C} \vee\text{-e}$$

$$\begin{array}{c}
\frac{}{x:A \vdash x:A} \text{ ax} \quad \frac{\Gamma \vdash t:A \quad \Gamma \vdash u:A}{\Gamma \vdash t+u:A} \text{ sum} \quad \frac{\Gamma \vdash t:A}{\Gamma \vdash a \bullet t:A} \text{ prod}(a) \\
\frac{}{\vdash a.\star : \top} \text{ T-i}(a) \quad \frac{\Gamma \vdash t:\top \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash \delta_{\top}(t, u):A} \text{ T-e} \quad \frac{\Gamma \vdash t:\perp}{\Gamma, \Delta \vdash \delta_{\perp}(t):C} \perp\text{-e}
\end{array}$$

$$\frac{\Gamma, x:A \vdash t:B}{\Gamma \vdash \lambda x.t : A \Rightarrow B} \Rightarrow\text{-i} \quad \frac{\Gamma \vdash t:A \Rightarrow B \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash t u : B} \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash t:A \quad \Gamma \vdash u:B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \wedge\text{-i}$$

$$\frac{\Gamma \vdash t:A \wedge B \quad \Delta, x:A \vdash u:C}{\Gamma, \Delta \vdash \delta_{\wedge}^1(t, x.u):C} \wedge\text{-e1} \quad \frac{\Gamma \vdash t:A \wedge B \quad \Delta, x:B \vdash u:C}{\Gamma, \Delta \vdash \delta_{\wedge}^2(t, x.u):C} \wedge\text{-e2}$$

$$\frac{\Gamma \vdash t:A}{\Gamma \vdash \text{inl}(t) : A \vee B} \vee\text{-i1} \quad \frac{\Gamma \vdash t:B}{\Gamma \vdash \text{inr}(t) : A \vee B} \vee\text{-i2}$$

$$\frac{\Gamma \vdash t:A \vee B \quad \Delta, x:A \vdash u:C \quad \Delta, y:B \vdash v:C}{\Gamma, \Delta \vdash \delta_{\vee}(t, x.u, y.v):C} \vee\text{-e}$$

$$\begin{array}{c}
\frac{}{x:A \vdash x:A} \text{ ax} \quad \frac{\Gamma \vdash t:A \quad \Gamma \vdash u:A}{\Gamma \vdash t+u:A} \text{ sum} \quad \frac{\Gamma \vdash t:A}{\Gamma \vdash a \bullet t:A} \text{ prod}(a) \\
\frac{}{\vdash a.\star : \overline{T}} \text{ T-i}(a) \quad \frac{\Gamma \vdash t:T \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash \delta_T(t,u):A} \text{ T-e} \quad \frac{\Gamma \vdash t:\perp}{\Gamma, \Delta \vdash \delta_\perp(t):C} \perp\text{-e}
\end{array}$$

$$\frac{\Gamma, x:A \vdash t:B}{\Gamma \vdash \lambda x.t : A \Rightarrow B} \Rightarrow\text{-i} \quad \frac{\Gamma \vdash t:A \Rightarrow B \quad \Delta \vdash u:A}{\Gamma, \Delta \vdash t u : B} \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash t:A \quad \Gamma \vdash u:B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \wedge\text{-i}$$

$$\frac{\Gamma \vdash t:A \wedge B \quad \Delta, x:A \vdash u:C}{\Gamma, \Delta \vdash \delta_\wedge^1(t, x.u) : C} \wedge\text{-e1} \quad \frac{\Gamma \vdash t:A \wedge B \quad \Delta, x:B \vdash u:C}{\Gamma, \Delta \vdash \delta_\wedge^2(t, x.u) : C} \wedge\text{-e2}$$

$$\frac{\Gamma \vdash t:A}{\Gamma \vdash \text{inl}(t) : A \vee B} \vee\text{-i1} \quad \frac{\Gamma \vdash t:B}{\Gamma \vdash \text{inr}(t) : A \vee B} \vee\text{-i2}$$

$$\frac{\Gamma \vdash t:A \vee B \quad \Delta, x:A \vdash u:C \quad \Delta, y:B \vdash v:C}{\Gamma, \Delta \vdash \delta_\vee(t, x.u, y.v) : C} \vee\text{-e}$$

$$\begin{aligned}
\delta_{\top}(a.\star, t) &\longrightarrow a \bullet t \\
(\lambda x.t) u &\longrightarrow (u/x)t \\
\delta_{\wedge}^1(\langle t, u \rangle, x.v) &\longrightarrow (t/x)v \\
\delta_{\wedge}^2(\langle t, u \rangle, x.v) &\longrightarrow (u/x)v \\
\delta_{\vee}(\text{inl}(t), x.v, y.w) &\longrightarrow (t/x)v \\
\delta_{\vee}(\text{inr}(u), x.v, y.w) &\longrightarrow (u/y)w
\end{aligned}$$

$$\begin{aligned}
a.\star + b.\star &\longrightarrow (a + b).\star \\
(\lambda x.t) + (\lambda x.u) &\longrightarrow \lambda x.(t + u) \\
\langle t, u \rangle + \langle v, w \rangle &\longrightarrow \langle t + v, u + w \rangle \\
\delta_{\vee}(t + u, x.v, y.w) &\longrightarrow \delta_{\vee}(t, x.v, y.w) + \delta_{\vee}(u, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
a \bullet b.\star &\longrightarrow (a \times b).\star \\
a \bullet \lambda x.t &\longrightarrow \lambda x.a \bullet t \\
a \bullet \langle t, u \rangle &\longrightarrow \langle a \bullet t, a \bullet u \rangle \\
\delta_{\vee}(a \bullet t, x.v, y.w) &\longrightarrow a \bullet \delta_{\vee}(t, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
\delta_{\top}(a.\star, t) &\longrightarrow a \bullet t \\
(\lambda x.t) u &\longrightarrow (u/x)t \\
\delta_{\wedge}^1(\langle t, u \rangle, x.v) &\longrightarrow (t/x)v \\
\delta_{\wedge}^2(\langle t, u \rangle, x.v) &\longrightarrow (u/x)v \\
\delta_{\vee}(\text{inl}(t), x.v, y.w) &\longrightarrow (t/x)v \\
\delta_{\vee}(\text{inr}(u), x.v, y.w) &\longrightarrow (u/y)w
\end{aligned}$$

$$\begin{aligned}
a.\star + b.\star &\longrightarrow (a + b).\star \\
(\lambda x.t) + (\lambda x.u) &\longrightarrow \lambda x.(t + u) \\
\langle t, u \rangle + \langle v, w \rangle &\longrightarrow \langle t + v, u + w \rangle \\
\delta_{\vee}(t + u, x.v, y.w) &\longrightarrow \delta_{\vee}(t, x.v, y.w) + \delta_{\vee}(u, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
a \bullet b.\star &\longrightarrow (a \times b).\star \\
a \bullet \lambda x.t &\longrightarrow \lambda x.a \bullet t \\
a \bullet \langle t, u \rangle &\longrightarrow \langle a \bullet t, a \bullet u \rangle \\
\delta_{\vee}(a \bullet t, x.v, y.w) &\longrightarrow a \bullet \delta_{\vee}(t, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
\delta_{\top}(a.\star, t) &\longrightarrow a \bullet t \\
(\lambda x.t) u &\longrightarrow (u/x)t \\
\delta_{\wedge}^1(\langle t, u \rangle, x.v) &\longrightarrow (t/x)v \\
\delta_{\wedge}^2(\langle t, u \rangle, x.v) &\longrightarrow (u/x)v \\
\delta_{\vee}(\text{inl}(t), x.v, y.w) &\longrightarrow (t/x)v \\
\delta_{\vee}(\text{inr}(u), x.v, y.w) &\longrightarrow (u/y)w
\end{aligned}$$

$$\begin{aligned}
a.\star + b.\star &\longrightarrow (a + b).\star \\
(\lambda x.t) + (\lambda x.u) &\longrightarrow \lambda x.(t + u) \\
\langle t, u \rangle + \langle v, w \rangle &\longrightarrow \langle t + v, u + w \rangle \\
\delta_{\vee}(t + u, x.v, y.w) &\longrightarrow \delta_{\vee}(t, x.v, y.w) + \delta_{\vee}(u, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
a \bullet b.\star &\longrightarrow (a \times b).\star \\
a \bullet \lambda x.t &\longrightarrow \lambda x.a \bullet t \\
a \bullet \langle t, u \rangle &\longrightarrow \langle a \bullet t, a \bullet u \rangle \\
\delta_{\vee}(a \bullet t, x.v, y.w) &\longrightarrow a \bullet \delta_{\vee}(t, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
\delta_{\top}(a.\star, t) &\longrightarrow a \bullet t \\
(\lambda x.t) u &\longrightarrow (u/x)t \\
\delta_{\wedge}^1(\langle t, u \rangle, x.v) &\longrightarrow (t/x)v \\
\delta_{\wedge}^2(\langle t, u \rangle, x.v) &\longrightarrow (u/x)v \\
\delta_{\vee}(\text{inl}(t), x.v, y.w) &\longrightarrow (t/x)v \\
\delta_{\vee}(\text{inr}(u), x.v, y.w) &\longrightarrow (u/y)w
\end{aligned}$$

$$\begin{aligned}
a.\star + b.\star &\longrightarrow (a + b).\star \\
(\lambda x.t) + (\lambda x.u) &\longrightarrow \lambda x.(t + u) \\
\langle t, u \rangle + \langle v, w \rangle &\longrightarrow \langle t + v, u + w \rangle \\
\delta_{\vee}(t + u, x.v, y.w) &\longrightarrow \delta_{\vee}(t, x.v, y.w) + \delta_{\vee}(u, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
a \bullet b.\star &\longrightarrow (a \times b).\star \\
a \bullet \lambda x.t &\longrightarrow \lambda x.a \bullet t \\
a \bullet \langle t, u \rangle &\longrightarrow \langle a \bullet t, a \bullet u \rangle \\
\delta_{\vee}(a \bullet t, x.v, y.w) &\longrightarrow a \bullet \delta_{\vee}(t, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
\delta_{\top}(a.\star, t) &\longrightarrow a \bullet t \\
(\lambda x.t) u &\longrightarrow (u/x)t \\
\delta_{\wedge}^1(\langle t, u \rangle, x.v) &\longrightarrow (t/x)v \\
\delta_{\wedge}^2(\langle t, u \rangle, x.v) &\longrightarrow (u/x)v \\
\delta_{\vee}(\text{inl}(t), x.v, y.w) &\longrightarrow (t/x)v \\
\delta_{\vee}(\text{inr}(u), x.v, y.w) &\longrightarrow (u/y)w
\end{aligned}$$

$$\begin{aligned}
a.\star + b.\star &\longrightarrow (a + b).\star \\
(\lambda x.t) + (\lambda x.u) &\longrightarrow \lambda x.(t + u) \\
\langle t, u \rangle + \langle v, w \rangle &\longrightarrow \langle t + v, u + w \rangle \\
\delta_{\vee}(t + u, x.v, y.w) &\longrightarrow \delta_{\vee}(t, x.v, y.w) + \delta_{\vee}(u, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
a \bullet b.\star &\longrightarrow (a \times b).\star \\
a \bullet \lambda x.t &\longrightarrow \lambda x.a \bullet t \\
a \bullet \langle t, u \rangle &\longrightarrow \langle a \bullet t, a \bullet u \rangle \\
\delta_{\vee}(a \bullet t, x.v, y.w) &\longrightarrow a \bullet \delta_{\vee}(t, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
\delta_{\top}(a.\star, t) &\longrightarrow a \bullet t \\
(\lambda x.t) u &\longrightarrow (u/x)t \\
\delta_{\wedge}^1(\langle t, u \rangle, x.v) &\longrightarrow (t/x)v \\
\delta_{\wedge}^2(\langle t, u \rangle, x.v) &\longrightarrow (u/x)v \\
\delta_{\vee}(\text{inl}(t), x.v, y.w) &\longrightarrow (t/x)v \\
\delta_{\vee}(\text{inr}(u), x.v, y.w) &\longrightarrow (u/y)w
\end{aligned}$$

$$\begin{aligned}
a.\star + b.\star &\longrightarrow (a + b).\star \\
(\lambda x.t) + (\lambda x.u) &\longrightarrow \lambda x.(t + u) \\
\langle t, u \rangle + \langle v, w \rangle &\longrightarrow \langle t + v, u + w \rangle \\
\delta_{\vee}(t + u, x.v, y.w) &\longrightarrow \delta_{\vee}(t, x.v, y.w) + \delta_{\vee}(u, x.v, y.w)
\end{aligned}$$

$$\begin{aligned}
a \bullet b.\star &\longrightarrow (a \times b).\star \\
a \bullet \lambda x.t &\longrightarrow \lambda x.a \bullet t \\
a \bullet \langle t, u \rangle &\longrightarrow \langle a \bullet t, a \bullet u \rangle \\
\delta_{\vee}(a \bullet t, x.v, y.w) &\longrightarrow a \bullet \delta_{\vee}(t, x.v, y.w)
\end{aligned}$$

Business as usual

Confluence

Termination

Introduction property

Subject reduction

Where multiplicative conjunction eliminated

$$\langle t, u \rangle \multimap \langle v, w \rangle \longrightarrow \langle t \multimap v, u \multimap w \rangle$$

Vectors

- ▶ One closed irreducible proof $a.\star$ of \top for each scalar a
Closed irreducible proofs of \top in one-to-one correspondence with \mathcal{S}
- ▶ Closed irreducible proofs of $\top \wedge \top$ in one-to-one correspondence with \mathcal{S}^2
- ▶ Closed irreducible proofs of $(\top \wedge \top) \wedge \top$ (and also $\top \wedge (\top \wedge \top)$) in one-to-one correspondence with \mathcal{S}^3
- ▶ ...

\mathcal{V} smallest set of propositions closed by \top and \wedge

Vectors

- ▶ A vector space of finite dimension n
- ▶ isomorphic to \mathcal{S}^n
- ▶ in one-to-one correspondence with the closed irreducible proofs $A \in \mathcal{V}$ with $n \top$

Each proof t represents a vector \underline{t}

Each vector u represents a proof \bar{u}

Basis-dependent (like matrices)

Then, $\underline{u + v} = \underline{u} + \underline{v}$ and $\underline{a \bullet u} = a \underline{u}$

$$\langle \underline{t}, \underline{u} \rangle + \langle \underline{v}, \underline{w} \rangle \longrightarrow \langle \underline{t + v}, \underline{u + w} \rangle$$

commutation rule between sum and \wedge -i

Also a vector calculation rule

Linear functions

$A, B \in \mathcal{V}$ with m and $n \top$

M a matrix with m columns and n lines

then there exists a closed proof t of $A \Rightarrow B$ such that, for all $u \in \mathcal{S}^m$

$$\underline{t} \bar{u} = Mu$$

Induction on A (m): blocks of columns

Example: $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ expressed as

$$t = \lambda x. (\delta_{\wedge}^1(x, y. \delta_{\top}(y, \langle a.\star, b.\star \rangle)) \mathbf{+} \delta_{\wedge}^2(x, z. \delta_{\top}(z, \langle c.\star, d.\star \rangle)))$$

And

$$t \langle e.\star, f.\star \rangle \longrightarrow^* \langle (a \times e + c \times f).\star, (b \times e + d \times f).\star \rangle$$

Converse: the Main theorem

$A \in \mathcal{V}$ and $B \in \mathcal{V}$

t closed proof of $A \Rightarrow B$

Then

$$t (u_1 \dagger u_2) \equiv (t u_1) \dagger (t u_2)$$

$$t (a \bullet u_1) \equiv a \bullet (t u_1)$$

Proof: read the paper

Future work: relax the condition $B \in \mathcal{V}$

Questions: `alejandro@diaz-caro.info`
`gilles.dowek@ens-paris-saclay.fr`