Linear lambda-calculus is linear

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Linear logic: something to do with algebraic linearity

A proof of a linear implication between A and B: ~~any~~ a linear function mapping proofs of A to proofs of B

Fruitfully exploited to build models of linear logic

Difficult to even formulate within the proof language:
$f(u+v) = f(u) + f(v)$, $f(a.u) = a.f(u)$
What are $+$ and $.$?

Changed with quantum programming languages (our motivation):
mix programming language constructions (abstraction, application) with algebraic operations ($+$, $.$)

This work

A extension of linear logic with $+$ and $.$

A linearity theorem: if f is a proof of an implication between two propositions (of some specific form), then

$$f(u+v) = f(u) + f(v)$$

$$f(a.u) = a.f(u)$$

A first step: interstitial rules

Deduction rule: premises = conclusion

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A}{\Gamma \vdash A}\ \text{sum} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A}\ \text{prod}$$

Constructors **+** and • in the proof language
That express the addition and the multiplication by a scalar

Trivial = internal

Commuting cuts

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash A} \quad \dfrac{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge\text{-i} \quad \dfrac{\dfrac{\pi_3}{\Gamma \vdash A} \quad \dfrac{\pi_4}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge\text{-i}}{\dfrac{\Gamma \vdash A \wedge B \qquad \dfrac{\pi_5}{\Gamma, A \vdash C}}{\Gamma \vdash C}\ \wedge\text{-e1}}\ \text{sum}$$

Needs to commute sum either with $\wedge$-e1 or with $\wedge$-i

Here commutation with introductions when possible (all but $\vee$)

Stronger introduction property

Second step: scalars

A field $\mathcal{S}$ of scalars

Replace

- the rule $\top$-i with a family of rules $\top$-i(a), one for each scalar
- the rule prod with a family of rules prod(a), one for each scalar

A choice of linear connectives

Intuitionistic linear logic: no multiplicative falsehood, no additive implication, no multiplicative disjunction

- multiplicative truth (additive truth not useful)
- additive falsehood
- multiplicative implication
- additive conjunction (multiplicative conjunction not possible)
- additive disjunction

Sum rule additive

Often written 1, 0, $\multimap$, &, and $\oplus$
We use the usual $\top$, $\bot$, $\Rightarrow$, $\wedge$, and $\vee$ (but notations are arbitrary)

$$\frac{}{x : A \vdash x : A}\ \text{ax} \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash t \mathbin{\boldsymbol{+}} u : A}\ \text{sum} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash a \bullet t : A}\ \text{prod}(a)$$

$$\frac{}{\vdash a.\star : \top}\ \top\text{-i}(a) \qquad \frac{\Gamma \vdash t : \top \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash \delta_\top(t, u) : A}\ \top\text{-e} \qquad \frac{\Gamma \vdash t : \bot}{\Gamma, \Delta \vdash \delta_\bot(t) : C}\ \bot\text{-e}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}\ \Rightarrow\text{-i} \qquad \frac{\Gamma \vdash t : A \Rightarrow B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\ u : B}\ \Rightarrow\text{-e}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \wedge B}\ \wedge\text{-i}$$

$$\frac{\Gamma \vdash t : A \wedge B \quad \Delta, x : A \vdash u : C}{\Gamma, \Delta \vdash \delta^1_\wedge(t, x.u) : C}\ \wedge\text{-e1} \qquad \frac{\Gamma \vdash t : A \wedge B \quad \Delta, x : B \vdash u : C}{\Gamma, \Delta \vdash \delta^2_\wedge(t, x.u) : C}\ \wedge\text{-e2}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash inl(t) : A \vee B}\ \vee\text{-i1} \qquad \frac{\Gamma \vdash t : B}{\Gamma \vdash inr(t) : A \vee B}\ \vee\text{-i2}$$

$$\frac{\Gamma \vdash t : A \vee B \quad \Delta, x : A \vdash u : C \quad \Delta, y : B \vdash v : C}{\Gamma, \Delta \vdash \delta_\vee(t, x.u, y.v) : C}\ \vee\text{-e}$$

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\begin{aligned}
\delta_\top(a.\star, t) &\longrightarrow a \bullet t \\
(\lambda x.t)\ u &\longrightarrow (u/x)t \\
\delta^1_\wedge(\langle t, u\rangle, x.v) &\longrightarrow (t/x)v \\
\delta^2_\wedge(\langle t, u\rangle, x.v) &\longrightarrow (u/x)v \\
\delta_\vee(inl(t), x.v, y.w) &\longrightarrow (t/x)v \\
\delta_\vee(inr(u), x.v, y.w) &\longrightarrow (u/y)w
\end{aligned}
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\begin{aligned}
a.\star \mathbin{\textbf{+}} b.\star &\longrightarrow (a+b).\star \\
(\lambda x.t) \mathbin{\textbf{+}} (\lambda x.u) &\longrightarrow \lambda x.(t \mathbin{\textbf{+}} u) \\
\langle t, u\rangle \mathbin{\textbf{+}} \langle v, w\rangle &\longrightarrow \langle t \mathbin{\textbf{+}} v, u \mathbin{\textbf{+}} w\rangle \\
\delta_\vee(t \mathbin{\textbf{+}} u, x.v, y.w) &\longrightarrow \delta_\vee(t, x.v, y.w) \mathbin{\textbf{+}} \delta_\vee(u, x.v, y.w)
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a.\star + b.\star &\longrightarrow (a+b).\star \\
(\lambda x.t) + (\lambda x.u) &\longrightarrow \lambda x.(t + u) \\
\langle t, u\rangle + \langle v, w\rangle &\longrightarrow \langle t + v, u + w\rangle \\
\delta_\vee(t + u, x.v, y.w) &\longrightarrow \delta_\vee(t, x.v, y.w) + \delta_\vee(u, x.v, y.w) \\
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Business as usual

Confluence

Normalization

Introduction property

Subject reduction
Where multiplicative conjunction eliminated

$$\langle t, u \rangle + \langle v, w \rangle \longrightarrow \langle t + v, u + w \rangle$$

Vectors

- One closed irreducible proof $a.\star$ of $\top$ for each scalar a
  Closed irreducible proofs of $\top$ in one-to-one correspondence with $\mathcal{S}$
- Closed irreducible proofs of $\top \wedge \top$ in one-to-one correspondence with $\mathcal{S}^2$
- Closed irreducible proofs of $(\top \wedge \top) \wedge \top$ (and also $\top \wedge (\top \wedge \top)$) in one-to-one correspondence with $\mathcal{S}^3$
- ...

$\mathcal{V}$ smallest set of propositions closed by $\top$ and $\wedge$

Vectors

- A vector space of finite dimension n
- isomorphic to $\mathcal{S}^n$
- in one-to-one correspondence with the closed irreducible proofs $A \in \mathcal{V}$ with n $\top$

Each proof t represents a vector $\underline{t}$
Each vector u represents a proof $\overline{\mathrm{u}}$
Basis-dependent (like matrices)

Then, $\underline{u \boldsymbol{+} v} = \underline{u} + \underline{v}$ and $\underline{a \bullet u} = a\underline{u}$

$$\langle t, u\rangle \boldsymbol{+} \langle v, w\rangle \longrightarrow \langle t \boldsymbol{+} v, u \boldsymbol{+} w\rangle$$

commutation rule between sum and $\wedge$-i
Also a vector calculation rule

Linear functions

$A, B \in \mathcal{V}$ with m and n $\top$

M a matrix with m columns and n lines

then there exists a closed proof t of $A \Rightarrow B$ such that, for all $\mathrm{u} \in \mathcal{S}^m$

$$\underline{t\ \overline{\mathrm{u}}} = M\mathrm{u}$$

Induction on A (m): blocks of columns

Example: $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ expressed as

$$t = \lambda x.(\delta^1_\wedge(x, y.\delta_\top(y, \langle a.\star, b.\star\rangle)) \mathbin{\textbf{+}} \delta^2_\wedge(x, z.\delta_\top(z, \langle c.\star, d.\star\rangle)))$$

And

$$t\ \langle e.\star, f.\star\rangle \longrightarrow^* \langle(a \times e + c \times f).\star, (b \times e + d \times f).\star\rangle$$

Converse: the Main theorem

A ~~$\in \mathcal{V}$~~ and $B \in \mathcal{V}$
t closed proof of $A \Rightarrow B$

Then

$$t\ (u_1 + u_2) \equiv (t\ u_1) + (t\ u_2)$$

$$t\ (a \bullet u_1) \equiv a \bullet (t\ u_1)$$

Proof sketch

By induction on the size $\mu(t)$ of the proof

Three nested case analyses

- t can be a variable, a sum, a product, an introduction, or an elimination
- If t is an elimination $t = K[v]$ K eliminations only and v not an elimination v is a variable, a sum, or a product
- If v is a variable last elimination of K can be $\top$-e, $\wedge$-e...

Do not generalize too much

If B not in $\mathcal{V}$: $A = \top$ and $B = (\top \Rightarrow \top) \Rightarrow \top$

$t = \lambda x.\lambda y.(y\ x)$ proof of $A \Rightarrow B$ but

$$t\ (1.\star + 2.\star) \longrightarrow^* \lambda y.(y\ 3.\star)$$

$$t\ 1.\star + t\ 2.\star \longrightarrow^* \lambda y.((y\ 1.\star) + (y\ 2.\star))$$

And chose your deductions rule carefully

If $\top$-i were additive, $\lambda x.\ (1.\star)$ would be a proof of $\top \Rightarrow \top$

If sum were multiplicative, $\lambda x.\ (x \mathbin{\boldsymbol{+}} 1.\star)$ would be a proof of $\top \Rightarrow \top$