Cut elimination for Zermelo set theory: Proof of 53 easy lemmas

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In this note we give the proof of 53 easy lemmas used to establish Theorem 1 in our paper entitled *Cut elimination for Zermelo set theory*.

Proposition 1. *The twenty seven formulæ of Table 1 are derivable.*

Proof. 1. The formula $x = x$ reduces to $\forall p\ (\mathrm{mem}(x,p) \Rightarrow \mathrm{mem}(x,p))$ that is derivable by purely logical means.

2. The formula $y = z$ rewrites to $\forall p\ (\mathrm{mem}(y,p) \Rightarrow \mathrm{mem}(z,p))$. We apply it to the term $g_{x,y_1,\ldots,y_n,P}(y_1,\ldots,y_n)$ and we get $(P(x \leftarrow y) \Rightarrow P(x \leftarrow z))$ (where $y_1,\ldots,y_n$ are the remaining variables of P).

3. The formula $a \approx a$ rewrites to

$$\begin{array}{ll}\exists r\ [\ \mathrm{rel}(\mathrm{root}(a),\mathrm{root}(a),r) & \wedge \\ \quad \forall x \forall x' \forall y\ ((x'\ \eta_a\ x \wedge \mathrm{rel}(x,y,r)) \Rightarrow \exists y'\ (y'\ \eta_a\ y \wedge \mathrm{rel}(x',y',r))) & \wedge \\ \quad \forall y \forall y' \forall x\ ((y'\ \eta_a\ y \wedge \mathrm{rel}(x,y,r)) \Rightarrow \exists x'\ (x'\ \eta_a\ x \wedge \mathrm{rel}(x',y',r)))\ ] & \end{array}$$

We prove it for the relation $g'_{x,y,x=y}$.

4. Assume $a \approx b$. There exists a relation r which is a bisimulation from a to b. Then take

$$r' = g'_{x,y,r,\mathrm{rel}(y,x,r)}(r)$$

as a bisimulation from b to a.

5. Assume $a \approx b$ and $b \approx c$. Consider bisimulations r and r' from a to b and from b to c, respectively. Then take

$$r'' = g'_{x,z,r,r',\exists y\ (\mathrm{rel}(x,y,r) \wedge \mathrm{rel}(y,z,r'))}(r,r')$$

as a bisimulation from a to c.

6. The formula $a \approx (a/\mathrm{root}(a))$ is convertible to the formula $a \approx a$, which holds since 3.

7.–13. For injectivity of S, i and j, we use the function symbols *Pred*, i' and j'. For non confusion, we use the predicate symbols *Null*, I and J.

14.–27. The formula $x\ \eta_{\bigcup(a)}\ i(y')$, that rewrites to

$$\begin{array}{l}(\exists y \exists z'\ (x = i(y) \wedge i(y') = i(z') \wedge y\ \eta_a\ z')) \\ \vee\ \ (\exists y \exists z\ (x = i(y) \wedge i(y') = o \wedge y\ \eta_a\ z \wedge z\ \eta_a\ \mathrm{root}(a)),\end{array}$$

Node identity

1. $x = x$
2. $y = z \Rightarrow (P(x \leftarrow y) \Rightarrow P(x \leftarrow z)) \quad (*)$

Bisimilarity

3. $a \approx a$
4. $a \approx b \Rightarrow b \approx a$
5. $(a \approx b \wedge b \approx c) \Rightarrow a \approx c$
6. $a \approx (a/\operatorname{root}(a))$

Injectivity and non confusion

7. $S(x) = S(y) \Rightarrow x = y$
8. $\neg 0 = S(x)$
9. $i(x) = i(y) \Rightarrow x = y$
10. $j(x) = j(y) \Rightarrow x = y$
11. $\neg i(x) = o$
12. $\neg j(x) = o$
13. $\neg i(x) = j(y)$

Eta simplification

14. $x\ \eta_{\bigcup(a)}\ i(y') \Leftrightarrow \exists y\ (x = i(y) \wedge y\ \eta_a\ y')$
15. $x\ \eta_{\bigcup(a)}\ o \Leftrightarrow \exists y\ \exists z\ (x = i(y) \wedge y\ \eta_a\ z \wedge z\ \eta_a\ \operatorname{root}(a))$
16. $x\ \eta_{\{a,b\}}\ i(y') \Leftrightarrow \exists y\ (x = i(y) \wedge y\ \eta_a\ y')$
17. $x\ \eta_{\{a,b\}}\ j(y') \Leftrightarrow \exists y\ (x = j(y) \wedge y\ \eta_b\ y')$
18. $x\ \eta_{\{a,b\}}\ o \Leftrightarrow (x = i(\operatorname{root}(a)) \vee x = j(\operatorname{root}(b)))$
19. $x\ \eta_{\mathfrak{P}(a)}\ i(y') \Leftrightarrow \exists y\ (x = i(y) \wedge y\ \eta_a\ y')$
20. $x\ \eta_{\mathfrak{P}(a)}\ j(\rho(c)) \Leftrightarrow \exists y\ (x = i(y) \wedge y\ \eta_a\ \operatorname{root}(a) \wedge (a/y) \in c)$
21. $x\ \eta_{\mathfrak{P}(a)}\ o \Leftrightarrow \exists c\ (x = j(\rho(c)))$
22. $x\ \eta_{f_{x,y_1,\ldots,y_n,P}(y_1,\ldots,y_n,a)}\ i(y') \Leftrightarrow \exists y\ (x = i(y) \wedge y\ \eta_a\ y')$
23. $x\ \eta_{f_{x,y_1,\ldots,y_n,P}(y_1,\ldots,y_n,a)}\ o \Leftrightarrow \exists y\ (x = i(y) \wedge y\ \eta_a\ \operatorname{root}(a) \wedge P(x \leftarrow (a/y)))$
24. $x\ \eta_{\Omega}\ i(y') \Leftrightarrow \exists y\ (x = i(y) \wedge y < y')$
25. $x\ \eta_{\Omega}\ o \Leftrightarrow \exists y\ (x = i(y) \wedge Nat(y))$
26. $x\ \eta_{\mathrm{Cl}(a)}\ i(y') \Leftrightarrow \exists y\ (x = i(y) \wedge y\ \eta_a\ y')$
27. $x\ \eta_{\mathrm{Cl}(a)}\ o \Leftrightarrow$
$$\exists y\ (x = i(y) \wedge \forall c\ [\forall z\ (z\ \eta_a\ \operatorname{root}(a) \Rightarrow \operatorname{mem}(z,c)) \wedge \forall z\ \forall z'\ ((z\ \eta_a\ z' \wedge \operatorname{mem}(z',c)) \Rightarrow \operatorname{mem}(z,c)) \ \Rightarrow\ \operatorname{mem}(y,c)])$$

$(*)$ Where P is any formula of the language of $\mathsf{IZ}^{\mathrm{mod}}$ that contains no function symbol of the form $g_{\ldots}$ or $g'_{\ldots}$.

Table 1.

Membership

28. $x\ \eta_a\ \ \text{root}(a) \Rightarrow (a/x) \in a$
29. $a \approx b \Rightarrow \forall x\ (x\ \eta_a\ \ \text{root}(a) \Rightarrow \exists y\ (y\ \eta_b\ \ \text{root}(b) \wedge (a/x) \approx (b/y)))$
30. $(a \in b \wedge a \approx c) \Rightarrow c \in b$
31. $(a \in b \wedge b \approx c) \Rightarrow a \in c$

Substitutivity

32. $(P(x \leftarrow a) \wedge a \approx b) \Rightarrow P(x \leftarrow b) \quad (*)$

Bisimilarity by relocation

33. $(\text{root}(b) = i(\text{root}(a)) \wedge \forall x \forall y'\ (y'\ \eta_b\ i(x) \Leftrightarrow \exists x'\ (y' = i(x') \wedge x'\ \eta_a\ x))) \Rightarrow a \approx b$
34. $(\text{root}(b) = j(\text{root}(a)) \wedge \forall x \forall y'\ (y'\ \eta_b\ j(x) \Leftrightarrow \exists x'\ (y' = j(x') \wedge x'\ \eta_a\ x))) \Rightarrow a \approx b$

Embedding

35. $\bigcup(a)/i(y) \approx (a/y)$
36. $(\{a,b\}/i(\text{root}(a))) \approx a$
37. $(\{a,b\}/j(\text{root}(b))) \approx b$
38. $\mathfrak{P}(a)/i(y) \approx (a/y)$
39. $f_{x,y_1,\ldots,y_p,P}(a_1,\ldots,a_p,b)/i(y) \approx (b/y)$
40. $\text{Cl}(a)/i(y) \approx (a/y)$

Extensionality

41. $P(c,d)$
$\wedge\ \ (\forall a \forall a' \forall b\ ((a' \in a \wedge P(a,b)) \Rightarrow \exists b'(b' \in b \wedge P(a',b'))))$
$\wedge\ \ (\forall a \forall b \forall b'\ ((b' \in b \wedge P(a,b)) \Rightarrow \exists a'(a' \in a \wedge P(a',b'))))$
$\Rightarrow\ \ (c \approx d) \quad (*)$

Finitary existence axioms

42. $c \in \bigcup(a) \Leftrightarrow \exists b\ (c \in b \wedge b \in a)$
43. $c \in \{a,b\} \Leftrightarrow (c \approx a \vee c \approx b)$
44. $a \in \mathfrak{P}(b) \Leftrightarrow \forall c\ (c \in a \Rightarrow c \in b)$
45. $a \in f_{x,y_1,\ldots,y_p,P}(y_1,\ldots,y_p,b) \Leftrightarrow a \in b \wedge P(x \leftarrow a) \quad (*)$

Infinity

46. $\neg a \in \varnothing$
47. $\varnothing \approx (\Omega/i(0))$
48. $(a \approx (\Omega/i(y))) \Rightarrow \bigcup(\{a,\{a\}\}) \approx (\Omega/i(S(y)))$
49. $\varnothing \in \Omega$
50. $a \in \Omega \Rightarrow \bigcup(\{a,\{a\}\}) \in \Omega$
51. $Ind(\Omega)$

Transitive closure

52. $a \in c \Rightarrow a \in \text{Cl}(c)$
53. $a \in b \Rightarrow b \in \text{Cl}(c) \Rightarrow a \in \text{Cl}(c)$

$(*)$ Where P is any formula expressed in the language $\approx$, $\in$ and where all the quantifiers are of sort G.

Table 2.

is equivalent, using injectivity of i (9.) and non confusion of i and o (11.) to

$$\begin{array}{l}(\exists y \exists z' \ (x = i(y) \wedge y' = z' \wedge y \ \eta_a \ z')) \\ \vee \ \ (\exists y \exists z \ (x = i(y) \wedge \bot \wedge y \ \eta_a \ z \wedge z \ \eta_a \ \operatorname{root}(a))\end{array}$$

Using trivial properties of connectors and quantifiers, this formula simplifies to

$$\exists y \exists z' \ (x = i(y) \wedge y' = z' \wedge y \ \eta_a \ z')$$

and finally to

$$\exists y \ (x = i(y) \wedge y \ \eta_a \ y')$$

The other formulæ are proved in the same way. □

Proposition 2. *The twenty-six formulæ of table 2 are derivable.*

Proof. 28. Assume $x \ \eta_a \ \operatorname{root}(a)$, the formula $(a/x) \in a$ rewrites to

$$\exists y \ (y \ \eta_a \ \operatorname{root}(a) \wedge (a/x) \approx (a/y))$$

that holds for $y = x$ using reflexivity of bisimilarity.

29. Assume $a \approx b$ and $x \ \eta_a \ \operatorname{root}(a)$. We want to prove

$$\exists y \ (y \ \eta_b \ \operatorname{root}(b) \wedge (a/x) \approx (b/y))$$

We have $a \approx b$ hence there is r such that

$$\begin{array}{rl}(i) & \operatorname{rel}(\operatorname{root}(a), \operatorname{root}(b), r) \\ (ii) & \forall x \forall x' \forall y \ (x' \ \eta_a \ x \wedge \operatorname{rel}(x, y, r) \Rightarrow \exists y' \ (y' \ \eta_b \ y \wedge \operatorname{rel}(x', y', r))) \\ (iii) & \forall y \forall y' \forall x \ (y' \ \eta_b \ y \wedge \operatorname{rel}(x, y, r) \Rightarrow \exists x' \ (x' \ \eta_a \ x \wedge \operatorname{rel}(x', y', r)))\end{array}$$

We apply (ii) to $\operatorname{root}(a)$, x, $\operatorname{root}(b)$, and to the assumptions $x \ \eta_a \ \operatorname{root}(a)$ and (i) and we get a node y such that $y \ \eta_b \ \operatorname{root}(b)$ and $\operatorname{rel}(x, y, r)$. We then have to prove

$$(a/x) \approx (b/y)$$

This is a consequence of (ii), (iii) and $\operatorname{rel}(x, y, r)$, which is equivalent to $\operatorname{rel}(\operatorname{root}(a/x), \operatorname{root}(b/y), r)$.

30. is a consequence of 29. and transitivity of bisimilarity.

31. is a consequence of symmetry and transitivity of bisimilarity.

32. By induction on the structure of P, using properties of bisimilarity and compatibility of $\in$ w.r.t. $\approx$.

33. Assume

$$\begin{array}{rl}(i) & \operatorname{root}(b) = i(\operatorname{root}(a)) \\ (ii) & \forall x \forall y' \ (y' \ \eta_b \ i(x) \Rightarrow \exists x' \ (y' = i(x') \wedge x' \ \eta_a \ x)) \\ (iii) & \forall x \forall y' \ ((\exists x' \ (y' = i(x') \wedge x' \ \eta_a \ x)) \Rightarrow y' \ \eta_b \ i(x))\end{array}$$

We prove $a \approx b$ by showing that the relation $g'_{x,y,y=i(x)}$ is a bisimulation. The fact that $\mathrm{root}(b) = i(\mathrm{root}(a))$ is hypothesis (i). We have to prove

$$\forall x \forall x' \ (x' \ \eta_a \ x \Rightarrow i(x') \ \eta_b \ i(x))$$

and

$$\forall x \forall y' \ (y' \ \eta_b \ i(x) \Rightarrow \exists x' \ (x' \ \eta_a \ x \wedge y' = i(x')))$$

The second formula is (ii). Let us prove the first. In (iii), we instantiate y' by $i(x')$ and x by x we obtain

$$\exists z \ (i(x') = i(z) \wedge z \ \eta_a \ x) \Rightarrow i(x') \ \eta_b \ i(x)$$

that is equivalent, using injectivity of i, to

$$x' \ \eta_a \ x \Rightarrow i(x') \ \eta_b \ i(x) \, .$$

34. Similar to 33.

35. We use 33. with the terms a/y and $\bigcup(a)/i(y)$. Thus we have to prove

$$\mathrm{root}(\bigcup(a)/i(y)) = i(\mathrm{root}(a/y))$$

and

$$y' \ \eta_{\bigcup(a)/i(y)} \ i(x) \Leftrightarrow \exists x' \ (y' = i(x') \wedge x' \ \eta_{a/y} \ x)$$

The first formula rewrites to $i(y) = i(y)$ and the second to

$$y' \ \eta_{\bigcup(a)} \ i(x) \Leftrightarrow \exists x' \ (y' = i(x') \wedge x' \ \eta_a \ x)$$

The formula $y' \ \eta_{\bigcup(a)} \ i(x)$ rewrites to the disjunction

$$\begin{array}{l} (\exists x' \exists w \ (y' = i(x') \wedge i(x) = i(w) \wedge x' \ \eta_a \ w)) \\ \vee \ (\exists x' \exists z \ (y' = i(x') \wedge i(x) = o \wedge x' \ \eta_a \ z \wedge z \ \eta_a \ \mathrm{root}(a))) \end{array}$$

whose second alternative vanishes (from non confusion) and whose remaining alternative is equivalent to

$$\exists x' \ (y' = i(x') \wedge x' \ \eta_a \ x) \, .$$

36.–40. are proved in a similar way, using 33. and 34.

41. Assume

(i) $P(c, d)$
(ii) $\forall a \forall a' \forall b \ ((a' \in a \wedge P(a, b)) \Rightarrow \exists b' \ (b' \in b \wedge P(a', b')))$
(iii) $\forall a \forall b \forall b' \ ((b' \in b \wedge P(a, b)) \Rightarrow \exists a' \ (a' \in a \wedge P(a', b')))$

We prove $c \approx d$ by showing that the relation $g'_{x,y,P(c/x,d/y)}$ is a bisimulation, *i.e.*

- $P(c/\,\mathrm{root}(c), d/\,\mathrm{root}(d))$
- $\forall x \ \forall x' \ \forall y \ (x' \ \eta_c \ x \wedge P(c/x, d/y) \ \Rightarrow \ \exists y' \ (y' \ \eta_d \ y \wedge P(c/x', d/y')))$

– $\forall y\ \forall y'\ \forall x\ (y'\ \eta_d\ y \wedge P(c/x, d/y)\ \Rightarrow\ \exists x'\ (x'\ \eta_c\ x \wedge P(c/x', d/y')))$

The first formula is a consequence of (i), 6. and 32. To prove the second, assume $x'\ \eta_c\ x$ and $P(c/x, d/y)$. We have to prove

$$\exists y'\ (y'\ \eta_d\ y \wedge P(c/x', d/y'))$$

We apply (ii) to (c/x), (c/x') and (d/y). We obtain

$$(c/x' \in c/x \wedge P(c/x, d/y)) \Rightarrow \exists b'\ (b' \in d/y \wedge P(c/x', b'))$$

By 28., $(c/x') \in (c/x)$ (since $x'\ \eta_c\ x$) and $P(c/x, d/y)$ is an hypothesis, thus there exists b' such that

$$b' \in d/y \wedge P(c/x', b').$$

The latter formula rewrites to

$$\exists y'\ (y'\ \eta_d\ y \wedge b' \approx (d/y')) \wedge P(c/x', b')\,.$$

We conclude with 32. The third item is proved in a similar way.

Equivalence by elementary means. In the remaining proofs we will need to use the same proof pattern several times. We have a formula of the form $t \in u$ where u is a term formed with a function symbol, for instance the formula $c \in \bigcup(a)$. We can first rewrite such a formula with the rule defining the symbol $\in$. In our example, this yields the formula

$$\exists x\ (x\ \eta_{\bigcup(a)}\ \text{root}(\bigcup(a)) \wedge c \approx (\bigcup(a)/x))$$

then we rewrite the term $\text{root}(\bigcup(a))$ to o

$$\exists x\ (x\ \eta_{\bigcup(a)}\ o \wedge c \approx (\bigcup(a)/x))$$

and we use 14.–37. (in this case 15.) to replace the formula $x\ \eta_{\bigcup(a)}\ o$ by an equivalent one involving simpler sets

$$\exists y \exists z\ (y\ \eta_a\ z \wedge z\ \eta_a\ \text{root}(a) \wedge c \approx (\bigcup(a)/i(y)))$$

Finally, we use embedding (35.–40.), the fact that bisimilarity is an equivalence relation (3.–5.) and that it is compatible with $\in$ (30.–31.) to simplify terms such as $\bigcup(a)/i(y)$ to a/y. In this case, we get

$$\exists y \exists z\ (y\ \eta_a\ z \wedge z\ \eta_a\ \text{root}(a) \wedge c \approx (a/y))$$

We shall say that the formula obtained this way is *equivalent by elementary means* to the formula we started with.

42. As we said above, $c \in \bigcup(a)$ is equivalent by elementary means to

$$\exists y \exists z\ (y\ \eta_a\ z \wedge z\ \eta_a\ \text{root}(a) \wedge c \approx (a/y))$$

Then it remains to prove that this formula is equivalent to

$$\exists b\ (c \in b \wedge b \in a)\,.$$

(Direct implication) First assume that there are y and z such that $y\ \eta_a\ z$, $z\ \eta_a\ \text{root}(a)$ and $c \approx (a/y)$. Take $b = a/z$. We check easily that $c \in b$ (by 28. and 30.) and $b \in a$ (by 28.).

(Converse implication) Assume that $c \in b$ and $b \in a$. As $b \in a$, there exists z such that $z\ \eta_a\ \text{root}(a)$ and $b \approx (a/z)$. Similarly, there exists x such that $x\ \eta_b\ \text{root}(b)$ and $c \approx (b/x)$. Applying 29. to b, (a/z) and x, we get that there exists y such that $y\ \eta_a\ z$ and $(b/x) \approx (a/y)$. Thus $c \approx (a/y)$.

43. The formula $c \in \{a, b\}$ is equivalent by elementary means to

$$c \approx a \vee c \approx b\,.$$

44. The formula $a \in \mathfrak{P}(b)$ is equivalent by elementary means to

$$\exists e\ (a \approx (\mathfrak{P}(b)/j(\rho(e))))$$

Using extensionality (that follows from 41. as noticed in section 1) this formula is equivalent to

$$\exists e \forall c\ (c \in a \Leftrightarrow c \in (\mathfrak{P}(b)/j(\rho(e))))$$

Then notice that this formula is equivalent by elementary means again to

$$\exists e \forall c\ (c \in a \Leftrightarrow \exists y\ (y\ \eta_b\ \text{root}(b) \wedge (b/y) \in e \wedge c \approx (b/y)))$$

This formula is equivalent by 30. to

$$\exists e \forall c\ (c \in a \Leftrightarrow (\exists y\ (y\ \eta_b\ \text{root}(b) \wedge c \in e \wedge c \approx (b/y))))$$

that is equivalent to

$$\exists e \forall c\ (c \in a \Leftrightarrow (c \in b \wedge c \in e))$$

that is logically equivalent to $\forall c\ (c \in a \Rightarrow c \in b)$.

45. First notice that the formula $a \in f_{x,y_1,\ldots,y_p,P}(y_1, \ldots, y_p, b)$ is equivalent by elementary means to

$$\exists y\ (y\ \eta_b\ \text{root}(b) \wedge P(x \leftarrow (b/y)) \wedge a \approx (b/y))$$

Using formula 32. this is equivalent to

$$\exists y\ (y\ \eta_b\ \text{root}(b) \wedge P(x \leftarrow a) \wedge a \approx (b/y))$$

and thus to $a \in b \wedge P(x \leftarrow a)$.

46. Is a consequence of 45.

47. Using extensionality, we prove that the formula $c \in (\Omega/i(0))$ is contradictory. First notice that this formula is equivalent by elementary means to

$$\exists y\ (y < 0 \wedge c \approx (\Omega/i(y)))$$

which rewrites to $\exists y\ (\bot \wedge c \approx (\Omega/i(y)))$.

48. Assume $a \approx (\Omega/i(y))$ we want to prove

$$\bigcup(\{a, \{a\}\}) \approx (\Omega/i(S(y)))$$

Using extensionality, 42. and 43. and 30. this is equivalent to

$$\forall c\ [(c \in a \vee c \approx a) \Leftrightarrow c \in (\Omega/i(S(y)))]$$

The formula

$$c \in (\Omega/i(S(y)))$$

is equivalent by elementary means to

$$\exists z\ (z < S(y) \wedge c \approx (\Omega/i(z)))$$

This rewrites to

$$\exists z\ ((z < y \vee z = y) \wedge c \approx (\Omega/i(z)))$$

which is equivalent to

$$\exists z\ (z < y \wedge c \approx \Omega/i(z)) \vee c \approx \Omega/i(y)$$

Notice that the formula $c \in \Omega/i(y)$ is equivalent by elementary means to

$$\exists z\ (z < y \wedge c \approx (\Omega/i(z)))$$

Hence the formula $c \in \Omega/i(S(y))$ is equivalent to

$$c \in \Omega/i(y) \vee c \approx \Omega/i(y)$$

and finally, by 31., to $c \in a \vee c \approx a$.

49. Using 47. and 30. it suffices to prove that $i(0)\ \eta_\Omega\ o$ which is equivalent by elementary means to $Nat(0)$, which rewrites to $\top$.

50. The formula $a \in \Omega$ is equivalent to

$$\exists y\ (Nat(y) \wedge a \approx (\Omega/i(y)))$$

consider such a y, we have

$$Nat(y)$$

and

$$a \approx (\Omega/i(y)))$$

From 48., we deduce

$$\bigcup(\{a, \{a\}\}) \approx (\Omega/i(S(y)))$$

and since $Nat(S(y))$ rewrites to $Nat(y)$, we have

$$\exists z\ (Nat(z) \wedge \bigcup\{a, \{a\}\} \approx (\Omega/z))$$

thus $\bigcup(\{a,\{a\}\}) \in \Omega$.

51. $Ind(\Omega)$ is equivalent to

$$\varnothing \in \Omega \wedge \forall a\ (a \in \Omega \Rightarrow \bigcup(\{a,\{a\}\}) \in \Omega))$$

that have been proved in 49. and 50.

52. Assume $a \in c$. There exists an x such that $x\ \eta_c\ \operatorname{root}(c)$ and $a \approx (c/x)$.

Then, let $H(c,e)$ be the formula

$$\forall z\ (z\ \eta_c\ \operatorname{root}(c) \Rightarrow \operatorname{mem}(z,e)) \wedge \forall z \forall z'\ (z\ \eta_c\ z' \wedge \operatorname{mem}(z',e) \Rightarrow \operatorname{mem}(z,e))$$

We have to prove the formula $a \in \operatorname{Cl}(c)$. This formula is equivalent by elementary means to

$$\exists y\ (\forall e\ (H(c,e) \Rightarrow \operatorname{mem}(y,e)) \wedge a \approx (c/y))\,.$$

We prove it for the object x. Thus we have to prove $\forall e\ (H(c,e) \Rightarrow \operatorname{mem}(x,e))$ and $a \approx (c/x)$. The second formula is already proved. To prove the first, consider e such that $H(c,e)$, we have to prove $\operatorname{mem}(x,e)$ that is a consequence of $H(c,e)$ and $x\ \eta_c\ \operatorname{root}(c)$.

53. Assume $a \in b$ and $b \in \operatorname{Cl}(c)$. Again, let $H(c,e)$ be the formula

$$\forall z\ (z\ \eta_c\ \operatorname{root}(c) \Rightarrow \operatorname{mem}(z,e)) \wedge \forall z \forall z'\ (z\ \eta_c\ z' \wedge \operatorname{mem}(z',e) \Rightarrow \operatorname{mem}(z,e))$$

The formula $b \in \operatorname{Cl}(c)$ is equivalent by elementary means to

$$\exists x\ (\forall e\ (H(c,e) \Rightarrow \operatorname{mem}(x,e)) \wedge b \approx (c/x))$$

Consider such an x, we have $\forall e\ (H(c,e) \Rightarrow \operatorname{mem}(x,e))$ and $b \approx (c/x)$.

As $a \in b$ there exists y such that $y\ \eta_b\ \operatorname{root}(b)$ and $a \approx (b/y)$. We have $b \approx (c/x)$ and $y\ \eta_b\ \operatorname{root}(b)$, hence by 29. there exists z such that $z\ \eta_{c/x}\ \operatorname{root}(c/x)$ and $(b/y) \approx ((c/x)/z)$. Thus $z\ \eta_c\ x$ and $(b/y) \approx (c/z)$.

We have to prove the formula $a \in \operatorname{Cl}(c)$. This formula is equivalent by elementary means to

$$\exists w\ (\forall e\ (H(c,e) \Rightarrow \operatorname{mem}(w,e)) \wedge a \approx (c/w))$$

we prove it for the object z. Thus we have to prove $\forall e\ (H(c,e) \Rightarrow \operatorname{mem}(z,e))$ and $a \approx (c/z)$. The second formula is a consequence of $a \approx (b/y)$, $(b/y) \approx (c/z)$ and transitivity of bisimilarity (5.). To prove the first, consider e such that $H(c,e)$. We have to prove $\operatorname{mem}(z,e)$. From $H(c,e)$ and $\forall e\ (H(c,e) \Rightarrow \operatorname{mem}(x,e))$ we deduce $\operatorname{mem}(x,e)$. Then, from $H(c,e)$, $z\ \eta_c\ x$ and $\operatorname{mem}(x,e)$, we deduce $\operatorname{mem}(z,e)$. □