A constructive proof of Skolem theorem for constructive logic

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Abstract. If the sequent $\Gamma \vdash \forall \overline{x} \exists y\ A$ is provable in first order constructive natural deduction, then the theory $\Gamma, \forall \overline{x}\ (f(\overline{x})/y)A$, where f is a new function symbol, is a conservative extension of Γ.

Skolem theorem asserts that if the sequent $\Gamma \vdash \forall \overline{x} \exists y\ A$ is provable in first order constructive logic, then the theory $\Gamma, \forall \overline{x}\ (f(\overline{x})/y)A$, where f is a new function symbol, is a conservative extension of Γ. That is if the sequent $\Gamma, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B$ is provable and f does not occur in B, then the sequent $\Gamma \vdash B$ is also provable. We give in this note a constructive proof, providing an algorithm transforming proofs of one sequent into proofs of the other. Our proof follows the lines of [2] with two differences. First, our logic is first-order constructive logic and not classical, second we use a formulation of the deduction rules based on natural deduction and not on sequent calculus.

We shall use two results about constructive natural deduction. First, the cut elimination theorem: when a sequent has a proof it also has a proof free of cuts, including the so-called commutative cuts. Second, the subformula property: a cut free proof of a sequent contains only subformulas of this sequent. Recall that the subformulas of a formula B are defined as follows. If B is atomic, then its only subformula is B itself. If B is $C \Rightarrow D$, $C \wedge D$ or $C \vee D$, then the subformulas of B are B and the subformulas of C and D. It B is $\forall x\ C$ or $\exists x\ C$, then the subformulas of B are B and the subformulas of all the propositions of the form $(t/x)C$ for some term t. The rules of constructive natural deduction and proofs of both results can be found, for instance, in [5].

We shall also use the admissibility of the weakening rule, i.e. the fact that if the sequent $\Gamma \vdash C$ is provable, then so is the sequent $\Gamma, B \vdash C$. This lemma can be proved by a simple induction on proof structure.

Let $\overline{x} = x_1, \ldots, x_n$ be a finite sequence of distinct variables and C a proposition, we write $\forall \overline{x}\ C$ for $\forall x_1 \ldots \forall x_n\ C$. Similarly, if $\overline{t} = t_1, \ldots, t_n$ is a finite sequence of terms and f a function symbol of arity n, we write $f(\overline{t})$ for the term $f(t_1, \ldots, t_n)$. Finally, we write $\overline{t}/\overline{x}$ for the simultaneous substitution $t_1/x_1, \ldots, t_n/x_n$.

Throughout this note, we consider a fixed proposition A where the symbol f does not occur and whose free variables are among $x_1, \ldots, x_n, y$. A *partial* *instance* is a proposition of the form $(t_1/x_1, \ldots, t_i/x_i)\forall x_{i+1} \ldots \forall x_n \ (f(\overline{x})/y)A$ with $0 \leq i < n$. A *total instance* is a proposition of the form $(\overline{t}/\overline{x})(f(\overline{x})/y)A$.

A f*-term* is a term of the form $f(\overline{t})$. The occurrence of a subterm u is said to be *frozen* in a proposition C if all variable occurrences of u are free in C. As a consequence of the subformula property, if in a sequent $\Gamma \vdash B$ all f-terms are frozen, then all f-terms are also frozen in a cut free proof of this sequent. Another consequence of the subformula property is that if all f-terms are frozen in Γ and B, then in a cut free proof of the sequent $\Gamma, \forall \overline{x} \ (f(\overline{x})/y)A \vdash B$, any proposition C is either a partial instance or all f-terms are frozen in C.

Proposition 1 (Proof pruning). *Let Γ be a context and B be a proposition where all f-terms are frozen. If the sequent $\Gamma \vdash B$ is provable, then so is the sequent $\sigma\Gamma \vdash \sigma B$, where σ is the function mapping terms to terms and propositions to propositions replacing a f-term $f(\overline{u})$ by a variable z not occurring in Γ and B.*

Proof. The sequent $\Gamma \vdash B$ is provable, thus it has a cut free proof and all the f-terms are frozen in this proof. Without loss of generality, we can assume that the variable z does not appear in this proof. By induction on the structure of this proof, we build a proof of the sequent $\sigma\Gamma \vdash \sigma B$ using, for quantifier rules, the fact that $\sigma((t/x)P) = (\sigma t/x)\sigma P$ when x occurs in no f-term of P.

Proposition 2 (Elimination of hypotheses). *Let Γ be a context and B be a proposition where all f-terms are frozen. Let $\overline{u}$ be a term sequence such that the term $f(\overline{u})$ does not occur in Γ and B. If the sequents $\Gamma \vdash \forall \overline{x} \exists y \ A$ and $\Gamma, (\overline{u}/\overline{x}, f(\overline{u})/y)A \vdash B$ are provable then so is the sequent $\Gamma \vdash B$.*

Proof. Let z be a fresh variable. Let σ be the function replacing the subterm $f(\overline{u})$ by the variable z. By Proposition 1, the sequent $\Gamma, (\overline{u}/\overline{x}, z/y)A \vdash B$ is provable. From the proofs of the sequent $\Gamma \vdash \forall \overline{x} \exists y \ A$ and $\Gamma, (\overline{u}/\overline{x}, z/y)A \vdash B$ we build a proof of the sequent $\Gamma \vdash B$.

Proposition 3 (Partial instances). *Let Γ be a context and B be a proposition where all f-terms are frozen. Let π be a cut free proof of the sequent $\Gamma, \forall \overline{x} \ (f(\overline{x})/y)A \vdash B$. If the proof π contains an occurrence of a sequent $\Delta \vdash C$, where C is a partial instance of $\forall \overline{x} \ (f(\overline{x})/y)A$, then this sequent is the premise of an elimination of the universal quantifier.*

Proof. By induction on the depth of the occurrence of the sequent $\Delta \vdash C$ in the proof. This sequent is not the conclusion of the proof as all f-terms are frozen in B, thus it is the premise of some rule.

If this rule is an introduction rule then the conclusion is a proposition where f occurs with a bound variable as one of its arguments, thus it is a partial instance of $\forall \overline{x} \ (f(\overline{x})/y)A$. Applying the induction hypothesis, the next rule is an elimination of the universal quantifier, contradicting the fact that the proof is cut free.

The sequent $\Delta \vdash C$ cannot be the minor premise of a elimination of the implication because the major premise would then be a proposition that is not a partial instance but where f occurs with a bound variable as one of its arguments.

If it is a minor premise of an elimination of the disjunction or of the existential quantifier then the conclusion of this elimination rule is of the form $\Delta' \vdash C$. Applying the induction hypothesis, the next rule is $\forall$-elim, contradicting the fact that the proof is free of commutative cuts.

Thus this sequent is the major premise of an elimination rule and this rule is the elimination of the universal quantifier.

Theorem 1. *Let Γ be a context and B be a proposition where the symbol f does not occur. If the sequents $\Gamma \vdash \forall \overline{x} \exists y\ A$ and $\Gamma, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B$ are provable, then the sequent $\Gamma \vdash B$ is provable.*

Proof. We prove, more generally, that if Γ is a context and B is a proposition where all f-terms are frozen and $f(\overline{u}_1), \ldots, f(\overline{u}_k)$ are the f-terms occurring in Γ and B, then if the sequents $\Gamma \vdash \forall \overline{x} \exists y\ A$ and $\Gamma, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B$ are provable then the sequent $\Gamma, \Delta \vdash B$ where $\Delta = (\overline{u}_1/\overline{x}, f(\overline{u}_1)/y)A, \ldots, (\overline{u}_k/\overline{x}, f(\overline{u}_k)/y)A$ is provable.

The proof proceeds by induction on the structure of a cut free proof of $\Gamma, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B$. The last rules derives $\Gamma, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B$ from premises $\Gamma_1, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B_1, \ldots, \Gamma_p, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B_p$.

- If one of the propositions $B_1, \ldots, B_p$ is a partial instance of $\forall \overline{x}\ (f(\overline{x})/y)A$, then by Proposition 3 the last rule is an elimination of the universal quantifier, the proposition B has the form $(\overline{t}/\overline{x}, f(\overline{t})/y)A$ and Δ contains B. Thus the sequent $\Gamma, \Delta \vdash B$ is provable with the axiom rule.
- Otherwise, all the f-terms are frozen in all the B_i's. We check, by a case analysis on the rule used, that all the f-terms are frozen in all the Γ_i's. By induction hypothesis, the sequents $\Gamma_1, \Delta_1 \vdash B_1, \ldots, \Gamma_p, \Delta_p \vdash B_p$ are provable where Δ_i is the set of total instances corresponding to the frozen f-terms of Γ_i and B_i. Let Δ' be the union of all these sequences. By weakening, the sequents $\Gamma_1, \Delta' \vdash B_1, \ldots, \Gamma_p, \Delta' \vdash B_p$ are provable.

  We now check that the sequent $\Gamma, \Delta' \vdash B$ can be proved from $\Gamma_1, \Delta' \vdash B_1$, $\ldots, \Gamma_p, \Delta' \vdash B_p$. The only point to check is that the eigenvariable conditions are verified in the case of the introduction rule of the universal quantifier and the elimination of the existential quantifier. In the first case, the proof has the form

$$\dfrac{\dfrac{\cdots}{\Gamma_1, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B_1}}{\Gamma_1, \forall \overline{x}\ (f(\overline{x})/y)A \vdash \forall z\ B_1}$$

  and the variable z is not free in $\Gamma_1, \forall \overline{x}\ (f(\overline{x})/y)A$. Notice that, by hypothesis, the proposition $\forall z\ B_1$ is not a partial instance. Hence, the variable z does not occur in an argument of f in B_1. As moreover z is not free in Γ_1, it is not free in Δ_1 and from $\Gamma_1, \Delta_1 \vdash B_1$, we can deduce $\Gamma_1, \Delta_1 \vdash \forall z\ B_1$.

In the second case, the proof has the form

$$\dfrac{\dfrac{\dots}{\Gamma_1, \forall \overline{x}\ (f(\overline{x})/y)A \vdash \exists z\ B_1} \quad \dfrac{\dots}{\Gamma_1, \forall \overline{x}\ (f(\overline{x})/y)A, B_1 \vdash B_2}}{\Gamma_1, \forall \overline{x}\ (f(\overline{x})/y)A \vdash B_2}$$

and the variable z is not free in $\Gamma_1, \forall \overline{x}\ (f(\overline{x})/y)A$ and B_2. Notice that the variable z does not appear in any f-term of B_1 as all f-terms are frozen in $\exists z\ B_1$. The variables free in Δ' are free in Γ_1, in B_1 or in B_2. Hence z is not free in Δ' and from $\Gamma_1, \Delta' \vdash \exists z\ B_1$ and $\Gamma_1, \Delta', B_1 \vdash B_2$ we can deduce $\Gamma_1, \Delta' \vdash B_2$.

From the proof of $\Gamma, \Delta' \vdash B$ we eliminate one by one all total instances of A corresponding to terms not occurring in Γ and B using Proposition 2, starting from the largest. Finally, we add, by weakening, the total instances corresponding to terms occurring in Γ and B that would not be in Δ' and we get this way a proof of $\Gamma, \Delta \vdash B$.

Remark. If the proposition $\forall \overline{x} \exists y\ A$ is an axiom of Γ, then we obtain a conservative extension of Γ by adding the axiom $\forall \overline{x}\ (f(\overline{x})/y)A$ to Γ. The axiom $\forall \overline{x} \exists y\ A$, that it is a consequence of $\forall \overline{x}\ (f(\overline{x})/y)A$, is then redundant and can be dropped. Thus, we obtain also a conservative extension if we replace the axiom $\forall \overline{x} \exists y\ A$ by $\forall \overline{x}\ (f(\overline{x})/y)A$.

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