Naive set theory

I. What we have seen so far

Basic notions: proof, theory, model

An example of theory: arithmetic

Not every mathematical statement can be expressed in arithmetic:
there is no bijection between $\mathbb{N}$ and $\mathbb{R}$

Theories where every mathematical statement can be expressed

Back in time

What is in a M1 course (predicate logic, proof, cut, model, completeness, incompleteness, undecidability): logic of the thirties: golden age of logic

What is in a (this) M2 course: from the seventies to now: interaction between logic and computer science

Today: back to the beginning of the 20th century: the foundational crisis

What do we need to express mathematics?

Natural numbers, sets and functions

Integers, rational numbers, real numbers, points, lines, vectors, etc. can be built from natural numbers, sets and functions
E.g. Real numbers built as sets of functions mapping natural numbers to rational numbers

Sets and functions are not both needed, sets: characteristic function, function: relation, i.e. set of ordered pairs

II. Application and membership

In arithmetic S ($+$, $\times$, *Pred*, etc.) expresses a function
But S is not a term, $S(S(0))$ is
Make S a constant thus a term

No way to build the term $S(0)$, a new symbol α for function application $\alpha(S, 0)$

Functions of several arguments

Similar function symbols α_2, α_3, ...: $\alpha_2(+, x, 0)$

Can be avoided: a function f of n arguments = a function of one argument mapping x to the function mapping $x_2, ..., x_n$ to $f(x, x_2, ..., x_n)$

$\alpha_n(f, x_1, ..., x_n)$ becomes $\alpha(...\alpha(f, x_1)..., x_n)$

$\alpha(f, x)$ often written $(f\ x)$
$(...(f\ x_1)...x_n)$ often written $(f\ x_1\ ...\ x_n)$

From classes to sets

even(0) becomes 0 ϵ *even*
when *even* becomes a constant (of sort κ)
Copula ϵ: similar to α

With classes, no classes of classes
Generalization, sets of sets ϵ becomes $\in$

From relations to propositional content

For relations: $\in_2$, $\in_3$, ...: $\in_2 (\leq, x, y)$

A symbol $\in_0$, also written ε

Build the proposition $\varepsilon(t)$ from term t expressing a relation with no arguments

The term t is the propositional content of the proposition $\varepsilon(t)$

(It is true | I know | I wish) that the sky is blue

Sets as functions

A set (a relation) can be defined as its characteristic function

E: the function mapping its argument x to the propositional content of the proposition expressing that x is an element of E

$x \in E$ written $\varepsilon(E\ x)$
$\in_2 (R, x, y)$ written $\varepsilon(R\ x\ y)$

$\in$, $\in_2$, ... not needed anymore

α and ε

III. Building functions and sets

Building a function

Informally: $3 \times x$
But ambiguous: $3 \times x$ is a multiple of 3, $3 \times x$ is monotone

$$x \mapsto 3 \times x$$

Often, only in definitions

$$f = (x \mapsto 3 \times x)$$

$(f\ 4)$, $\int_0^1 f$, etc. here also $((x \mapsto 3 \times x)\ 4)$, $\int_0^1 (x \mapsto 3 \times x)$

Building / naming

Combinators

For each term t, whose free variables are among $x_1, ..., x_n$ a constant $x_1, ..., x_n \mapsto t$

Building sets and relations (in comprehension):
For each proposition A, whose free variables are among $x_1, ..., x_n$, a constant $\{x_1, ..., x_n \mid A\}$

Axioms and rules

Apply the function $x \mapsto (x \times x) + 2$ to 7: want $(7 \times 7) + 2$
Apply the set $\{x \mid \exists y\ (x = 2 \times y)\}$ to 7: want $\exists y\ (7 = 2 \times y)$

Conversion axioms:

$$\forall x_1 ... \forall x_n\ (((x_1, ..., x_n \mapsto t)\ x_1\ ...\ x_n) = t)$$

$$\forall x_1 ... \forall x_n\ (\varepsilon(\{x_1, ..., x_n \mid A\}\ x_1\ ...\ x_n) \Leftrightarrow A)$$

In Deduction modulo theory: conversion rules

$$((x_1, ..., x_n \mapsto t)\ x_1\ ...\ x_n) \longrightarrow t$$

$$\varepsilon(\{x_1, ..., x_n \mid A\}\ x_1\ ...\ x_n) \longrightarrow A$$

Another variation: comprehension axioms

For each term t of the language, the axiom

$$\exists f \ \forall x_1 \ ... \ \forall x_n \ ((f \ x_1 \ ... \ x_n) = t)$$

For each proposition A, the axiom

$$\exists E \ \forall x_1 \ ... \ \forall x_n \ (\varepsilon(E \ x_1 \ ... \ x_n) \Leftrightarrow A)$$

IV. Russell's paradox

(Modulo minor variations) invented many times: naive set theory

Unfortunately: inconsistent

$R = \{x \mid \neg\varepsilon(x\ x)\}$ set of the sets that are not elements of themselves
A the proposition *the set R is an element of R*:
$\varepsilon(R\ R) = \varepsilon(\{x \mid \neg\varepsilon(x\ x)\}\ R)$
A reduces to $\neg\varepsilon(R\ R)$ i.e. $\neg A$
Thus, prove $\neg A$, then A, and $\perp$

Type theory and set theory

- every predicate is an object
- every predicate can be applied to every object

Abandon the first principle: set theory
Abandon the second: simple type theory

V. Set theory

Functions are relations
Relations are sets of ordered pairs
Only primitive notion: set

Only predicate symbols: $=$, $\in$
Russell's paradox: $R = \{x \mid \neg x \in x\}$
$R \in R \longrightarrow \neg R \in R$

Set theory: not always possible to build the set $\{x \mid A\}$

Possible in four cases
- E, F sets, pair containing E and F
- E set, union of the elements of E
- E set, powerset of E
- E a set and A proposition in the language $=, \in$, subset of E of elements verifying A

Subset of E of the elements verifying A
Convenient to introduce a sort κ for classes of sets

A comprehension scheme: every proposition in the language $=, \in$ defines a class in comprehension
Subset of E of the elements in c

An axiomatic theory

Function symbols $\{,\}$, $\bigcup$, $\mathcal{P}$, $\{|\}$, and $f_{x_1,...,x_n,y,A}$

$$\forall E \forall F \forall x\ (x \in \{E, F\} \Leftrightarrow (x = E \vee x = F))$$

$$\forall E \forall x\ (x \in \bigcup(E) \Leftrightarrow \exists y\ (x \in y \wedge y \in E))$$

$$\forall E \forall x\ (x \in \mathcal{P}(E) \Leftrightarrow \forall y\ (y \in x \Rightarrow y \in E))$$

$$\forall E \forall c \forall x (x \in \{E \mid c\} \Leftrightarrow (x \in E \wedge x \epsilon c))$$

$$\forall x_1 ... \forall x_n \forall y\ (y \epsilon f_{x_1,...,x_n,y,A}(x_1, ..., x_n) \Leftrightarrow A)$$

Russell's paradox avoided

No set of sets that are not element of themselves
Whether a set is an element of itself: always well-formed question
E set, $c = f_{y,\perp}$ empty class
$\{E \mid c\}$ empty subset of E is not an element of itself

$$\neg(\{E \mid c\} \in \{E \mid c\})$$

$$\neg(\{E \mid c\} \in E \wedge \perp)$$

provable

Reduction rules

$$x \in \{E, F\} \longrightarrow (x = E \vee x = F)$$

$$x \in \bigcup(E) \longrightarrow \exists y\ (y \in E \wedge x \in y)$$

$$x \in \mathcal{P}(E) \longrightarrow \forall y\ (y \in x \Rightarrow y \in E)$$

$$x \in \{E \mid c\} \longrightarrow (x \in E \wedge x \ \epsilon \ c)$$

$$y \ \epsilon \ f_{x_1, \ldots, x_n, y, A}(x_1, \ldots, x_n) \longrightarrow A$$

Another formulation: existence axioms

$$\forall E \forall F \exists G \forall x \ (x \in G \Leftrightarrow (x = E \vee x = F))$$

$$\forall E \exists G \forall x \ (x \in G \Leftrightarrow \exists y \ (y \in E \wedge x \in y))$$

$$\forall E \exists G \forall x \ (x \in G \Leftrightarrow \forall y \ (y \in x \Rightarrow x \in E))$$

$$\forall E \forall c \exists G \forall x \ (x \in G \Leftrightarrow (x \in E \wedge x \ \epsilon \ c))$$

$$\forall x_1 \ \ldots \ \forall x_n \exists c \forall y \ (y \ \epsilon \ c \Leftrightarrow A)$$

More axioms

Extensionality

$$\forall E \forall F \ ((\forall x \ (x \in E \Leftrightarrow x \in F)) \Rightarrow E = F)$$

Replacement
Choice
etc.

Cuts in set theory

The class of sets that are not elements of themselves $f_{x,\neg(x \in x)}$
E set $C = \{E \mid f_{x,\neg x \in x}\}$ subset of E of elements that are not elements of themselves
$A = C \in C$ reduces to $C \in E \wedge \neg C \in C$, i.e. $B \wedge \neg A$

No contradiction but ...

jeopardizes the termination of proof reduction

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{B, A \vdash B \wedge \neg A}}{B, A \vdash \neg A}
\quad
\dfrac{}{B, A \vdash A}
}{B, A \vdash \bot}
}{B \vdash \neg A}
\Rightarrow\text{-intro}
\qquad
\dfrac{
\dfrac{}{B \vdash B}
\quad
\dfrac{
\dfrac{
\dfrac{\dfrac{}{B, A \vdash B \wedge \neg A}}{B, A \vdash \neg A}
\quad
\dfrac{}{B, A \vdash A}
}{B, A \vdash \bot}
}{B \vdash \neg A}
}{B \vdash A}
}{
\dfrac{B \vdash \bot}{\vdash \neg B}
}
\Rightarrow\text{-elim}
$$

reduces to itself in two steps $\qquad$ $\neg B$ a proof, but no cut-free proof

Natural numbers

Only one base object: empty set

Natural numbers need to be constructed
Cantor numbers:
Axiom stating the existence of an infinite set B

Natural numbers as finite cardinals in B
Elements of the powerset of the powerset of B

Peano numbers:
Axiom stating the existence of an infinite set B
S a non surjective injection on B, 0 an element not in its image

Von Neumann numbers:
n set of numbers strictly less than n
$0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$, ...
Need an axiom for the set of natural numbers

Always an axiom asserting the existence of an infinite set:
otherwise models where all sets are finite

Next time

Simple type theory