Arithmetic

I. What we have seen so far

The notion of proof: constructivity, witness property, termination of proof reduction

The notion of theory: axioms spoil the last rule property, replace them by a congruence

The notion of model: many valued, constructive proofs, deduction modulo theory, super-consistency

Super-consistency

A theory is consistent if it has a model valued in some non-trivial algebra

A theory is super-consistent if it has a model valued in all (full, ordered, and complete) pre-Heyting algebras
Example: $P \longrightarrow (Q \Rightarrow Q)$
In any $\mathcal{B}$ a model: $\hat{Q} = \tilde{\top}$, $\hat{P} = (\tilde{\top} \tilde{\Rightarrow} \tilde{\top})$

Full, ordered, and complete

Full: the domains $\mathcal{A}$ of $\tilde{\forall}$ and $\mathcal{E}$ of $\tilde{\exists}$ is $\mathcal{P}^+(\mathcal{B})$

Ordered pre-Heyting algebra: pre-Heyting algebra equipped with an extra order relation $\sqsubseteq$ such that $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\forall}$, and $\tilde{\exists}$ are monotone, $\tilde{\Rightarrow}$ is left anti-monotone and right monotone

A ordered pre-Heyting algebra is complete if every subset of $\mathcal{B}$ has a greatest lower bound for $\sqsubseteq$

Super-consistency implies termination of proof-reduction

Today and in the next lectures

Examples of theories

Arithmetic, set theory, simple type theory

II. Arithmetic

Examples of propositions

$$\forall x \exists y \ (x = 2 \times y \vee x = 2 \times y + 1)$$

$$\exists y \ (4 = 2 \times y)$$

$$\exists x \exists y \ (7 = (x + 2) \times (y + 2))$$

$$\forall x \exists y \ (y \geq x \wedge prime(y))$$

$\geq$, *prime*?

2, 4, etc.

not constants
not terms expressing numbers in binary or decimal notation

Terms expressing numbers in unary notation: with a constant 0 and a unary function symbol S

4 is $S(S(S(S(0))))$

Several axiomatic theories

Classical logic: Peano arithmetic (PA)
Constructive logic: Heyting arithmetic (HA)

Several formulations:
with or without a sort κ for classes
with or without a predicate symbol N for natural numbers

Our goal: $HA^{\kappa N}$ both κ and N (back to Peano)
Transformed into a purely computational theory
Full witness property

III. HA^{κ}

$0,\ S,\ Pred,\ +,\ \times,\ Null,\ =$

$$Pred(0) = 0$$

$$\forall x\ (Pred(S(x)) = x)$$

$$\forall y\ (0 + y = y)$$

$$\forall x\ \forall y\ (S(x) + y = S(x + y))$$

$$\forall y\ (0 \times y = 0)$$

$$\forall x\ \forall y\ (S(x) \times y = (x \times y) + y)$$

$$Null(0)$$

$$\forall x\ \neg Null(S(x))$$

Induction

No other numbers than those constructed with 0 and S
Every class containing 0 and closed by S contains everything
Besides ι, a sort κ for classes, a predicate symbol ϵ

$$\forall c \ (0 \ \epsilon \ c \Rightarrow \forall x \ (x \ \epsilon \ c \Rightarrow S(x) \ \epsilon \ c) \Rightarrow \forall y \ y \ \epsilon \ c)$$

Comprehension axiom scheme: existence of some classes

$$\forall x_1 ... \forall x_n \exists c \forall y \ (y \ \epsilon \ c \Leftrightarrow A)$$

if A does not contain ϵ (predicative arithmetic)

Equality

Classes also used to express the properties of equality

$$\forall x \forall y \ (x = y \Leftrightarrow \forall c \ (x \ \epsilon \ c \Rightarrow y \ \epsilon \ c))$$

Exercise: prove reflexivity, symmetry, transitivity, and substitutivity

How to use these axioms to prove $\forall y\ (y + 0 = y)$?

High school proof:
$0 + 0 = 0$
$\forall x\ (x + 0 = x \Rightarrow S(x) + 0 = S(x))$
hence $\forall y\ (y + 0 = y)$

Using the axioms

$$\forall y\ (0 + y = y)$$

$$\forall x\ \forall y\ (S(x) + y = S(x + y))$$

How do we know

$$0 + 0 = 0 \Rightarrow \forall x\ (x + 0 = x \Rightarrow S(x) + 0 = S(x)) \Rightarrow \forall y\ (y + 0 = y)\ ?$$

How do we know

$$0 + 0 = 0 \Rightarrow \forall x\ (x + 0 = x \Rightarrow S(x) + 0 = S(x)) \\ \Rightarrow \forall y\ (y + 0 = y)\ ?$$

$$\forall c\ (0 \epsilon c \Rightarrow \forall x\ (x \epsilon c \Rightarrow S(x) \epsilon c) \Rightarrow \forall y\ y \epsilon c)$$

How do we know

$$0+0=0 \Rightarrow \forall x\ (x+0=x \Rightarrow S(x)+0=S(x)) \Rightarrow \forall y\ (y+0=y)\ ?$$

$$\forall c\ (0 \epsilon c \Rightarrow \forall x\ (x \epsilon c \Rightarrow S(x) \epsilon c) \Rightarrow \forall y\ y \epsilon c)$$

$$\exists c \forall y\ (y \in c \Leftrightarrow y+0=y)$$

Another exercise

Prove

$$\forall x \forall y\ (S(x) = S(y) \Rightarrow x = y)$$

$$\forall x\ \neg(0 = S(x))$$

HA: avoiding classes

Instead of a sort κ, a comprehension scheme, an induction axiom:
an induction scheme

$$\forall x_1 ... \forall x_n \; ((0/y)A \Rightarrow \forall p \; ((p/y)A \Rightarrow (S(p)/y)A) \Rightarrow \forall q \; (q/y)A)$$

(same thing for equality)
For instance $A = y + 0 = y$:
$$0 + 0 = 0 \Rightarrow \forall p \; (p + 0 = p \Rightarrow S(p) + 0 = S(p)) \Rightarrow \forall q \; (q + 0 = q)$$

Equivalent

Equivalent: in what sense?

A proposition A provable in HA iff it is provable in HA^κ

No way: the language of HA^κ contains more symbols

If A in the language of HA: A is provable in HA iff provable in HA^κ

If A provable in HA then A is provable in HA^κ easy (extension)

If A provable in HA^κ then provable in HA (conservative extension): not so easy

Conservative extension of an axiomatic theory

$\mathcal{L} \subseteq \mathcal{L}'$
$\mathcal{T}$ in $\mathcal{L}$, $\mathcal{T}'$ in $\mathcal{L}'$

$\mathcal{T}'$ is an extension of $\mathcal{T}$ if all propositions provable in $\mathcal{T}$ are provable in $\mathcal{T}'$

$\mathcal{T}'$ is a conservative extension of $\mathcal{T}$ if all the propositions of $\mathcal{L}$ provable in $\mathcal{T}'$ provable in $\mathcal{T}$

To prove that a theory is a conservative extension of another: extension of a model

$\mathcal{L} \subseteq \mathcal{L}'$
$\mathcal{M}$ model of $\mathcal{L}$ and $\mathcal{M}'$ of $\mathcal{L}'$

$\mathcal{M}'$ is an extension of $\mathcal{M}$ if for all sorts and symbols of $\mathcal{L}$ interpreted in the same way in both models

If for all models $\mathcal{M}$ of $\mathcal{T}$, there exists an extension $\mathcal{M}'$ of $\mathcal{M}$ that is a model of $\mathcal{T}'$, then $\mathcal{T}'$ conservative extension of $\mathcal{T}$

A proposition in $\mathcal{L}$ provable in $\mathcal{T}'$
We want: A provable in $\mathcal{T}$, i.e. A valid in all models of $\mathcal{T}$

$\mathcal{M}$ any model of $\mathcal{T}$
There exists $\mathcal{M}'$ model of $\mathcal{T}'$ extension of $\mathcal{M}$
A is valid in $\mathcal{M}'$ ($\mathcal{M}'$ model of $\mathcal{T}'$)
Same interpretation of A in $\mathcal{M}$ and $\mathcal{M}'$ ($\mathcal{M}'$ extension of $\mathcal{M}$)
A valid in $\mathcal{M}$

HA^κ is a conservative extension of HA
Any model of HA extends to a model of HA^κ
Need to define $\mathcal{M}_\kappa$ and $\hat{\epsilon}$

First idea $\mathcal{M}_\kappa$: the set of all functions from $\mathcal{M}_\iota$ to $\mathcal{B}$
No way to prove the validity of the induction axiom

$$\forall c\ (0 \mathrel{\epsilon} c \Rightarrow \forall x\ (x \mathrel{\epsilon} c \Rightarrow S(x) \mathrel{\epsilon} c) \Rightarrow \forall y\ y \mathrel{\epsilon} c)$$

$\mathcal{M}_\kappa$: the set of definable functions from $\mathcal{M}_\iota$ to $\mathcal{B}$, i.e. of the form $a \mapsto [\![A]\!]_{\phi, x=a}$ for some A (not using ϵ) and ϕ
Validity of HA-induction scheme: validity of HA^κ-induction axiom

IV. Peano’s predicate symbol

Induction axiom: all objects of sort ι are natural numbers
Alternative: not all objects are natural numbers, a predicate symbol N for the natural numbers

$$\forall c\ (0 \epsilon c \Rightarrow \forall x\ (x \epsilon c \Rightarrow S(x) \epsilon c) \Rightarrow \forall y\ (N(y) \Rightarrow y \epsilon c))$$

or even (equivalent)

$$\forall c\ (0 \epsilon c \Rightarrow \forall x\ (N(x) \Rightarrow x \epsilon c \Rightarrow S(x) \epsilon c) \Rightarrow \forall y\ (N(y) \Rightarrow y \epsilon c))$$

More axioms

$$N(0)$$

$$\forall x\ (N(x) \Rightarrow N(S(x)))$$

$$\forall y \; (N(y) \Rightarrow \forall c \; (0 \; \epsilon \; c \Rightarrow \forall x \; (N(x) \Rightarrow x \; \epsilon \; c \Rightarrow S(x) \; \epsilon \; c) \Rightarrow y \; \epsilon \; c))$$

Converse provable (with $N(0)$ and $\forall x \; (N(x) \Rightarrow N(S(x)))$)
Alternative:

$$\forall y \; (N(y) \Leftrightarrow \forall c \; (0 \; \epsilon \; c \Rightarrow \forall x \; (N(x) \Rightarrow x \; \epsilon \; c \Rightarrow S(x) \; \epsilon \; c) \Rightarrow y \; \epsilon \; c))$$

($N(0)$ and $\forall x \; (N(x) \Rightarrow N(S(x)))$ dropped)
$\mathrm{HA}^{\kappa N}$

A conservative extension of HA^κ?

Not even an extension

$$\forall x\ (x = 0 \vee \exists y\ (x = S(y)))$$

provable in HA^κ (by induction), but not in $HA^{\kappa N}$

$$\forall x\ (N(x) \Rightarrow (x = 0 \vee \exists y\ (x = S(y))))$$

is

Translation

$|\forall x\ A| = \forall x\ (N(x) \Rightarrow |A|)$
$|\exists x\ A| = \exists x\ (N(x) \wedge |A|)$

$|P| = P$, if P is atomic, $|A \wedge B| = |A| \wedge |B|$, etc. $|\forall c\ A| = \forall c\ |A|$, $|\exists c\ A| = \exists c\ |A|$

A closed proposition in the language of HA^κ

If A provable in HA^κ then $|A|$ provable in $\mathsf{HA}^{\kappa N}$ ($\simeq$ extension)

If $|A|$ provable in $\mathsf{HA}^{\kappa N}$ then A provable in HA^κ ($\simeq$ conservative extension)

What is so great about Peano predicate symbol N?

(as we shall see) $\mathrm{HA}^{\kappa N}$: disjunction and witness property
HA^{κ}: restricted to closed propositions

$$\forall x\ (x = 0 \lor \exists y\ (x = S(y)))$$

$$x = 0 \lor \exists y\ (x = S(y))$$

but neither $x = 0$ nor $\exists y\ (x = S(y))$ provable

In $HA^{\kappa N}$

$$\forall x \ (x = 0 \vee \exists y \ (x = S(y)))$$

not provable

$$\forall x \ (N(x) \Rightarrow (x = 0 \vee \exists y \ (x = S(y))))$$

$$N(x) \Rightarrow (x = 0 \vee \exists y \ (x = S(y)))$$

provable but not disjunctions
HA^{κ} cannot be transformed into a purely computational theory where proof reduction terminates
$HA^{\kappa N}$ can

V. Arithmetic as a purely computational theory

$$Pred(0) \longrightarrow 0$$

$$Pred(S(x)) \longrightarrow x$$

$$0 + y \longrightarrow y$$

$$S(x) + y \longrightarrow S(x + y)$$

$$0 \times y \longrightarrow 0$$

$$S(x) \times y \longrightarrow (x \times y) + y$$

$$Null(0) \longrightarrow \top$$

$$Null(S(x)) \longrightarrow \bot$$

$$x = y \longrightarrow \forall c \; (x \in c \Rightarrow y \in c)$$

$$N(y) \longrightarrow \forall c \; (0 \in c \Rightarrow \forall x \; (N(x) \Rightarrow x \in c \Rightarrow S(x) \in c) \Rightarrow y \in c)$$

The comprehension scheme

$$\forall x_1 ... \forall x_n \exists c \forall y \ (y \ \epsilon \ c \Leftrightarrow A)$$

Introduce a notation for this class: $f_{x_1,...,x_n,y,A}(x_1, ..., x_n)$

$$\forall x_1 ... \forall x_n \forall y \ (y \ \epsilon \ f_{x_1,...,x_n,y,A}(x_1, ..., x_n) \Leftrightarrow A)$$

$$y \ \epsilon \ f_{x_1,...,x_n,y,A}(x_1, ..., x_n) \longrightarrow A$$

$HA^{\longrightarrow}$ conservative extension of $HA^{\kappa N}$

VI. Models of arithmetic

A model valued in the algebra $\{0, 1\}$

$\mathcal{M}_\iota = \mathbb{N}$, $\mathcal{M}_\kappa = \mathbb{N} \to \{0, 1\}$

$\hat{0}$, $\hat{S}$, $\hat{Pred}$, $\hat{+}$, $\hat{\times}$, $\hat{Null}$: obvious way
$\hat{\epsilon}$: function mapping the number n and the function g of $\mathbb{N} \to \{0, 1\}$ to $g(n)$
$\hat{=}$: function mapping n and p to 1 if $n = p$ and to 0 otherwise
$\hat{N}$: constant function equal to 1
$\hat{f}_{x_1,...,x_n,y,A}$: function mapping $a_1, ..., a_n$ to function mapping b to $[\![A]\!]_{x_1=a_1,...,x_n=a_n,y=b}$

Super-consistency

$\mathcal{B}$ a full, ordered and complete pre-Heyting algebra
build a model whose pre-Heyting algebra is $\mathcal{B}$:

$\mathcal{M}_\iota = \mathbb{N}$
$\mathcal{M}_\kappa = \mathbb{N} \rightarrow \mathcal{B}$,

$\hat{0}$, $\hat{S}$, $\hat{Pred}$, $\hat{+}$, $\hat{\times}$, obvious way
$\hat{Null}$ function mapping 0 to $\tilde{\top}$ and the other numbers to $\tilde{\bot}$
$\hat{\epsilon}$ function mapping n and g to $g(n)$

Remain to be interpreted: $=$, N, and $f_{x_1,...,x_n,y,A}$
Interpretation must validate the rules

$$x = y \longrightarrow \forall c\ (x \epsilon c \Rightarrow y \epsilon c)$$

$$N(y) \longrightarrow \forall c\ (0 \epsilon c \Rightarrow \forall x\ (N(x) \Rightarrow x \epsilon c \Rightarrow S(x) \epsilon c) \Rightarrow y \epsilon c)$$

$$y \epsilon f_{x_1,...,x_n,y,A}(x_1, ..., x_n) \longrightarrow A$$

$$x = y \longrightarrow \forall c\ (x \,\epsilon\, c \Rightarrow y \,\epsilon\, c)$$

definition: interpret the left-hand side like the right-hand side

$\hat{=}$ function mapping n and p to $[\![\forall c\ (x \,\epsilon\, c \Rightarrow y \,\epsilon\, c)]\!]_{x=n,y=p}$
i.e. $\tilde{\forall}\ \{f(n) \tilde{\Rightarrow} f(p) \mid f \in \mathbb{N} \rightarrow \mathcal{B}\}$

This cannot be done for the induction rule

$$N(y) \longrightarrow \forall c \; (0 \; \epsilon \; c \Rightarrow \forall x \; (N(x) \Rightarrow x \; \epsilon \; c \Rightarrow S(x) \; \epsilon \; c) \Rightarrow y \; \epsilon \; c)$$

Super-consistency: ordered and complete pre-Heyting algebras
For each function f of $\mathbb{N} \rightarrow \mathcal{B}$: $\mathcal{M}_f$ where N interpreted by f
Φ mapping f to the function mapping the natural number n to

$$[\![\forall c \; (0 \in c \Rightarrow \forall x \; (N(x) \Rightarrow x \in c \Rightarrow S(x) \in c) \Rightarrow y \in c)]\!]^{\mathcal{M}_f}_{n/y}$$

The order on $\mathbb{N} \rightarrow \mathcal{B}$ complete, Φ monotone, fixed-point g, $\hat{N} = g$

$f_{x,y_1,\ldots,y_n,A}$ obvious way

$HA^{\longrightarrow}$ super-consistent

After the break

Naive set theory