

The Notion of model

I. What we have seen last time

A theory = a set of axioms + a decidable and non confusing **congruence** (often defined by a reduction system)

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-intro}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash C} \wedge\text{-intro if } C \equiv A \wedge B$$

Proof reduction does not always terminate

But **all** (1) purely computational theories where (2) proof reduction terminates have the witness property

Next

Examples of theories

Prove **termination** of proof-reduction for some of these theories
For this: the notion of model

II. Models valued in $\{0, 1\}$

The algebra $\{0, 1\}$

$$\mathcal{B} = \{0, 1\}$$

\leq : natural order on this set

$$\tilde{\top} = 1, \tilde{\perp} = 0$$

$\tilde{\wedge}$, function from $\{0, 1\} \times \{0, 1\}$ to $\{0, 1\}$

$\tilde{\wedge}$	0	1
0	0	0
1	0	1

$\tilde{\forall}$ and $\tilde{\Rightarrow}$ similar

$\tilde{\forall}$ and $\tilde{\exists}$ functions from $\mathcal{P}^+(\{0, 1\})$ to $\{0, 1\}$

$\tilde{\forall}$	{0}	{0, 1}	{1}
	0	0	1

$\tilde{\exists}$	{0}	{0, 1}	{1}
	0	1	1

Models valued in $\{0, 1\}$

A **model** for a language \mathcal{L} is formed with

- ▶ for each sort s , a non empty set \mathcal{M}_s
- ▶ for each function symbol f of arity $\langle s_1, \dots, s_n, s' \rangle$, a function \hat{f} from $\mathcal{M}_{s_1} \times \dots \times \mathcal{M}_{s_n}$ to $\mathcal{M}_{s'}$
- ▶ for each predicate symbol P of arity $\langle s_1, \dots, s_n \rangle$, a function \hat{P} from $\mathcal{M}_{s_1} \times \dots \times \mathcal{M}_{s_n}$ to \mathcal{B}

Interpretation in a model

$\llbracket \cdot \rrbracket$ maps every term t of sort s , to an element $\llbracket t \rrbracket$ of \mathcal{M}_s
every proposition A to an element $\llbracket A \rrbracket$ of \mathcal{B}

Morphism

$$\llbracket f(t_1, \dots, t_n) \rrbracket = \hat{f}(\llbracket t_1 \rrbracket, \dots, \llbracket t_n \rrbracket)$$

$$\llbracket P(t_1, \dots, t_n) \rrbracket = \hat{P}(\llbracket t_1 \rrbracket, \dots, \llbracket t_n \rrbracket)$$

$$\llbracket A \wedge B \rrbracket = \llbracket A \rrbracket \tilde{\wedge} \llbracket B \rrbracket, \text{ etc.}$$

Completely defined by its image on the variables

Valuations

Function ϕ of finite domain associating to the variables x_1, \dots, x_n of sorts s_1, \dots, s_n elements a_1, \dots, a_n of $\mathcal{M}_{s_1}, \dots, \mathcal{M}_{s_n}$

Any valuation ϕ **extends** to a morphism $\llbracket \cdot \rrbracket_\phi$ between

- ▶ the terms and the propositions whose free variables are in the domain of ϕ
- ▶ and the model \mathcal{M}

- ▶ $\llbracket x \rrbracket_\phi = \phi(x)$
- ▶ $\llbracket f(t_1, \dots, t_n) \rrbracket_\phi = \hat{f}(\llbracket t_1 \rrbracket_\phi, \dots, \llbracket t_n \rrbracket_\phi)$
- ▶ $\llbracket P(t_1, \dots, t_n) \rrbracket_\phi = \hat{P}(\llbracket t_1 \rrbracket_\phi, \dots, \llbracket t_n \rrbracket_\phi)$
- ▶ $\llbracket \top \rrbracket_\phi = \tilde{\top}$, $\llbracket \perp \rrbracket_\phi = \tilde{\perp}$
- ▶ $\llbracket A \wedge B \rrbracket_\phi = \llbracket A \rrbracket_\phi \tilde{\wedge} \llbracket B \rrbracket_\phi$, $\llbracket A \vee B \rrbracket_\phi = \llbracket A \rrbracket_\phi \tilde{\vee} \llbracket B \rrbracket_\phi$
- ▶ $\llbracket A \Rightarrow B \rrbracket_\phi = \llbracket A \rrbracket_\phi \tilde{\Rightarrow} \llbracket B \rrbracket_\phi$
- ▶ $\llbracket \forall x A \rrbracket_\phi = \tilde{\forall} \{ \llbracket A \rrbracket_{\phi, x=a} \mid a \in \mathcal{M}_s \}$
- ▶ $\llbracket \exists x A \rrbracket_\phi = \tilde{\exists} \{ \llbracket A \rrbracket_{\phi, x=a} \mid a \in \mathcal{M}_s \}$

Validity

A **valid** in \mathcal{M} if for all ϕ , $\llbracket A \rrbracket_{\phi} \geq \tilde{\top}$

$A_1, \dots, A_n \vdash B$ **valid** in \mathcal{M} if the proposition $(A_1 \wedge \dots \wedge A_n) \Rightarrow B$ is

\mathcal{T} **valid** in \mathcal{M} if all its axioms are

Soundness: If the proposition A has a classical proof in \mathcal{T} , then it is valid in all models of \mathcal{T}

Completeness: If the proposition A is valid in all models of \mathcal{T} , then it has a classical proof in \mathcal{T}

Contrapositive of soundness

If a model \mathcal{M} of \mathcal{T} s.t. A not valid in \mathcal{M} , then A not provable in \mathcal{T}

Exercise: two proposition symbols P and Q

A single axiom P

Q is not provable

$\neg Q$ is not provable

III. Many valued models

Four problems

Adapt the notion of model to **prove indep. of Q with single model**

Adapt the notion of model to **constructive provability**

Adapt the notion of model to **Deduction modulo theory**

Adapt the notion of model to **prove termination of proof-reduction**

One solution: **many valued models**

Algebras

A set \mathcal{B}

a binary relation \leq on \mathcal{B}

two elements $\tilde{\top}$ and $\tilde{\perp}$ of \mathcal{B}

three functions $\tilde{\wedge}$, $\tilde{\vee}$, and $\tilde{\Rightarrow}$ from $\mathcal{B} \times \mathcal{B}$ to \mathcal{B}

a subset \mathcal{A} of $\mathcal{P}^+(\mathcal{B})$, a function $\tilde{\nabla}$ from \mathcal{A} to \mathcal{B}

a subset \mathcal{E} of $\mathcal{P}^+(\mathcal{B})$, a function $\tilde{\exists}$ from \mathcal{E} to \mathcal{B}

Models

A model for a language \mathcal{L} is formed with

- ▶ for each sort s , a non empty set \mathcal{M}_s
- ▶ an algebra $\mathcal{B} = \langle \mathcal{B}, \leq, \tilde{\top}, \tilde{\perp}, \tilde{\wedge}, \tilde{\vee}, \mathcal{A}, \tilde{\forall}, \mathcal{E}, \tilde{\exists}, \tilde{\Rightarrow} \rangle$,
- ▶ for each function symbol f of arity $\langle s_1, \dots, s_n, s' \rangle$, a function \hat{f} from $\mathcal{M}_{s_1} \times \dots \times \mathcal{M}_{s_n}$ to $\mathcal{M}_{s'}$
- ▶ for each predicate symbol P of arity $\langle s_1, \dots, s_n \rangle$, a function \hat{P} from $\mathcal{M}_{s_1} \times \dots \times \mathcal{M}_{s_n}$ to \mathcal{B}

valued in the algebra \mathcal{B}

Valuation (as above): a function ϕ of finite domain associating to the variables x_1, \dots, x_n of sorts s_1, \dots, s_n elements a_1, \dots, a_n of $\mathcal{M}_{s_1}, \dots, \mathcal{M}_{s_n}$

Interpretation (as above):

- ▶ $\llbracket x \rrbracket_\phi = \phi(x)$, $\llbracket f(t_1, \dots, t_n) \rrbracket_\phi = \hat{f}(\llbracket t_1 \rrbracket_\phi, \dots, \llbracket t_n \rrbracket_\phi)$
- ▶ $\llbracket P(t_1, \dots, t_n) \rrbracket_\phi = \hat{P}(\llbracket t_1 \rrbracket_\phi, \dots, \llbracket t_n \rrbracket_\phi)$
- ▶ $\llbracket A \wedge B \rrbracket_\phi = \llbracket A \rrbracket_\phi \tilde{\wedge} \llbracket B \rrbracket_\phi$, etc.
- ▶ $\llbracket \forall x A \rrbracket_\phi = \tilde{\forall} \{ \llbracket A \rrbracket_{\phi, x=a} \mid a \in \mathcal{M}_s \}$

Validity (as above): for all ϕ , $\llbracket A \rrbracket_\phi \geq \tilde{\top}$

Examples of algebras

$\{0, 1\}$

But also:

$$\mathcal{B} = \mathcal{P}(\{3, 4\}) = \{\emptyset, \{3\}, \{4\}, \{3, 4\}\}, \leq = \subseteq,$$

$$\tilde{\top} = \{3, 4\}, \tilde{\perp} = \emptyset,$$

$$a \tilde{\wedge} b = a \cap b, a \tilde{\vee} b = a \cup b, a \tilde{\Rightarrow} b = (\{3, 4\} \setminus a) \cup b,$$

$$\tilde{\forall} E = \bigcap_{x \in E} x, \tilde{\exists} E = \bigcup_{x \in E} x$$

$$\text{Note } \tilde{\neg} a = a \tilde{\Rightarrow} \tilde{\perp} = \{3, 4\} \setminus a$$

$$\hat{P} = \tilde{T} = \{3, 4\}$$

$$\hat{Q} = \{4\}$$

Neither Q nor $\neg Q$ is valid

Aggregates two models in one

$$\mathcal{M}_3 : \hat{P} = 1, \hat{Q} = 0$$

$$\mathcal{M}_4 : \hat{P} = 1, \hat{Q} = 1$$

$$\mathcal{M} : \hat{A} = \{i \mid A \text{ valid in } \mathcal{M}_i\}$$

From $\mathcal{P}(\{3, 4\})$ to pre-Boolean algebras

Models where \mathcal{B} is a powerset

Generalize: models where \mathcal{B} is a **Boolean algebra**

A set with, an order, greatest lowerbounds ($\tilde{\top}$, $\tilde{\wedge}$, $\tilde{\vee}$) and least upperbounds ($\tilde{\perp}$, $\tilde{\vee}$, $\tilde{\exists}$) and a relative complement \Rightarrow

Generalize further: order: reflexive, antisymmetric, transitive

Antisymmetry useless and complicates proofs: drop it

Intuition: $A \leq B$ if $A \Rightarrow B$ provable: reflexive, transitive, not antisymmetric

Pre-Boolean algebras

Set \mathcal{B} , binary relation \leq , $\tilde{\top}$ and $\tilde{\perp}$ elements of \mathcal{B} , $\tilde{\wedge}$, $\tilde{\vee}$, and $\tilde{\Rightarrow}$ binary functions, $\tilde{\nabla}$ function from a subset \mathcal{A} of $\mathcal{P}^+(\mathcal{B})$ to \mathcal{B} , and $\tilde{\exists}$ function from a subset \mathcal{E} of $\mathcal{P}^+(\mathcal{B})$ to \mathcal{B}

$a \leq a$, if $a \leq b$ and $b \leq c$ then $a \leq c$

$a \tilde{\wedge} b \leq a$, $a \tilde{\wedge} b \leq b$, if $c \leq a$ and $c \leq b$ then $c \leq a \tilde{\wedge} b$

etc.

$a \leq b \tilde{\Rightarrow} c$ if and only if $a \tilde{\wedge} b \leq c$

$\tilde{\top} \leq (a \tilde{\nabla} (a \tilde{\Rightarrow} b))$

Examples of pre-Boolean algebras

$\{0, 1\}$

$\mathcal{P}(\{3, 4\})$

but also $\{0\}$

and any set equipped with the full relation and any operations

Soundness and completeness

Soundness: If the proposition A has a classical proof in \mathcal{T} , then it is valid in all models of \mathcal{T}

Completeness: If the proposition A is valid in all models of \mathcal{T} , then it has a classical proof in \mathcal{T}

More models: soundness **stronger**, completeness **weaker**

IV. Models and constructive proofs

The validity of the excluded-middle

Models **valued in $\{0, 1\}$** are all models of the excluded-middle

$$\llbracket A \vee \neg A \rrbracket_{\phi} = \llbracket A \rrbracket_{\phi} \tilde{\vee} \tilde{\neg} \llbracket A \rrbracket_{\phi} = \max(\llbracket A \rrbracket_{\phi}, 1 - \llbracket A \rrbracket_{\phi}) = 1$$

Models **valued in $\mathcal{P}(E)$** also

$$\llbracket A \vee \neg A \rrbracket_{\phi} = \llbracket A \rrbracket_{\phi} \tilde{\vee} \tilde{\neg} \llbracket A \rrbracket_{\phi} = \llbracket A \rrbracket_{\phi} \cup (E \setminus \llbracket A \rrbracket_{\phi}) = E$$

Models **valued in a (pre-)Boolean algebra** also

$$\tilde{\top} \leq (a \tilde{\vee} (a \tilde{\Rightarrow} b))$$

Valid in all models but no constructive proof

From pre-Boolean algebra to pre-Heyting algebras

Just drop the condition

$$\tilde{\top} \leq (a \tilde{\vee} (a \tilde{\Rightarrow} b))$$

pre-Heyting algebra

A pre-Heyting algebra that is not a pre-Boolean algebra

Instead of $\mathcal{P}(\mathbb{R})$, the **open sets** only

Pre-order \subseteq (antisymmetric in this case)

Everything works (open sets stable by unions, finite intersections)
except infinite intersections and complement

In this case take the interior

$$\hat{P} = (-\infty, 0)$$

$$\tilde{\neg}\hat{P} = [0, +\infty) = (0, +\infty)$$

$$\hat{P} \tilde{\vee} \tilde{\neg}\hat{P} = (-\infty, 0) \cup (0, +\infty) = \mathbb{R} \setminus \{0\}$$

Another pre-Heyting algebra that is not pre-Boolean

$\{0, 1/2, 1\}$

natural order

$$\begin{aligned} \tilde{\top} &= 1, \tilde{\perp} = 0, a \tilde{\wedge} b = \text{glb}(a, b), a \tilde{\vee} b = \text{lub}(a, b), \\ \tilde{\forall} A &= \text{glb}_{x \in AX}, \tilde{\exists} E = \text{lub}_{x \in EX} \end{aligned}$$

$a \tilde{\Rightarrow} b = b$ if $a > b$, and 1 otherwise

$a \leq (b \tilde{\Rightarrow} c)$ if and only if $(a \tilde{\wedge} b) \leq c$ (three cases: $b \leq c$, $b > c$ and $a \leq c$, and $b > c$ and $a > c$)

$$1/2 \tilde{\vee} (1/2 \tilde{\Rightarrow} 0) = 1/2 \tilde{\vee} 0 = 1/2$$

Soundness

If the proposition A has a constructive proof in \mathcal{T} , then it is valid in all models of \mathcal{T}

Lemma: If $\Gamma \vdash A$ has a constructive proof, then it is valid in all pre-Heyting models (By induction over proof structure)

Completeness

If the proposition A is valid in all models of \mathcal{T} , then it has a constructive proof in \mathcal{T}

Weak

A **simple** proof: build a single model where valid propositions are exactly those that have a constructive proof in \mathcal{T}

The Lindenbaum model

Idea: Interpret each term (resp. proposition) by itself

\mathcal{M}_s = set of terms (of sort s), \mathcal{B} = set of propositions

Closed terms and prop. of $\mathcal{L} \cup S$, S infinite set of constants

$A \leq B$ if $A \Rightarrow B$ has a constructive proof in \mathcal{T}

The operations $\tilde{\top}$, $\tilde{\perp}$, $\tilde{\wedge}$, $\tilde{\vee}$, and $\tilde{\Rightarrow}$, are \top , \perp , \wedge , \vee , and \Rightarrow

$\mathcal{A} = \mathcal{E}$ set of subsets of \mathcal{B} of the form $\{(t/x)A \mid t \in \mathcal{M}_s\}$ for some A

A unique

$$\tilde{\forall} \{(t/x)A \mid t \in \mathcal{M}_s\} = (\forall x A)$$

$$\tilde{\exists} \{(t/x)A \mid t \in \mathcal{M}_s\} = (\exists x A)$$

\hat{f} : function mapping t_1, \dots, t_n to $f(t_1, \dots, t_n)$

\hat{P} : function mapping t_1, \dots, t_n to $P(t_1, \dots, t_n)$

Algebra of Lindenbaum model of \mathcal{T} : a pre-Heyting algebra

Lindenbaum model of \mathcal{T} : model of \mathcal{T}

A valid in the Lindenbaum model of \mathcal{T} then A has a constructive proof in \mathcal{T}

V. Models and Deduction modulo theory

Validity of a theory in Deduction modulo theory

\equiv **valid** in \mathcal{M} if for all A and B such that $A \equiv B$, for all ϕ
 $\llbracket A \rrbracket_{\phi} = \llbracket B \rrbracket_{\phi}$

\mathcal{T}, \equiv **valid** in \mathcal{M} if all axioms of \mathcal{T} and \equiv are valid in \mathcal{M}

Soundness

If the proposition A has a constructive proof in \mathcal{T} , then it is valid in all models of \mathcal{T}

Lemma: If $\Gamma \vdash A$ has a constructive proof, then it is valid in all pre-Heyting models

(By induction over proof structure using the fact that the model is a model of the congruence to justify the replacement of a proposition by a congruent one in each rule)

Completeness

Lindenbaum model:

Replace terms by **classes of terms modulo \equiv**

Replace propositions by **classes of propositions modulo \sim**

Only difficulty \sim not \equiv : $\{(t/x)A \mid t \in \mathcal{M}_s\}$ does not uniquely define A

A reason to drop antisymmetry

If $A \Leftrightarrow B$ provable in $\mathcal{T}, \equiv,$

for all ϕ , $\llbracket A \rrbracket_{\phi} \leq \llbracket B \rrbracket_{\phi}$ and $\llbracket B \rrbracket_{\phi} \leq \llbracket A \rrbracket_{\phi}$

If $A \equiv B$, then for all ϕ , $\llbracket A \rrbracket_{\phi} = \llbracket B \rrbracket_{\phi}$

With antisymmetry: same notion

Without

$4 = 4$ and $2 + 2 = 4$ same interpretation

Fermat's little theorem and Fermat's last theorem different $\leq \geq$
interpretations

Consistency

If \mathcal{T} has a model if and only if \mathcal{T} consistent

Here: even non consistent theories have models

But: A pre-Heyting algebra is **trivial** if $a \leq b$ always

A theory \mathcal{T}, \equiv is consistent if and only if it has a model whose pre-Heyting algebra is non trivial

VI. Super-consistency

An exercise

A model valued in $\{0, 1\}$ of the congruence defined by the reduction rule

$$P \longrightarrow (Q \Rightarrow Q)$$

That is: find \hat{P} and \hat{Q} such that $\hat{P} = (\hat{Q} \Rightarrow \hat{Q})$

A solution: $\hat{Q} = 1$ and $\hat{P} = (1 \Rightarrow 1) = 1$

But ...

No property of the algebra $\{0, 1\}$ really used

$\hat{Q} = \tilde{T}$ and $\hat{P} = (\tilde{T} \rightrightarrows \tilde{T})$ works in any pre-Heyting algebra \mathcal{B}

Thus, the congruence \equiv has a model valued in $\{0, 1\}$ and also in **any** algebra \mathcal{B}

Super-consistency

A theory is **super-consistent** if it has a model valued in any (full, ordered, and complete) pre-Heyting algebra

Why do we care: as we shall see super-consistency **implies** termination of proof-reduction, hence the witness property

Next time

Arithmetic