Proofs in theories

Why do proofs matter to computer scientists?

Church's theorem: undecidability of provability (1936)

Proofs and algorithms are two completely different things

Method to judge a proposition true: build a proof
Algorithms can only be used for very specific decidable problems

But...

1. Computers are truth judgment machines

The 100th decimal of π is a 9

2. Proof-checking and proof-search algorithms

Provability undecidable
But correctness of proof decidable: proof-checking algorithms
and provability semi-decidable: proof-search semi-algorithms

3. Proofs of algorithms and programs

Critical systems: transportation, energy, medicine...
A way to avoid bugs

Prove your programs correct

Programs: do, do, do... what for?

4. Constructivity and Brouwer-Heyting-Kolmogorov interpretation

Constructive proofs are algorithms

The language of (constructive) proofs is a programming language where all programs terminate

5. Theories

Proofs are not purely logical objects

Theories: arithmetic, set theory, type theory, etc.

Theories: sets of axioms, some theories algorithms

This course: proofs in theories

$$2 + 2 = 4 \Rightarrow 2 + 2 = 4$$

$$n + 1 = p + 1 \Rightarrow n = p$$

Proof theory: proofs in pure logic
Then proofs in some specific theories (Arithmetic, Simple type theory...)

Here: an arbitrary theory as long as we can

This course: proof-reduction and models

Two notions of truth: proofs, models
But (more and more) convergence

Key results in proof-theory: termination of proof-reduction
Proving termination of proof-reduction $\simeq$ building a model

Structure of this course
(11 courses + 4 exercises sessions + 1 master class)

1, 2, 3: basic notions (proof, theory, many-valued model...)

4, 5, 6: examples of theories

7, 8: proof reduction

9, 10, 11: unified formalisms ($\lambda\Pi$-calculus, $\lambda\Pi$-calculus modulo theory, Martin-Löf type theory, the Calculus of Constructions)

Along the way: Proof-checking systems

Simple type theory: HOL, HOL-light, Isabelle/HOL, PVS

$\lambda\Pi$-calculus: Twelf

$\lambda\Pi$-calculus modulo theory: Dedukti

Martin-Löf's type theory: Agda

The Calculus of constructions: Coq

What you are supposed to know

The notion of inductive definition

The notions of free and bound variable, alphabetic equivalence, and substitution

The syntax of (many-sorted) predicate logic

The natural deduction

The untyped and simply typed lambda-calculi

The expression of computable functions in arithmetic, in the language of rewrite rules and in the lambda-calculus

The Natural Deduction

I. The Natural Deduction Rules

The set of provable proposition

An inductive definition

$$\frac{A \Rightarrow B \quad A}{B}$$

$$\frac{}{P \Rightarrow Q \Rightarrow R}$$

$$\frac{}{P}$$

$$\frac{}{Q}$$

But not so comfortable

To prove $A \Rightarrow B$, assume A and prove B

Do not deduce propositions but pairs formed with hypotheses and a conclusion, sequents, $\Gamma \vdash A$

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{}{\Gamma, A \vdash A}$$

An exercise

Prove $P \vdash Q \Rightarrow P$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\text{-intro}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{-elim}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{-elim}$$

The classification of the rules

These three rules mention only the connective $\wedge$

Most rules mention only one connective: the rules of $\wedge$, the rules of $\vee$, etc.

Either in the conclusion or in the premises

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim}$$

introduction / elimination

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\text{-intro}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee\text{-intro}$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ \vee\text{-elim}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow\text{-intro}$$

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow\text{-elim}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x\ A}\ \forall\text{-intro if } x \notin FV(\Gamma)$$

$$\frac{\Gamma \vdash \forall x\ A}{\Gamma \vdash (t/x)A}\ \forall\text{-elim}$$

$$\frac{\Gamma \vdash (t/x)A}{\Gamma \vdash \exists x\ A}\ \exists\text{-intro}$$

$$\frac{\Gamma \vdash \exists x\ A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\ \exists\text{-elim if } x \notin FV(\Gamma, B)$$

$$\frac{}{\Gamma \vdash \top}\ \top\text{-intro}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ \bot\text{-elim}$$

$$\frac{}{\Gamma \vdash A}\ \text{axiom if } A \in \Gamma$$

$$\frac{}{\Gamma \vdash A \vee \neg A}\ \text{excluded-middle}$$

$\neg$, $\Leftrightarrow$

No rules for $\neg$ and $\Leftrightarrow$

$\neg A$ abbreviation for $A \Rightarrow \perp$
$A \Leftrightarrow B$ abbreviation for $(A \Rightarrow B) \wedge (B \Rightarrow A)$

Proofs

A sequent $\Gamma \vdash A$ is provable iff it has a derivation (proof)

A tree where nodes are labelled with sequents

Root labelled by $\Gamma \vdash A$

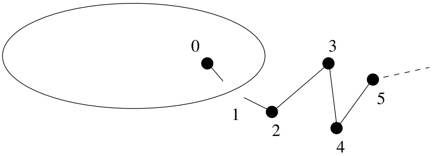
If node labelled by $\Delta \vdash B$ and children labelled by $\Sigma_1 \vdash C_1$, ..., $\Sigma_n \vdash C_n$ then a Natural deduction rule deduces $\Delta \vdash B$ from $\Sigma_1 \vdash C_1, \ldots, \Sigma_n \vdash C_n$

Proof of a proposition, proof in an axiomatic theory

A proposition A is provable (without any axioms), if $\vdash A$ is

Axiomatic theory $\mathcal{T}$: set of closed propositions (axioms)
A provable in $\mathcal{T}$ if finite subset Γ of $\mathcal{T}$, $\Gamma \vdash A$ provable

II. Constructive proofs



$0 \in P$ and $2 \notin P$

Does there exists n such that $n \in P$ and $n+1 \notin P$?

$P(0), \neg P(S(S(0))) \vdash \exists x\ (P(x) \wedge \neg P(S(x)))$

π_1

$$\dfrac{\dfrac{\dfrac{}{\Gamma, P(S(0)) \vdash P(S(0))} \quad \dfrac{}{\Gamma, P(S(0)) \vdash \neg P(S(S(0)))}}{\Gamma, P(S(0)) \vdash P(S(0)) \wedge \neg P(S(S(0)))}}{\Gamma, P(S(0)) \vdash \exists x\ (P(x) \wedge \neg P(S(x)))}$$

where $\Gamma = \{P(0), \neg P(S(S(0)))\}$

π_2

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \neg P(S(0)) \vdash P(0)} \quad \dfrac{}{\Gamma, \neg P(S(0)) \vdash \neg P(S(0))}}{\Gamma, \neg P(S(0)) \vdash P(0) \wedge \neg P(S(0))}}{\Gamma, \neg P(S(0)) \vdash \exists x \ (P(x) \wedge \neg P(S(x)))}$$

Finally

$$\dfrac{\dfrac{}{\Gamma \vdash P(S(0)) \vee \neg P(S(0))} \quad \dfrac{\pi_1}{\Gamma, P(S(0)) \vdash A} \quad \dfrac{\pi_2}{\Gamma, \neg P(S(0)) \vdash A}}{\Gamma \vdash A}$$

where $A = \exists x \ (P(x) \wedge \neg P(S(x)))$

We can prove

$$\exists x\ (P(x) \wedge \neg P(x))$$

Can we prove

$$P(n) \wedge \neg P(S(n))$$

for some natural number n?

No: easy to prove that for each number n

$$P(0), \neg P(S(S(0))) \vdash P(n) \wedge \neg P(S(n))$$

not provable

Without any axioms

We can prove

$$\exists x \; (P(0) \Rightarrow \neg P(S(S(0))) \Rightarrow (P(x) \wedge \neg P(S(x))))$$

We can prove

$$P(0) \Rightarrow \neg P(S(S(0))) \Rightarrow (P(n) \wedge \neg P(S(n)))$$

for no natural number n

The notion of witness

E has the witness property if
when $\exists x\ A$ is in E, there exists t such that $(t/x)A$ is in E

The set of provable propositions: no witness property

How is this possible?

Only one possibility to prove $\exists x\ A$: prove $(t/x)A$ and then use the $\exists$-intro rule

Example π_1 and π_2

Then a proof by case

$$\dfrac{\dots \quad \dfrac{\pi_1}{\Gamma, P(S(0)) \vdash A} \quad \dfrac{\pi_2}{\Gamma, \neg P(S(0)) \vdash A}}{\Gamma \vdash A}$$

0 or $S(0)$?

But still needs to prove $P(S(0)) \vee \neg P(S(0))$

The excluded-middle rule

$(A \vee \neg A)$ without knowing which of A or $\neg A$ holds

The notion of constructive proof

A proof that does not use the excluded-middle rule

As we shall see: if a proposition $\exists x\ A$ has a constructive proof, without any axioms, then there exists a term t such that $(t/x)A$ has a proof

Algorithm to extract witness from proof: proof reduction

Extends to many theories

Programming with proofs

A constructive proof π of

$$\forall x \exists y \; (x = 2 \times y \vee x = 2 \times y + 1)$$

A proof of the proposition

$$\exists y \; (25 = 2 \times y \vee 25 = 2 \times y + 1)$$

Extract a witness from this proof
By construction, correct with respect to specification

$$x = 2 \times y \vee x = 2 \times y + 1$$

III. Cuts and proof reduction

Cuts

A proof ending with an elimination rule whose main premise is proved by an introduction rule on the same symbol
For instance

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \quad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro}}{\Gamma \vdash A}\ \wedge\text{-elim}$$

Seven cases

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B}\Rightarrow\text{-intro} \qquad \dfrac{\pi'}{\Gamma \vdash A}}{\Gamma \vdash B}\Rightarrow\text{-elim}$$

Proof reduction

Contains a cut: a sub-tree of the proof is a cut
Proof reduction: replace this sub-tree with another

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \quad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge\text{-intro}}{\Gamma \vdash A}\ \wedge\text{-elim}$$

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B}\Rightarrow\text{-intro} \qquad \dfrac{\pi'}{\Gamma \vdash A}}{\Gamma \vdash B}\Rightarrow\text{-elim}$$

Eliminating a cut is easy
Eliminating a cut may create others: termination?

Technically: a major topic of this course

Why do we care?

Cut-free: contains no cut

A proof π that is (1.) constructive, (2.) cut-free, and (3.) without any axioms ends with an introduction rule.

A proof π of $\exists x\ A$ that is (1.) constructive, (2.) cut-free, and (3.) without any axioms ends with a $\exists$-intro rule: witness property

After the break

The notion of theory