MPRI
Foundations of proof systems 3
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Exercise 1 *What is the proof-term associated to the proof*

$$\dfrac{\dfrac{\dfrac{}{\exists x\,(P(x) \Rightarrow P(x)) \vdash \exists x\,(P(x) \Rightarrow P(x))}\textit{axiom}}{\vdash \exists x\,(P(x) \Rightarrow P(x)) \Rightarrow \exists x\,(P(x) \Rightarrow P(x))}\Rightarrow\textit{-intro} \quad \dfrac{\dfrac{\dfrac{}{P(c) \vdash P(c)}\textit{axiom}}{\vdash P(c) \Rightarrow P(c)}\Rightarrow\textit{-intro}}{\vdash \exists x\,(P(x) \Rightarrow P(x))}\exists\textit{-intro}}{\vdash \exists x\,(P(x) \Rightarrow P(x))}\Rightarrow\textit{-elim}$$

? Reduce it to its irreducible form.

Exercise 2 *Leivant, Krivine, and Parigot have proposed to extend arithmetic with rules similar to those of addition and multiplication for other functions. For instance the rules*

$$\chi(0) \longrightarrow 0$$

$$\chi(S(n)) \longrightarrow S(0)$$

that define the function χ *that takes the value* 0 *at* 0 *and the value* 1 *at other natural numbers.*

1. *Consider an extension of* $HA^{\longrightarrow}$ *with a symbol* χ *and the two rules above. Prove that this extension of* $HA^{\longrightarrow}$ *is super-consistent. (Hint: just add a few characters to the proof of Theorem 4.10).*

2. *In this theory, give a proof-term* π *of type*

$$\forall x\,(N(x) \Rightarrow \exists y\,(N(y) \wedge y = \chi(x)))$$

3. *Consider a natural number* n. *What is the type of the proof-term* $(\pi\ S^n(0)\ \rho_n)$? *Show that its irreducible form has the form* $\langle t, \langle \pi_1, \pi_2 \rangle \rangle$ *What is* t? *What is* π_1? *What is* π_2?

4. *In this theory, give a proof-term* π *of type*

$$\forall x\,(N(x) \Rightarrow N(\chi(x)))$$

5. *Consider a natural number* n. *What is the type of the proof-term* $(\pi\ S^n(0)\ \rho_n)$? *What is its irreducible form* π_1?