Exercise 1  The simple types are inductively defined by two rules:

- \( \iota \) and \( o \) are simple types,
- if \( A \) and \( B \) are simple types, then \( A \to B \) is a simple type.

For each simple type, we consider an infinite set of variables of this type and possibly some constants. The Simply typed \( \lambda \)-terms are defined by

- variables and constants of type \( A \) are terms of type \( A \),
- if \( t \) is a term of type \( A \to B \) and \( u \) a term of type \( A \), then \( (tu) \) is a term of type \( B \),
- if \( x \) is a variable of type \( A \) and \( t \) a term of type \( B \), then \( \lambda x : A \; t \) is a term of type \( A \to B \).

1. The \( \beta \)-reduction rule is

\[
((\lambda x : A \; t) \; u) \longrightarrow (u/x)t
\]

Define the notion of one-step \( \beta \)-reduction, termination and strong termination of \( \beta \)-reduction.

2. Consider the following strong termination proof.

By induction of the structure of \( \lambda \)-terms

- variables and constants are irreducible, hence they strongly terminate,
- if \( t \) strongly terminates, then so does \( \lambda x \; t \),
- if \( t \) and \( u \) strongly terminate, then so does \( (tu) \).

Why is this proof wrong?

We define, by induction over the type \( A \), a set of terms \( R_A \).

- If \( A = \iota \) or \( A = o \), then a term \( t \) is an element of \( R_A \) if and only if it strongly terminates.
- If \( A = B \to C \), then a term \( t \) is an element of \( R_A \) if and only if it strongly terminates and whenever it reduces to a term of the form \( \lambda x : B \; u \), then for every term \( v \) in \( R_B \), \( (v/x)u \) is an element of \( R_C \).

3. Prove that if \( x \) is a variable or a constant, then \( x \in R_A \) for all \( A \).

4. Prove that if \( t \) is an element of \( R_A \) and \( t \) reduces to \( t' \), then \( t' \) is an element of \( R_A \).
5. Let \( t \) be a term of the form \((u_1 u_2)\) such that all the one-step reducts of \( t \) are in \( R_A \). We want to prove that \( t \) is in \( R_A \).

Prove that \( t \) strongly terminates.

Prove that if \( A = \iota \) or \( A = \circ \), then \( t \) is in \( R_A \).

Prove that if \( A = B \to C \), then \( t \) is in \( R_A \).

6. Let \( t_1 \) be a term in \( R_{A \to B} \) and \( t_2 \) a term in \( R_A \). We want to prove that \((t_1 t_2)\) is in \( R_B \).

Prove that the term \( t_1 \) and \( t_2 \) strongly terminates.

Let \( n_1 \) be the maximum length of a reduction sequence issued from the term \( t_1 \) and \( n_2 \) be the maximum length of a reduction sequence issued from \( t_2 \).

Prove by induction on \( n_1 + n_2 \) that \((t_1 t_2)\) is in the set \( R_B \).

7. Let \( t \) be a term of type \( A \) and \( \sigma \) be a substitution mapping each variable of type \( B \) to an element of \( R_B \). Prove that \( \sigma t \) is in \( R_A \).

8. Let \( t \) be a term of type \( A \). Prove that \( t \) strongly terminates.

Exercise 2 Consider the model of Simple type theory defined as follows \( \mathcal{M}_\iota = \{7\} \), \( \mathcal{M}_\circ = \{0, 1\} \), \( \mathcal{M}_{A \to B} = \mathcal{M}_A \to \mathcal{M}_B \), that is the set of all functions from \( A \) to \( B \).

- \( \hat{\varepsilon} \) is the identity function,
- \( \hat{\alpha}_{A,B} \) is the function mapping \( f \) and \( a \) to \( f(a) \),
- \( \hat{\kappa}_{A,B} \) is the function mapping \( a \) and \( b \) to \( a \),
- \( \hat{S}_{A,B,C} \) is the function mapping \( f, g \) and \( a \) to \( f \ a \ (g \ a) \),
- \( \hat{\top} = \top = 1 \),
- \( \hat{\bot} = \bot = 0 \),
- \( \hat{\lambda} = \lambda \),
- \( \hat{\forall} = \forall \),
- \( \hat{\exists} = \exists \),
- \( \hat{\forall}_A \) is the function mapping \( f \) to the minimum of \( f(a) \) for \( a \) in \( \mathcal{M}_A \),
- \( \hat{\exists}_A \) is the function mapping \( f \) to the maximum of \( f(a) \) for \( a \) in \( \mathcal{M}_A \).

1. Prove that \( \mathcal{M} \) is a model of Simple type theory.
2. Equality is defined by the rule

\[ x = y \rightarrow \forall c \, ((c \, x) \Rightarrow (c \, y)) \]

prove that \( \|\varepsilon(t = u)\|_\rho = 1 \) if and only if \( \|t\|_\rho = \|u\|_\rho \).

Let \( E \) be the extensionality axiom

\[ \forall f : (\iota \rightarrow \iota) \forall g : (\iota \rightarrow \iota)(\forall x : \iota \, (f \, x) = (g \, x) \Rightarrow f = g) \]

Prove that \( E \) is valid in this model

Prove that \( \neg E \) is not provable in Simple type theory.

3. Build a model of Simple type theory where \( E \) is not valid.

Prove that \( E \) is not provable in Simple type theory.