MPRI 2-7-1
Foundations of proof systems

Gilles Dowek

November, 28th 2022

1 hour and a half.
All documents can be used.

1

(5 points)

(a) Let P and Q be two proposition symbols. Give a cut-free proof in natural deduction of the proposition $P \Rightarrow (Q \Rightarrow P)$.

(b) Express this proof as a closed term in Simply typed lambda calculus.

(c) What is the type of this term?

(d) Show that an irreducible term in Simply typed lambda-calculus has the form $\lambda x_1 : A_1 ... \lambda x_n : A_n\ (x\ u_1\ ...\ u_p)$.

(e) How many closed irreducible terms of type $P \Rightarrow (Q \Rightarrow P)$ are there?

2

(5 points)

Let P and Q be two proposition symbols, and consider the theory in Deduction modulo defined by the rule

$$P \longrightarrow (P \Rightarrow Q)$$

(a) Give a proof of the proposition Q in this theory.

(b) Does the proposition Q have a cut free proof in this theory?

(c) Prove that this theory has a model valued in the algebra $\{0, 1\}$.

(d) Is this theory consistent? It is super-consistent?

(e) Give an example of algebra where this theory does not have a model.

3

(5 points)

In arithmetic, we consider the following alternative to the rule reducing the propositions of the form $N(y)$

$$N(y) \longrightarrow \forall c \ (0 \ \epsilon \ c \Rightarrow \forall x \ (x \ \epsilon \ c \Rightarrow S(x) \ \epsilon \ c) \Rightarrow y \ \epsilon \ c)$$

(a) Let c be a class, π be a proof-term of the proposition $0 \ \epsilon \ c$, and π' be a proof-term of the proposition $\forall x \ (x \ \epsilon \ c \Rightarrow S(x) \ \epsilon \ c)$.

Give a proof-term of the proposition

$$S(S(S(S(S(S(S(S(S(0))))))))) \ \epsilon \ c$$

(b) Give a proof-term of the proposition

$$N(S(S(S(S(S(S(S(S(S(0))))))))))$$

(c) Let c be a class, π be a proof-term of the proposition $0 \ \epsilon \ c$, and π' be a proof-term of the proposition $\forall x \ (x \ \epsilon \ c \Rightarrow S(x) \ \epsilon \ c)$.

Give a proof-term of the proposition

$$\forall y \ (N(y) \Rightarrow y \ \epsilon \ c)$$

(d) Using this proof-terms of questions (b) and (c), give a proof-term of the proposition

$$S(S(S(S(S(S(S(S(S(0))))))))) \ \epsilon \ c$$

(e) Reduce this proof-term to an irreducible form. Have you seen this proof-term before?

4

(5 points)

In $\lambda\Pi$-calculus, give closed irreducible terms of the following types.

(a) $P(c) \Rightarrow (P(d) \Rightarrow P(c))$

(b) $P(c) \Rightarrow (P(d) \Rightarrow P(d))$

(c) $\forall x \ ((\forall y \ P(y)) \Rightarrow P(x))$

(d) $(\forall x \forall y \ R(x, y)) \Rightarrow (\forall x \forall y \ R(y, x))$

(e) $\forall x \ P(x)$