MPRI 2-7-1
Fondements des systèmes de preuves

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1 hour and a half.
All documents can be used.

1

(5 points)

Let P be a proposition symbols (that is, a predicate symbol of arity 0) and let $n \geq 1$.

(a) Give a cut free proof in natural deduction of the proposition

$$\underbrace{P \Rightarrow P \Rightarrow ... \, P \Rightarrow}_{n \text{ occurrences of } P} P$$

(b) Express this proof as a term of simply typed λ-calculus.

(c) How many closed irreducible terms of this type are there?

2

(5 points)

(a) Give two propositions A and B such that

- x is not free in A,
- x is free in B,
- the proposition $\forall x\ (A \Rightarrow B)$ has a constructive proof.

(b) Let π be a constructive cut free proof of $\forall x\ (A \Rightarrow B)$. Show that this proof ends with two introduction rules.

(c) Show that the proposition $A \Rightarrow \forall x\ B$ has a constructive proof.

3

(5 points)

In this exercise, we can use without proof the following theorems:

- If E is a subset of $\mathbb{R}$, then the interior $\overset{\circ}{E}$ of E is an open set.
- If E is a subset of $\mathbb{R}$, then the interior $\overset{\circ}{E}$ of E is a subset of E, $\overset{\circ}{E} \subseteq E$.
- If E is a subset of $\mathbb{R}$ and both E and $\mathbb{R} \setminus E$ are open, then $E = \varnothing$ or $E = \mathbb{R}$.
- $\overset{\circ}{\mathbb{R}} = \mathbb{R}$.

(a) Prove that if E and F are two disjoint sets, $E \cap F = \varnothing$, then $(E \cup F) \setminus E = F$.

(b) Prove that if E is a subset of $\mathbb{R}$, and $E \cup (\overset{\circ}{\mathbb{R} \setminus E}) = \mathbb{R}$, then $(\overset{\circ}{\mathbb{R} \setminus E}) = \mathbb{R} \setminus E$.

(c) Prove that if E is a subset of $\mathbb{R}$, and $E \cup (\overset{\circ}{\mathbb{R} \setminus E}) = \mathbb{R}$, then $\mathbb{R} \setminus E$ is an open set.

(d) Prove that if E is an open subset of $\mathbb{R}$, and $E \cup (\overset{\circ}{\mathbb{R} \setminus E}) = \mathbb{R}$, then $E = \varnothing$ or $E = \mathbb{R}$.

(e) Consider a model of constructive predicate logic, where the propositions are interpreted in the Heyting algebra of the open subsets of $\mathbb{R}$, $[\![A \vee B]\!]_\rho = [\![A]\!]_\rho \cup [\![B]\!]_\rho$, $[\![\neg A]\!]_\rho = (\overset{\circ}{\mathbb{R} \setminus [\![A]\!]_\rho})$, and A is valid if for all ρ, $[\![A]\!]_\rho = \mathbb{R}$.

Prove that if A is a closed proposition and $A \vee \neg A$ is valid in this model then A is valid or $\neg A$ is valid.

4

(5 points)

In the $\lambda\Pi$-calculus modulo the rules of arithmetic, we have, among others, the rules

$$N(y) \longrightarrow \forall c\ (0 \ \epsilon\ c \Rightarrow \forall x\ (N(x) \Rightarrow x \ \epsilon\ c \Rightarrow S(x) \ \epsilon\ c) \Rightarrow y \ \epsilon\ c)$$

$$y \ \epsilon\ E \longrightarrow y + 0 = y$$

$$0 + y \longrightarrow y$$

$$S(x) + y \longrightarrow S(x + y)$$

(a) Give a closed term of type

$$0{+}0 = 0 \Rightarrow \forall x\ (N(x) \Rightarrow x{+}0 = x \Rightarrow S(x){+}0 = S(x)) \Rightarrow \forall y\ (N(y) \Rightarrow y{+}0 = y)$$

(b) Let π be a closed term of type $0 = 0$ and π' a closed term of type $\forall x\ (N(x) \Rightarrow x + 0 = x \Rightarrow S(x + 0) = S(x))$. Give a closed term of type

$$\forall y\ (N(y) \Rightarrow y + 0 = y)$$