MPRI 2-7-1
Fondements des systèmes de preuves

Gilles Dowek

Wednesday, November 21st, 2018

1 hour and a half.
All documents can be used.

1

(5 points)

Let P and Q be proposition symbols (that is, two predicate symbols of arity 0).

(a) Give a proof in natural deduction of the proposition

$$P \Rightarrow P \Rightarrow (P \Rightarrow Q) \Rightarrow Q$$

(b) Express this proof as a term of simply typed λ-calculus.

(c) What is the irreducible form of this term?

(d) How many closed irreducible terms of type

$$P \Rightarrow P \Rightarrow (P \Rightarrow Q) \Rightarrow Q$$

are there?

2

(5 points) Let A be the proposition

$$a \leq b$$

(a) In which language (predicate symbols and function symbols) is this proposition expressed?

(b) Give a model where the proposition A is not valid.

(c) Give a model where $\neg A$ is not valid.

(d) Give a model where neither A nor $\neg A$ are valid.

(e) Give a model where $A \vee \neg A$ is not valid.

3

(5 points)

(a) Let Q be an atomic proposition and A a non atomic one. Prove that a constructive cut free proof of the sequent $Q \vdash A$ ends with an introduction rule.

(b) Let Q be an atomic proposition. Prove that there is no constructive cut free proof of the sequent $Q \vdash \bot$.

(c) Let P and Q be two proposition symbols (that is two predicate symbols of arity 0) and consider the theory in Deduction modulo theory, formed with the rewrite rule

$$P \longrightarrow Q \wedge (P \Rightarrow \bot)$$

Prove that there is no constructive cut free proof of the sequent $Q \vdash \bot$ in this theory.

(d) Give a constructive proof of the sequent $Q \vdash \bot$ in this theory.

(e) Is this theory super-consistent?

4

(5 points)

Consider a language with three sorts of terms: *point*, *line* and *scalar*, two predicate symbols $=$ with arity $\langle scalar, scalar \rangle$ and $\in$ with arity $\langle point, line \rangle$ and two function symbols d, *distance*, with arity $\langle point, point, scalar \rangle$ and b, *bisector*, with arity $\langle point, point, line \rangle$.

Consider the theory formed with the rewrite rule

$$x \in b(y, z) \longrightarrow d(x, y) = d(x, z)$$

and the axiom

$$\forall x \forall y \forall z \ (x = y \Rightarrow y = z \Rightarrow x = z)$$

Let A a proposition stating that if two bisectors of the triangle xyz intersect at a point w, then the three bisectors intersect at this point:

$$\forall w \forall x \forall y \forall z \ (w \in b(x, y) \Rightarrow w \in b(y, z) \Rightarrow w \in b(x, z))$$

(a) What is the irreducible form of this proposition?

(b) Give a proof of this the proposition A in this theory.

(c) Is this theory super-consistent?

(d) Express the proof built at question (b) as a term of the $\lambda\Pi$-calculus modulo theory.