MPRI 2-7-1
Fondements des systèmes de preuves

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1 hour and a half.
All documents can be used.

1

(5 points)

Let P and Q be proposition symbols (that is, predicate symbols of arity 0). And let $\mathcal{B}$ be a pre-Heyting algebra whose domain is $\{0, 1/2, 1\}$ and such that $\tilde{\Rightarrow}(a, b) = b$ if $a > b$, and 1 otherwise.

(a) Build a model valued in this algebra where the proposition $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ is not valid.

(b) Does the proposition $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ have a constructive proof?

2

(6 points)

(a) Let P and Q be proposition symbols. Give a non cut-free proof in natural deduction of the proposition $P \Rightarrow (Q \Rightarrow P)$.

(b) Express this proof by a closed term in Simply typed lambda calculus.

(c) What is the type of this term?

(d) Is this term normal? What is its normal form?

(e) Show that a normal closed term in Simply typed lambda-calculus has the form $\lambda x_1 : A_1 ... \lambda x_n : A_n\ (x\ u_1\ ...\ u_p)$.

(f) How many normal closed terms of type $P \Rightarrow (Q \Rightarrow P)$ are there?

3

(4 points)

Let P and Q be proposition symbols.

Consider the theory in Deduction modulo, formed with the rewrite rule

$$P \longrightarrow (Q \Rightarrow Q)$$

(a) Is this theory super-consistent?

(b) Is it consistent?

(c) What about the theory formed with the rewrite rule

$$P \longrightarrow ((P \Rightarrow Q)) \Rightarrow Q$$

?

4

(5 points)

In the $\lambda\Pi$-calculus modulo the rules of arithmetic, we have, among others, the rules

$$N(y) \longrightarrow \forall c\ (0 \ \epsilon \ c \Rightarrow \forall x\ (N(x) \Rightarrow x \ \epsilon \ c \Rightarrow S(x) \ \epsilon \ c) \Rightarrow y \ \epsilon \ c)$$

$$y \ \epsilon \ E \longrightarrow y + 0 = y$$

$$0 + y \longrightarrow y$$

$$S(x) + y \longrightarrow S(x + y)$$

(a) Give a closed term of type

$$0{+}0 = 0 \Rightarrow \forall x\ (N(x) \Rightarrow x{+}0 = x \Rightarrow S(x){+}0 = S(x)) \Rightarrow \forall y\ (N(y) \Rightarrow y{+}0 = y)$$

(b) Let π be a closed term of type $0 = 0$ and π' a closed term of type $\forall x\ (N(x) \Rightarrow x + 0 = x \Rightarrow S(x + 0) = S(x))$. Give a closed term of type

$$\forall y\ (N(y) \Rightarrow y + 0 = y)$$