MPRI 2-7-1
Foundations of proof systems

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1 hour and a half.
All documents can be used.

1

(6 pts)

Let P, Q and R be three proposition symbols, and consider the theory in Deduction modulo defined by the rule

$$P \longrightarrow (P \Rightarrow Q)$$

(a) Give a proof of the proposition Q in this theory.

(b) Does the proposition Q have a cut free proof in this theory?

(c) Prove that this theory has a model valued in $\{0, 1\}$.

(d) Is the proposition R provable in this theory.

(e) Is this theory consistent? It is super-consistent?

(f) Give an example of algebra where this theory does not have a model.

2

(4 pts)

(a) Give a proof in Natural Deduction of the proposition $((P \Rightarrow P) \Rightarrow P) \Rightarrow P$.

(b) Express this proof as a term of simply typed lambda-calculus.

(c) What is the type of this term?

3

(5 pts)

In $\lambda\Pi$-calculus, give irreducible terms of the following types.

(a) $P(c) \Rightarrow (P(d) \Rightarrow P(c))$

(b) $\forall x\ ((\forall y\ P(y)) \Rightarrow P(x))$

(c) $(\forall x \forall y\ R(x, y)) \Rightarrow (\forall x \forall y\ R(y, x))$

(d) $P(c) \Rightarrow (\forall x\ (P(x) \Rightarrow P(f(x)))) \Rightarrow P(f(f(c)))$

(e) $\forall x\ P(x)$

4

(5 pts) Consider $\lambda\Pi$-calculus modulo the rule

$$N(y) \longrightarrow \forall c\ (0\ \epsilon\ c \Rightarrow \forall x\ (N(x) \Rightarrow x\ \epsilon\ c \Rightarrow S(x)\ \epsilon\ c) \Rightarrow y\ \epsilon\ c)$$

(a) Give a term of type $N(0)$.

(b) Give a term of type $N(S(0))$.