Proof search – Sequent calculus – Unification
0. Summary of previous episodes
The notion of proof
The notions of model and validity
A provable in $\mathcal{T}$ if and only if $A$ valid in all models of $\mathcal{T}$
Examples of theories: arithmetic, set theory
Undecidability and incompleteness
Constructive logic (without the excluded middle)
Positive use of negative results

Church’s theorem: predicate logic undecidable

But

1. requires a binary predicate: if only unary predicates: decidable
2. may become decidable if we add axioms: identify decidable theories Presburger, Skolem, Tarski, ...
3. semi-decidable: proof search methods
Semi-decidability

Enumerate all the trees
If a proof of $A$ exists, it will eventually show up
Otherwise no termination ($\mu$)

A very general method to solve problems: generate and test
How to write a good novel

a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t u, v, w, x, y, z, aa, ab, ac, ad, ae, af, ag, ah, ai, aj, ak, al, am...

Eventually *Mrs Dalloway* will show up
If you can recognize a good novel, you can write one (but it takes longer)
Enumeration

Theoretical use (semi-decidability, Gödel’s theorem)
But no practical use

However: the idea of enumeration and test has a practical use
I. Proof search in natural deduction
Enumerate the rules

Enumerate the rules that may apply at each node (bottom up)

\[ P \vdash Q \Rightarrow (P \land Q) \, ? \]

Let us start with an introduction rule

How many options?
Enumerate the rules

Enumerate the rules that may apply at each node (bottom up)

\[
\begin{array}{c}
P, Q \vdash P \land Q \\
P \vdash Q \Rightarrow (P \land Q) \\
\end{array}
\]

\Rightarrow\text{-intro}

One more introduction rule
How many options?
Enumerate the rules

Enumerate the rules that may apply at each node (bottom up)

\[
\frac{P, Q \vdash P}{P, Q \vdash P \land Q} \quad \frac{P, Q \vdash Q}{P, Q \vdash P \land Q} \quad \frac{P, Q \vdash P \land Q}{P \vdash Q \Rightarrow (P \land Q)}
\]

∧-intro \quad ⇒-intro

One more introduction rule
Enumerate the rules that may apply at each node (bottom up)

\[
\begin{align*}
\frac{P, Q \vdash P}{P, Q \vdash P \land Q} & \quad \text{axiom} \\
\frac{P, Q \vdash Q}{P, Q \vdash P \land Q} & \quad ? \quad \land\text{-intro} \\
\frac{P \vdash Q \Rightarrow (P \land Q)}{P \vdash Q \Rightarrow (P \land Q)} & \quad \Rightarrow\text{-intro}
\end{align*}
\]
Enumerate the rules that may apply at each node (bottom up)

\[
\begin{align*}
P, Q &\vdash P \quad \text{axiom} \\
P, Q &\vdash Q \quad \text{axiom} \\
P, Q &\vdash P \land Q \quad \land\text{-intro} \\
P &\vdash Q \implies (P \land Q) \quad \implies\text{-intro}
\end{align*}
\]
The elimination rules

A less pleasant situation

\[ \Gamma \vdash A \land B \]
\[ \vdash \land \text{-elim} \]

\[ \Gamma \vdash A \]

Always applies

Moreover we need to guess \( B \) that does not appear in the conclusion (that is: enumerate all the possible \( B \)'s)
The elimination rules

\[ P \land Q \vdash P \]

?-elim
The elimination rules

\[
\begin{align*}
P \land Q & \vdash P \land Q \\
P \land Q & \vdash P \quad \land\text{-elim}
\end{align*}
\]

How did you guess \( \land \) and \( Q \)?
An asymmetry

In natural deduction

The shape of the conclusion of the sequent guides the choice of the introduction rules

The shape of the hypotheses of the sequent does not guide the choice of the elimination rules
II. The (cut free) sequent calculus
The idea

Introduction rules (that work): kept: **right rules**
Elimination rules: replaced by introduction rules applied to the hypotheses of the sequent: **left rules**

For instance

\[
\Gamma, A \land B \vdash \mathcal{C} \quad \text{\textasciitilde-left}
\]

What can we do with this hypothesis?
The idea

Introduction rules (that work): kept: right rules
Elimination rules: replaced par introduction rules applied to the hypotheses of the sequent: left rules

For instance

\[
\begin{array}{c}
\Gamma, A, B \vdash C \\
\Gamma, A \land B \vdash C
\end{array}
\quad \text{\^{\land}-left}
\]
Back to our example

\[ P \land Q \vdash P \]
Back to our example

\[
\begin{align*}
  & P, Q \vdash P \\
  \hline
  & P \\
  & Q \vdash P \\
  \hline
  & P, Q \vdash P \\
  & P \land Q \vdash P \quad \text{\texttt{\textsf{\texttt{\texttt{-left}}}}} \\
\end{align*}
\]
Other rules

\[
\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C} \quad \Rightarrow\text{-left}
\]

\[
\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B} \quad \neg\text{-left}
\]

\[
\frac{\Gamma, (t/x)A \vdash B}{\Gamma, \forall x A \vdash B} \quad \forall\text{-left}
\]

\[
\frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B} \quad \exists\text{-left} \quad x \notin FV(\Gamma, B)
\]
The contraction rule

In natural deduction: preservation of hypotheses: multiple use
In sequent calculus

\[ \Gamma, (t/x)A \vdash B \]
\[ \frac{}{\Gamma, \forall x A \vdash B} \text{ \(\forall\)-left} \]

the hypothesis \(\forall x A\) disappears

Saving a copy of the hypothesis

\[ \Gamma, A, A \vdash B \]
\[ \frac{}{\Gamma, A \vdash B} \text{ contraction-left} \]

multi-set
III. The excluded middle in sequent calculus
Like in natural deduction

Three solutions:

special rule

\[
\Gamma \vdash A \lor \neg A
\]

special rule

\[
\begin{align*}
\Gamma & \vdash \neg \neg A \\
\Gamma & \vdash A
\end{align*}
\]

Or...
An example

\[ \begin{array}{c}
\vdash \neg(P \Rightarrow Q) \\
\neg\neg(P \Rightarrow Q), \ P \vdash Q \\
\end{array} \]

\(-\text{left}\)
A better idea

\[
\begin{align*}
\neg
(\neg\neg(P \Rightarrow Q), P, \neg Q \mid \bot) & \quad \text{\neg-left} \\
\neg
(\neg\neg(P \Rightarrow Q), P \mid \neg\neg Q) & \quad \text{\neg-right} \\
\neg
(\neg\neg(P \Rightarrow Q), P \mid Q) & \quad \text{excluded middle}
\end{align*}
\]
A better idea

\[
\begin{align*}
  & P \vdash P \quad \text{axiom} \\
  & P, Q \vdash Q \quad \text{axiom} \\
  & \frac{P, P \Rightarrow Q \vdash Q}{P, P \Rightarrow Q \vdash Q} \quad \Rightarrow\text{-left} \\
  & \frac{P, \neg Q, P \Rightarrow Q \vdash \bot}{P, \neg Q \vdash \neg (P \Rightarrow Q)} \quad \neg\text{-left} \\
  & \frac{\neg(P \Rightarrow Q), P \vdash \bot}{\neg(P \Rightarrow Q), P \vdash \neg Q} \quad \neg\text{-right} \\
  & \frac{\neg(P \Rightarrow Q), P \vdash \neg Q}{\neg(P \Rightarrow Q), P \vdash Q} \quad \text{excluded middle}
\end{align*}
\]
Save $\mathcal{Q}$ on the left

\[
\begin{align*}
\frac{P \vdash P \text{ axiom}}{P, P \Rightarrow Q \vdash Q \Rightarrow-	ext{left}} & \quad \frac{P, Q \vdash Q \text{ axiom}}{P, P \Rightarrow Q \vdash Q \Rightarrow-	ext{left}} \\
\frac{P, \neg Q, P \Rightarrow Q \vdash \bot \neg-	ext{left}}{P, \neg Q \vdash \neg(P \Rightarrow Q) \neg-	ext{right}} & \quad \frac{P, \neg Q \vdash \neg(P \Rightarrow Q) \neg-	ext{left}}{P, \neg Q \vdash \neg\neg Q \neg-	ext{right}} \\
\frac{\neg\neg(P \Rightarrow Q), P, \neg Q \vdash \bot \neg-	ext{right}}{\neg\neg(P \Rightarrow Q), P \vdash \neg\neg Q \text{ excluded middle}} & \quad \frac{\neg\neg(P \Rightarrow Q), P \vdash Q}{\neg\neg(P \Rightarrow Q), P \vdash Q}
\end{align*}
\]
Or keep it on the right

Sequents with several propositions on the right
Sequents with several propositions on the right

\[
\frac{P \vdash P, Q}{P, P \Rightarrow Q \vdash Q} \quad \text{\(\text{axiom}\)}
\]

\[
\frac{P, Q \vdash Q}{P \vdash P, Q \Rightarrow Q} \quad \Rightarrow\text{-left}
\]

\[
\frac{P \vdash \neg (P \Rightarrow Q), Q}{P \vdash \neg \neg (P \Rightarrow Q), P \vdash Q} \quad \neg\text{-right}
\]

\[
\frac{\neg \neg (P \Rightarrow Q), P \vdash Q}{\neg \neg (P \Rightarrow Q), P \vdash Q} \quad \neg\text{-left}
\]
The excluded middle

Just as you have a left contraction rule, you have a right one

\[
\Gamma \vdash A, A, \Delta \\
\hfill \text{contraction-right} \\
\hfill \Gamma \vdash A, \Delta
\]

Now you can prove the excluded middle

\[
A \vdash A \quad \text{axiom} \\
\hfill \neg\text{-right} \\
\vdash A, \neg A \\
\hfill \lor\text{-right} \\
\vdash A, A \lor \neg A \\
\hfill \lor\text{-right} \\
\vdash A \lor \neg A, A \lor \neg A \\
\hfill \text{contraction-right} \\
\hfill \vdash A \lor \neg A
\]
IV. Proof search in (cut free) sequent calculus
The choices

No need to enumerate propositions

But ... some choices left
1. The choice of the sequent

\[
\frac{P, Q \vdash P \quad P, Q \vdash Q}{P, Q \vdash P \land Q} \quad \land\text{-right}
\]

2. The choice of the proposition

\[P \land Q \vdash Q \lor R\]

3. The choice of the rule: logic, contraction (or axiom)

4. The choice of the term

\[
\frac{\Gamma, (t/x)A \vdash B}{\Gamma, \forall x\ A \vdash B} \quad \forall\text{-left}
\]
Mazes and deviled eggs

Don’t know choices
Either A or B: you chose A, if A fails you chose B (general case)

Don’t care choices
Either A or B: whatever you chose, the result will be the same (sequencing of independent tasks)
1. The choice of the sequent

\[ P, Q \vdash P \quad ? \quad P, Q \vdash Q \quad ? \quad P, Q \vdash P \land Q \quad \land\text{-right} \]

2. The choice of the proposition

\[ \forall x (P(x) \land Q(x)) \vdash \forall x (P(x)) \]

3. The choice of the rule: logic, contraction (or axiom)

\[ \forall x (P(x) \land \neg P(S(x))) \vdash \]

4. The choice of the term

\[ P(f(f(c))) \vdash \exists x P(f(x)) \]
Finite and infinite don’t know choices

Choice of the proposition: finite

Choice of the rule: finite

Choice of the term: infinite
Avoiding the choice of the term $\exists$-right

$\frac{P(f(f(c))) \vdash \exists x \ P(f(x))}{\exists$-right}$

try $c$, $f(c)$, $f(f(c))$, ...
Which term to chose? How did you guess?
Try instead to delay the choice of the term

\[
\begin{align*}
P(f(f(c))) & \vdash P(f(X)) \\
P(f(f(c))) & \vdash \exists x \ P(f(x)) \quad \exists\text{-right}
\end{align*}
\]
Try instead to delay the choice of the term

\[
\begin{align*}
P(f(f(c))) & \vdash P(f(X)) & \text{axiom} \\
P(f(f(c))) & \vdash \exists x \ P(f(x)) & \exists\text{-right} 
\end{align*}
\]

Proof scheme:

- \exists\text{-right} and \forall\text{-left restricted to a (meta)-variable}
- the \textit{axiom} rule permits to prove any sequent
In a second step

\[
\begin{align*}
P(f(f(c))) \vdash P(f(X)) & \quad \text{axiom} \\
P(f(f(c))) \vdash \exists x \ P(f(x)) & \quad \exists\text{-right}
\end{align*}
\]

Find a substitution that completes this proof scheme (that is, that transforms it into a proof)

Comparing \( P(f(f(c))) \) and \( P(f(X)) \)
Completing a proof scheme

In each sequent proved by the (pseudo) axiom rule

Chose an atomic proposition on the left and on the right

And search for a substitution that make all these pairs identical: the unification algorithm

(then check the conditions: $x$ does not occur in $\Gamma \Delta$)
The unification algorithm: an example

The solutions of the problem

\[ P(f(X)) = P(f(f(c))) \]

are the same as those of the problem

\[ f(X) = f(f(c)) \]

that are the same as those of the problem

\[ X = f(c) \]

and this problem has a solution that is the substitution \( f(c)/X \)
The unification algorithm: general case

Chose an equation in the system

- \[ f(t_1, \ldots, t_n) = f(u_1, \ldots, u_n) \rightarrow \text{replace it with} \]
  \[ t_1 = u_1, \ldots, t_n = u_n \]
- \[ f(t_1, \ldots, t_n) = g(u_1, \ldots, u_m) \rightarrow \text{fail} \]
- \[ X = X \rightarrow \text{delete} \]
- \[ X = t \ (\text{or } t = X), \ X \text{ occurs in } t, \ t \text{ different from } X, \ \text{fail} \]
- \[ X = t \ (\text{or } t = X), \ X \text{ does not occur in } t, \ \text{substitute } t \text{ for } X \text{ in rest of the system solve} \rightarrow \text{substitution } \sigma, \ \text{return } \sigma \cup \{\sigma t/X\} \]
The unification algorithm

I think I have seen this algorithm somewhere
Next time: The equivalence of (cut free) sequent calculus and natural deduction