The completeness theorem
0. Summary of previous episodes
The notion of proof
The notion of model
Examples de theories
The soundness theorem - Independence
I. The soundness theorem
The soundness theorem

If a sequent is provable, then it is valid in all models
The soundness theorem

A mere induction on proof structure

\[
\begin{array}{c}
\pi_1 & \pi_2 \\
\hline
\Gamma \vdash A & \Gamma \vdash B \\
\hline
\Gamma \vdash A \land B
\end{array}
\]

Induction hypothesis: \( \Gamma \vdash A \) and \( \Gamma \vdash B \) valid in all models
\( \Gamma = \{ G_1, ..., G_n \} \) and \( G = G_1 \land ... \land G_n \)
\( G \Rightarrow A \) and \( G \Rightarrow B \) valid in all models hence \( G \Rightarrow (A \land B) \) valid in all models
Same for the other rules
A corollary

Soit

- $\mathcal{T}$ be a theory
- $\mathcal{M}$ be a model in which all the axioms of $\mathcal{T}$ are valid
- $A$ a proposition

If $A$ is provable in $\mathcal{T}$, then $A$ is valid in $\mathcal{M}$

There exists a finite subset $\Gamma$ of $\mathcal{T}$ such that $\Gamma \vdash A$ provable
$\Gamma \vdash A$ valid in $\mathcal{M}$ hence $A$ valid in $\mathcal{M}$
Let

- $\mathcal{T}$ be a theory
- $\mathcal{M}$ be a model in which all the axioms of $\mathcal{T}$ are valid
- $A$ a proposition

If $A$ is not valid in $\mathcal{M}$ then $A$ is not provable in $\mathcal{T}$
The three forms of the soundness theorem

1. If $A$ provable in $\mathcal{T}$ then, $A$ valid in all models of $\mathcal{T}$

2. If there exists a model of $\mathcal{T}$ that is not a model of $A$, then $A$ is not provable in $\mathcal{T}$

3. If $\mathcal{T}$ has a model then $\mathcal{T}$ is consistent
A method to prove that $A$ is not provable in $\mathcal{T}$

Find a model $\mathcal{M}$ in which

all the axioms of $\mathcal{T}$ are valid

$A$ is not valid
An example

A theory $\mathcal{T}$ containing one axiom: $P(c) \lor Q(c)$

Prove that $P(c)$ is not provable in $\mathcal{T}$
Prove that $Q(c)$ is not provable in $\mathcal{T}$
Another example

A binary function symbol +, binary predicate symbol =

\((\mathbb{N}, \text{addition on } \mathbb{N}, \text{equality on } \mathbb{N})\) Is \(\forall x \forall y \exists z (x + z = y)\) valid? Same question for \(\mathbb{Z}\) equipped with addition and equality on \(\mathbb{Z}\)? Is the proposition \(\forall x \forall y \exists z (x + z = y)\) provable?

Is the proposition \(\forall x \forall y (x + y = y + x)\) valid in these models? In all models?
The axiom of parallels

(after 22 centuries) not provable from the other axioms of geometry
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The axiom of parallels

(after 22 centuries) not provable from the other axioms of geometry
The axiom of infinity

- $V_0 = \emptyset,$
- $V_{i+1} = \mathcal{P}(V_i)$
- $V_\omega = \bigcup_i V_i.$

All the elements of $V_\omega$ are finite

- $\mathcal{M} = V_\omega$
- $\hat{=} \text{ function from } \mathcal{M} \times \mathcal{M} \text{ to } \{0, 1\} \text{ such that } \hat{=}(a, b) = 1 \text{ if } a \text{ is equal to } b \text{ and } \hat{=}(a, b) = 0 \text{ otherwise}$
- $\hat{\in} \text{ function from } \mathcal{M} \times \mathcal{M} \text{ to } \{0, 1\} \text{ such that } \hat{\in}(a, b) = 1 \text{ is } a \text{ is an element of } b \text{ nd } \hat{\in}(a, b) = 0 \text{ otherwise}$

Model of the axioms of pairing, union axiom, power set, subset, and extensionality
But not of the axiom of infinity
II. The completeness theorem
What is the universality of this method?

Each time we have a proposition that is not provable in a theory $\mathcal{T}$, can we always find a model that separates $\mathcal{T}$ from $A$? Does the soundness theorem have a converse?

Yes: Gödel’s completeness theorem
If $A$ is valid in all models of $\mathcal{T}$ then $A$ provable in $\mathcal{T}$
The three forms of the completeness theorem

1. If $A$ is valid in all models of $T$, then $A$ is provable in $T$

2. If $A$ is not provable in $T$, then it exists a model of $T$ that is not a model of $A$

3. If $T$ is consistent, then $T$ has a model
1. If $A$ is valid in all models of $\mathcal{T}$, then $A$ is provable in $\mathcal{T}$

2. If $A$ is not provable in $\mathcal{T}$, then there exists a model of $\mathcal{T}$ that is not a model of $A$

3. If $\mathcal{T}$ is consistent, then $\mathcal{T}$ has a model

1. and 2. equivalent: trivial

2. implies 3.: trivial

3. implies 2.
1. If $A$ is valid in all models of $\mathcal{T}$, then $A$ is provable in $\mathcal{T}$
2. If $A$ is not provable in $\mathcal{T}$, then there exists a model of $\mathcal{T}$ that is not a model of $A$
3. If $\mathcal{T}$ is consistent, then $\mathcal{T}$ has a model

1. and 2. equivalent: trivial
2. implies 3.: trivial
3. implies 2.
   - $A$ not provable in $\mathcal{T}$
   - $\mathcal{T}, \neg A$ consistent
   - $\mathcal{T}, \neg A$ has a model $\mathcal{M}$
   - $\mathcal{M}$ model of $\mathcal{T}$ but not of $A$
III. The proof of the completeness theorem
3. If $\mathcal{T}$ is consistent, then $\mathcal{T}$ has a model

A language $\mathcal{L}$, a consistent theory $\mathcal{T}$
We want to build a model $\mathcal{M}$

What can we choose for the elements of $\mathcal{M}$?
Not much to get your teeth into: $\mathcal{L}$, its symbols, its terms and its propositions, $\mathcal{T}$, its axioms...
The closed terms of the language of the theory $\mathcal{T}$
A first attempt

\[ \mathcal{M} \text{ set of closed terms of the language} \]

\[ \hat{f} \text{ function mapping } t_1, \ldots, t_n \text{ to } f(t_1, \ldots, t_n) \]
(\text{if } t \text{ closed } \llbracket t \rrbracket = t)\]

\[ \hat{P} \text{ function mapping } t_1, \ldots, t_n \text{ to 1 if } P(t_1, \ldots, t_n) \text{ is provable, and to 0 otherwise} \]
One axiom $P(c) \lor Q(c)$

$P(c), \neg P(c), Q(c), \neg Q(c)$ all non provable

$\mathcal{M} = \{c\}$, $\hat{P}(c) = 0$, $\hat{Q}(c) = 0$

thus $P(c) \lor Q(c)$ not valid in $\mathcal{M}$

Neither $P(c)$ nor $\neg P(c)$ is provable: no reason to chose 0 rather than 1 for $\hat{P}(c)$
There can be no middle course

We have to make a decision: either $P(c)$ or $\neg P(c)$

If we add the axiom $P(c)$, then $\hat{P}(c) = 1$
If we add the axiom $\neg P(c)$, then, as $P(c) \lor Q(c)$, $Q(c)$ provable and $\hat{Q}(c) = 1$
Adding axioms... and also constants

A constant $c$
Two axioms $\neg P(c)$ and $\exists x \ P(x)$

$\mathcal{M} = \{c\}$
$\hat{P}(c) = 0$
$\exists x \ P(x)$ is not valid in this model

Add a constant $d$ and an axiom $P(d)$, yielding $\mathcal{M} = \{c, d\}$
The constant $d$: (Henkin’s) witness of the existence of an objet verifying $P$
The completion of a theory

Language $\mathcal{L}$, consistent theory $\mathcal{T}$ in $\mathcal{L}$
There exists $\mathcal{L}' (\supseteq \mathcal{L})$ and $\mathcal{U}$ in $\mathcal{L}' (\mathcal{U} \supseteq \mathcal{T})$ s.t.

1. $\mathcal{U}$ is consistent
2. $A$ (closed) or $\neg A$ is provable (even an axiom) in $\mathcal{U}$
3. If $\exists x \ A$ provable in $\mathcal{U}$ there exists $c$ s.t. $(c/x)A$ provable in $\mathcal{U}$
We examine one closed proposition after the other

1. If $A$ is provable, we take it as an axiom
2. If $\neg A$ is provable, we take it as an axiom
3. If neither $A$ nor $\neg A$ is provable, we choose $A$ as an axiom

If $\exists x \ B$ we add an axiom $(c/x)B$ where $c$ is a new constant
\(\mathcal{H} = \{c_i\}\) an infinite number of constants \(c_0, c_1, c_2\ldots\)

\(\mathcal{L}' = (\mathcal{F} \cup \mathcal{H}, \mathcal{P})\)

Closed propositions in \(\mathcal{L}'\) countable: enumeration \(A_0, A_1, A_2,\ldots\)

Family of theories \(\mathcal{U}_n\)

\(\mathcal{U}_0 = \mathcal{T}\)

1. If \(A_n\) provable in \(\mathcal{U}_n\), we let \(B = A_n\)
2. If \(\neg A_n\) provable in \(\mathcal{U}_n\), we let \(B = \neg A_n\)
3. If neither \(A_n\) nor \(\neg A_n\) provable in \(\mathcal{U}_n\), we let \(B = A_n\)

If \(B\) not of the form \(\exists x\ C\), we let \(\mathcal{U}_{n+1} = \mathcal{U}_n \cup \{B\}\)

If \(B = \exists x\ C\) then, we let \(\mathcal{U}_{n+1} = \mathcal{U}_n \cup \{B, (c_i/x)C\}\) (\(c_i\) smallest constant not in \(\mathcal{U}_n\))
\[ U = \bigcup U_i \]

1. \( U \) is consistent
2. \( A \) (closed) or \( \neg A \) is provable (even an axiom) in \( U \)
3. If \( \exists x \ A \) provable in \( U \) there exists \( c \) s.t. \( (c/x)A \) provable in \( U \)

A “very infinite” theory, even not always an algorithm to recognize the axioms
The properties of the theory $\mathcal{U}$

$\neg A$ provable if and only if $A$ non provable

$A \land B$ provable if and only if $A$ provable and $B$ provable

$A \lor B$ if and only if $A$ provable or $B$ is provable

$A \Rightarrow B$ provable if and only if if $A$ provable then $B$ provable

$\forall x \ A$ provable if and only if for all closed term $t$, $(t/x)A$ provable

$\exists x \ A$ is provable if and only if there exists a closed term $t$, such that $(t/x)A$ provable

The case of the $\lor$: if $A \lor B$ provable, then

$A$ provable or $\neg A$ provable

if $\neg A$ provable, then $B$ provable
(Finally) the completeness theorem

\( \mathcal{M} \) set of closed terms of \( \mathcal{L'} \)
\( \hat{f} \) function mapping \( t_1, \ldots, t_n \) to \( f(t_1, \ldots, t_n) \)
\( \hat{P} \) function mapping \( t_1, \ldots, t_n \) to 1 if \( P(t_1, \ldots, t_n) \) is provable, and to 0 otherwise

A valid in \( \mathcal{M} \) if and only if \( A \) provable in \( \mathcal{U} \) (induction on the structure of formulas)
\( \mathcal{M} \) model of \( \mathcal{U} \) hence of \( \mathcal{T} \)
This proof works only when $\mathcal{L}$ is finite or countable (enumeration)

Extension to non countable language (transfinite enumeration, axiom of choice)
Soundness + completeness

A provable in $\mathcal{T}$
if and only if
$A$ is valid in all the models of $\mathcal{T}$

A miracle: finite / infinite
In TD: more examples

Next time: the undecidability theorem