Exercise 1. (© Ivan Varzinczak 2018) Let $N_C = \{\text{Man, Woman, Parent, Grandparent}\}$ and $N_R = \{\text{marriedTo, hasChild}\}$. Formalize in $\mathcal{ALC}$ the following concepts expressed in natural language.

1. Married women.
2. Fathers married to married women.
3. Men who are single or have unmarried daughters.
4. Men who are single or have unmarried daughters that do not have married sons.
5. Men who have only unmarried daughters.
6. Mothers married to married men or single men who are not parents.
7. Men married to a woman who has only married daughters.
8. Men who are fathers of women who are not married but are mothers.
9. Men or women who are grandparents whose children are married men.

Exercise 2. Propose concept names, role names and individual names to express in some knowledge base the properties below. Any feature for description logics is allowed but try to minimize what is out of $\mathcal{ALC}$.

1. Employed students are students and employees.
2. Students are not taxpayers.
3. Employed students are taxpayers.
4. Employed students who are parents are not taxpayers.
5. To work for is to be employed by.
6. John is an employed student, John works for IBM.

Exercise 3. Show that if the concept $C$ in $\mathcal{ALC}$ is valid, then $\forall r.C$ is valid for all role names $r$. 

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Exercise 4. Determine which concepts below are satisfiable.

1. \((\neg\exists r. (A \sqcup \neg A)) \sqcup \exists s. \bot\).
2. \((\neg \forall r. A) \sqcup \exists r. A\).
3. \((\exists r. A) \cap (\exists s. \neg A)\).
4. \((\exists r. C) \cap (\forall r. \neg C)\).

Exercise 5. Consider the interpretation \(I\) below.

\[\begin{array}{ccc}
\downarrow & s & \downarrow \\
A & \rightarrow s & \rightarrow r & \rightarrow r \downarrow \\
\end{array}\]

For the concepts \(C\) below, compute \(C^I\).

- \(\neg\exists r. (\neg A) \sqcap (\neg B) \sqcap \forall s. \neg A \sqcap \exists s. \exists s. \exists s. \exists s. A\)
- \(\forall s. \exists s. \exists s. A \sqcap (\exists s. (A \sqcap \neg\forall s. \neg B)) \sqcap (\neg\forall r. (\exists r. (A \sqcup \neg A)))\)

Exercise 6. Show that \((\mathcal{T}, A) \models a : C\) iff \((\mathcal{T}, A \cup \{a : \neg C\})\) is not consistent.

Exercise 7. (Infinite models) Let \(\mathcal{ALCIN}\) be the extension of \(\mathcal{ALC}\) with unqualified number restrictions and inverse roles. Let \(C = \neg A \sqcap \exists r. A\) and \(\mathcal{T} = \{A \sqsubseteq \exists r. A, T \sqsubseteq (\leq 1 \ r^-)\}\). Show that for all interpretations \(\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})\) such that \(C^\mathcal{I} \neq \emptyset\) and \(\mathcal{I} \models \mathcal{T}\), the interpretation domain \(\Delta^\mathcal{I}\) is infinite.

Exercise 8. Let \(\mathcal{K} = (\mathcal{T} \cup \{A \sqsubseteq C\}, \cdot, A)\) be a knowledge base such that \(A\) is a concept name and \(B\) does not occur in \(\mathcal{K}\). Show that \(\mathcal{K}\) is consistent iff \(\mathcal{K}' = (\mathcal{T} \cup \{A \equiv B \sqcap C\}, \cdot, A)\) is consistent.

Exercise 9. (Tree interpretation property) A tree is understood below as a directed graph \((V, E)\) such that there is a unique root \(r\) such that there is no \(v \in V\) with \((v, r) \in E\) and for every node \(v \in V \setminus \{r\}\), there is a unique node \(v' \in V\) such that \((v', v) \in E\).

Let \(C\) be an \(\mathcal{ALC}\) concept and \(\mathcal{T}\) be a TBox. An interpretation \(\mathcal{I}\) is a tree model for \(C\) with respect to \(\mathcal{T}\) iff the conditions below hold:

- \(\mathcal{I}_\mathcal{I} = (\Delta^\mathcal{I}, \cup_r v^\mathcal{I})\) is a tree,
- the root of \(\mathcal{I}_\mathcal{I}\) belongs to \(C^\mathcal{I}\),
Given an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, a path in $\mathcal{I}$ is a finite sequence $(a_1, \ldots, a_n) \in (\Delta^\mathcal{I})^+$ such that for all $i \in [1, n - 1]$, we have $(a_i, a_{i+1}) \in \bigcup_r r^\mathcal{I}$. An $a$-path is a path such that $a_1 = a$. Given an interpretation $\mathcal{I}$, its unravelling at $a \in \Delta^\mathcal{I}$ is the interpretation $\mathcal{U} = (\Delta^\mathcal{U}, \cdot^\mathcal{U})$ such that

- $\Delta^\mathcal{U}$ is the set of $a$-paths in $\mathcal{I}$.
- For all concept names $A$, we have $A^\mathcal{U} \overset{\text{def}}{=} \{(a_1, \ldots, a_n) \in \Delta^\mathcal{U} \mid a_n \in A^\mathcal{I}\}$.
- For all role names $r$, we have $r^\mathcal{U} \overset{\text{def}}{=} \{((a_1, \ldots, a_n), (a_1, \ldots, a_n, a_{n+1})) \mid (a_n, a_{n+1}) \in r^\mathcal{I}\}$.

1. Show that $\mathcal{U}$ is a tree model for $\top$ with respect to the empty TBox.
2. Show that for all concepts $C$ and all $(a_1, \ldots, a_n) \in \Delta^\mathcal{U}$, we have $(a_1, \ldots, a_n) \in C^\mathcal{U}$ iff $a_n \in C^\mathcal{I}$.
3. Conclude that if $C$ is satisfiable with respect to the TBox $\mathcal{T}$, then $C$ has a tree interpretation with respect to $\mathcal{T}$.
4. Determine whether if $C$ is satisfiable with respect to the TBox $\mathcal{T}$, then $C$ has always a finite tree interpretation with respect to $\mathcal{T}$.

**Exercise 10.** (Disjoint unions) Let $X$ be an index set and $(\mathcal{I}_i)_{i \in X}$ be a family of interpretations $\mathcal{I}_i = (\Delta^{\mathcal{I}_i}, \cdot^{\mathcal{I}_i})$. The disjoint union $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ is defined as follows.

- $\Delta^\mathcal{J} \overset{\text{def}}{=} \bigcup_{i \in X} \{i\} \times \Delta^{\mathcal{I}_i}$.
- $A^\mathcal{J} \overset{\text{def}}{=} \bigcup_{i \in X} \{i\} \times A^{\mathcal{I}_i}$.
- $r^\mathcal{J} \overset{\text{def}}{=} \bigcup_{i \in X} \{(i, a), (i, b)\} \mid (a, b) \in r^{\mathcal{I}_i}$.

1. Show that for all $i \in X$, $a \in \Delta^{\mathcal{I}_i}$ and $\text{ALC}$ concepts $C$, we have $a \in C^{\mathcal{I}_i}$ iff $(i, a) \in C^\mathcal{J}$.
2. Given an $\text{ALC}$ TBox $\mathcal{T}$, assume that for all $i \in X$, we have $\mathcal{I}_i \models \mathcal{T}$. Conclude that $\mathcal{J} \models \mathcal{T}$.
3. Show that there is no $\text{ALC}$ concept $C$ that is satisfiable but only in finite interpretations.
4. Propose an extension of disjoint unions to handle individual names.

5. We have seen the relation \( \mathcal{K} \models C \subseteq D \) where \( \mathcal{K} \) is a knowledge base. Given a TBox \( \mathcal{T}, \mathcal{T} \models C \subseteq D \) is defined similarly: \( \mathcal{T} \models C \subseteq D \) iff for all interpretations \( \mathcal{I} \) such that \( \mathcal{I} \models \mathcal{T} \), we have \( C^\mathcal{I} \subseteq D^\mathcal{I} \). Given a knowledge base \( \mathcal{K} = (\mathcal{T}, \mathcal{A}) \), show that \( \mathcal{K} \models C \subseteq D \) iff \( \mathcal{T} \models C \subseteq D \).

**Exercise 11.** (© Meghyn Bienvenu) Consider the knowledge base \( \mathcal{K} = (\mathcal{T}, \mathcal{A}) \) with

- \( \mathcal{T} = \{ A \sqsubseteq \forall r.B, B \sqsubseteq \neg F, E \sqsubseteq G, A \sqsubseteq D \sqcup E, D \sqsubseteq \exists r.F, \exists r.\neg B \sqsubseteq G \} \)
- \( \mathcal{A} = \{ a : A, (a, b) : r, b : F \} \).

1. Do we have \( \mathcal{T} \models A \sqsubseteq \exists r.B \)?

2. Is \( A \sqcap \forall r.\neg B \) satisfiable with respect to \( \mathcal{T} \)?

3. Classify \( \mathcal{T} \): state which atomic concept inclusions are entailed from \( \mathcal{T} \).

4. Is \( \mathcal{K} \) satisfiable/consistent?