** Exercises related to previous sessions **

**Exercise 1.** Let $K = (T, A)$ be a knowledge base for $ALC$ with role axioms, with $T = T_{GCI} \cup (T_{RA} \cup \{r \circ s \equiv s \circ r\})$. $T_{GCI}$ is made of GCIs and $T_{RA}$ is made of role axioms (whose except format is unspecified). Show that $K$ is consistent iff $(T_{GCI} \cup (T_{RA} \cup T'_{RA}), A)$ is consistent with $T'_{RA}$ equal to $\{r \circ s \sqsubseteq t, t \sqsubseteq r \circ s, s \circ r \sqsubseteq t, t \sqsubseteq s \circ r\}$ for some new role name $t$. To do so, use minimal assumptions about role axioms in $T_{RA}$.

**Exercise 2.** $EL$ is a fragment of $ALC$ in which the $EL$ concept are defined from $C ::= \top | A | C \sqcap D | \exists r.C$. $EL$ knowledge bases are defined as for $ALC$ except that only $EL$ concepts are allowed. Show that every $EL$ knowledge base is consistent.

**Exercise 3.** (© Franz Baader 2017) Let $K = (T, A)$ be a knowledge base for $ALC$ with concept names in NNF. A precompletion of $K$ is defined as a clash-free $ABox A'$ obtained from $A$ by applying all possible tableaux rules except the $\exists$-rule.

1. Prove that $(T, A)$ is consistent iff there is a precompletion $A'$ such that $(T, A')$ is consistent.

2. Let $X$ be an index set and $(I_i)_{i \in X}$ be a family of interpretations $I_i = (\Delta_{I_i}, \cdot_{I_i})$. The disjoint union $J = (\Delta^J, \cdot^J)$ is defined as follows.

   - $\Delta^J \overset{\text{def}}{=} \bigcup_{i \in X} \{i\} \times \Delta_{I_i}$.
   - $A^J \overset{\text{def}}{=} \bigcup_{i \in X} \{i\} \times A_{I_i}$.
   - $r^J \overset{\text{def}}{=} \bigcup_{i \in X} \{(i, a), (i, b)\} \mid (a, b) \in r_{I_i}$.
   - The interpretation of the individual names is arbitrary.

   (2.1) Show that for all $i \in X$, $a \in \Delta_{I_i}$ and $ALC$ concepts $C$, we have $a \in C_{I_i}$ iff $(i, a) \in C^J$.
2.2 Given an $\mathcal{ALC}$ TBox $\mathcal{T}$, assume that for all $i \in X$, we have $\mathcal{I}_i \models \mathcal{T}$. Conclude that $\mathcal{J} \models \mathcal{T}$.

3. Show that $\mathcal{K}$ is consistent iff there is a precompletion $\mathcal{A'}$ such that for all individual names $a$ occurring in $\mathcal{A}$, the concept $\bigcap_{a : C \in \mathcal{A}'}$ is satisfiable with respect to $\mathcal{T}$.

** Exercises related to today session**

**Exercise 4.** The set of formulae for QBF (“Quantified Boolean Formula”) is defined as follows

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \exists p \, \varphi \mid \forall p \, \varphi$$

Formulae are interpreted on valuations $v : \text{PROP} \to \{T, \bot\}$ with the following clauses

- $v \models \exists p \, \varphi \iff v[p \leftarrow \bot] \models \varphi \lor v[p \leftarrow T] \models \varphi$.
- $v \models \forall p \, \varphi \iff v[p \leftarrow \bot] \models \varphi \land v[p \leftarrow T] \models \varphi$.

A QBF formula $\varphi$ is satisfiable $\iff$ there is a valuation $v$ such that $v \models \varphi$. For instance, $\forall p \, \forall q \, p \iff q$ is not QBF satisfiable whereas $\exists p \, \exists q \, p \iff q$ is QBF satisfiable. The satisfiability problem for QBF is known to be PSPACE-complete even for the restriction to the QBF formulae below

$$\forall p_{2n} \exists p_{2n-1} \cdots \exists p_2 \exists p_1 \varphi,$$

where $\varphi$ is a (quantifier-free) propositional formula built over the propositional variables $p_1, \ldots, p_{2n}$.

Define a reduction from QBF satisfiability (for the restricted form) to $\mathcal{ALC}$ concept satisfiability, possibly using (sub)concepts enforcing an exponential number of individuals in the interpretations (as seen previously). It is not requested to show the correctness of the reduction.

**Exercise 5.** In the correctness proof for the reduction from the $(\infty \times \infty)$-tiling problem to concept satisfiability of $\mathcal{ALC}$ with local role value maps, show that for every $C \subseteq D \in \mathcal{T}$, we have $\mathcal{J} \models C \subseteq D$. 
**Exercise 6.** Let $T$ be an $\mathcal{EL}$ TBox, $C$, $D$ be $\mathcal{EL}$ concepts and $A$, $B$ be new concept names not occurring in $T$, $C$, $D$. Show the equivalence between the statements below:

1. $T \models C \subseteq D$,

2. $T \cup \{A \subseteq C, D \subseteq B\} \models A \subseteq B$.

Does the equivalence hold if we replace $\mathcal{EL}$ by the more expressive description logic $\mathcal{ALC}$?