Exercise 1. Let us consider the CGS $\mathcal{M}$ below with two agents.

Show that $\mathcal{M}, s_2 \models \langle\langle 1 \rangle \rangle (GF\ p \land GF\ q)$ and $\mathcal{M}, s_2 \not\models \langle\langle 2 \rangle \rangle (GF\ p \land GF\ q)$.

Exercise 2. Show that $\langle\langle \emptyset \rangle \rangle G(\psi \Rightarrow (\varphi \land \langle\langle A \rangle \rangle X\psi)) \Rightarrow \langle\langle \emptyset \rangle \rangle G(\psi \Rightarrow \langle\langle A \rangle \rangle G\varphi)$ is valid in ATL.

Exercise 3. Given a finite interpretation $I$, an individual $a \in \Delta_T$ and an $ALC$ concept $C$, we have seen that checking whether $a \in C_T$ can be checked in polynomial time. Below, we aim at getting this result by using the decision procedure dedicated to the model-checking problem for ATL (written MC(Atl)), known to be in PTIME too. In this exercise, the role names are among $r_1, \ldots, r_\alpha$, the concept names are among $A_1, \ldots, A_\beta$ with fixed $\alpha, \beta \geq 1$. For the sake of simplicity, we exclude $\top$ and $\bot$ and the only concept constructors are restricted to $\neg$, $\sqcap$ and $\exists r_\alpha$, unless otherwise stated. Similarly for ATL, we restrict ourselves to the propositional variables $p_1, \ldots, p_\beta$ and $q_1, \ldots, q_\alpha$.

Given an interpretation $I = (\Delta_T, \Delta)$ with finite domain $\Delta_T$, we associate the finite CGS $\mathcal{M}_I = (\text{Agt}, S, Act, \text{act}, \delta, L)$ as follows:

- $\text{Agt} \overset{\text{def}}{=} \{1\}$, $S \overset{\text{def}}{=} \Delta_T \cup \{(a, b, i) | i \in [1, \alpha], a, b \in \Delta_T, (a, b) \in r_1^I\}$,

- $Act \overset{\text{def}}{=} \{(a, b, i) | i \in [1, \alpha], a, b \in \Delta_T, (a, b) \in r_1^I\} \cup \{\varepsilon\}$.

- For all $s \in S$ such that $s = (a, b, i)$, we have $\text{act}(1, s) \overset{\text{def}}{=} \{\varepsilon\}$.

- For all $s \in S$ such that $s = a \in \Delta_T$, we have $\text{act}(1, s) \overset{\text{def}}{=} \{(a, b, i) | i \in [1, \alpha], a, b \in \Delta_T, \text{ such that } (a, b) \in r_1^I\}$.

- As there is a unique agent, we can assume that $\delta$ is defined for a subset of $S \times Act (a, b \in \Delta_T, i \in [1, \alpha])$.

- $\delta(a, (a, b, i)) \overset{\text{def}}{=} (a, b, i); \delta((a, b, i), \varepsilon) \overset{\text{def}}{=} b$,

- for all other pairs in $S \times Act, \delta$ is undefined.

- For all $a \in S, L(a) \overset{\text{def}}{=} \{p_i | i \in [1, \beta], a \in A_1^T\};$ for all $(a, b, i) \in S, L((a, b, i)) \overset{\text{def}}{=} \{q_i\}$.

Here is the graphical representation of an interpretation $I$ (left) and its associated CGS $\mathcal{M}_I$ (right) for $\alpha = 2$ and $\beta = 1$. 
1. Assume that the size of $\mathcal{I}$ is defined as $\text{card}(\Delta^\mathcal{I}) \times \beta + \sum_{i=1}^\alpha \text{card}(r_i^\mathcal{I})$ (written $|\mathcal{I}|$) and the size of $\mathcal{M}_I$ is defined as $\text{card}(S) \times \beta + \text{card}(S)^2 \times \text{card}(\text{Act})$ (written $|\mathcal{M}_I|$), show that $|\mathcal{M}_I|$ is polynomial in $|\mathcal{I}|$.

2. Let us define the translation map $t$ from $\mathcal{ALC}$ concepts to $\mathcal{ATL}$ formulae:

- $t(A_i) \equiv p_i$; $t(\neg D) \equiv \neg t(D)$; $t(D_1 \cap D_2) \equiv t(D_1) \land t(D_2)$.
- $t(\exists r_i.D) \equiv \{\{1\}\}X (q_i \land \{\{1\}\}X t(D))$.

Show that for all $a \in \Delta^\mathcal{I}$ and for all $\mathcal{ALC}$ concepts $C$, we have $a \in C^\mathcal{I}$ (in $\mathcal{ALC}$) iff $\mathcal{M}_I, a \models t(C)$ (in $\mathcal{ATL}$).

3. Using the known results about $\text{MC}(\mathcal{ATL})$, conclude that checking whether $a \in C^\mathcal{I}$ (for $\mathcal{ALC}$) can be done in $\text{PTIME}$. 

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