**Exercises related to previous sessions**

**Exercise 1.** Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be an interpretation (for $\mathcal{ALC}$), $a \in \Delta^\mathcal{I}$, and $(a, 1), (a, 2) \not\in \Delta^\mathcal{I}$ (fresh individuals).

1. Given $X \subseteq \Delta^\mathcal{I}$, we write $[X]$ to denote the set below
   
   $$(X \setminus \{a\}) \cup \{(a, i) \mid i \in \{1, 2\}, \ a \in X\}$$

   $([X]$ is obtained from $X$ by replacing $a$ by $(a, 1), (a, 2)$)

   a. Show that $X$ is non-empty iff $[X]$ is non-empty.

   b. Show that $X \subseteq Y$ implies $[X] \subseteq [Y]$.

2. Let $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ be the interpretation defined from $\mathcal{I}$ as follows:

   - $\Delta^\mathcal{J} \overset{\text{def}}{=} [\Delta^\mathcal{I}]$.
   - $A^\mathcal{J} \overset{\text{def}}{=} [A^\mathcal{I}]$, for all concept names $A$.
   - For all $b \in \Delta^\mathcal{J} \cap \Delta^\mathcal{I}$, $r^\mathcal{J}(b) \overset{\text{def}}{=} [r^\mathcal{I}(b)]$, for all role names $r$.
   - For all $(a, i) \in \Delta^\mathcal{J}$ ($i \in \{1, 2\}$), $r^\mathcal{J}((a, i)) \overset{\text{def}}{=} [r^\mathcal{I}(a)]$, for all role names $r$.
   - $\mathcal{I}$ and $\mathcal{J}$ agree on the interpretation of the individual names when the value is in $\Delta^\mathcal{I} \setminus \{a\}$.

   Show that for all concepts $C$, we have $C^\mathcal{J} = [C^\mathcal{I}]$.

3. Prove that if $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is consistent, then there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$ and for all distinct individual names $a$ and $b$ occurring in $\mathcal{A}$, we have $a^\mathcal{I} \neq b^\mathcal{I}$.

**Exercise 2.** $\mathcal{EL}$ is a fragment of $\mathcal{ALC}$ in which the $\mathcal{EL}$ concept are defined from $C ::= \top \mid A \mid C \cap D \mid \exists r.C$. $\mathcal{EL}$ knowledge bases are defined as for $\mathcal{ALC}$ except that only $\mathcal{EL}$ concepts are allowed. Show that every $\mathcal{EL}$ knowledge base is consistent.
Exercise 3. (from exam 2020/2021) An $\mathcal{ALC}$ TBox $\mathcal{T}$ is said to be simple if it contains GCIs of the form

$$A \sqsubseteq B \quad A_1 \sqcap A_2 \sqsubseteq B \quad A \sqsubseteq \exists r. B \quad \exists r. A \sqsubseteq B$$

where $A$, the $A_i$'s and $B$ are arbitrary concept names or $\top$ and $r$ is an arbitrary role name. In the sequel, by convention, we consider that $\top$ is a special concept name (instead of a truth constant) whose interpretation is always the full interpretation domain $\Delta^I$ (assuming that the interpretation is $I = (\Delta^I, \cdot^I)$). Given a simple TBox $\mathcal{T}$, we write $\text{voc}(\mathcal{T})$ to denote the set of concept and role names occurring in $\mathcal{T}$ with the addition of $\top$ (when $\top$ is not already in $\mathcal{T}$).

1. We introduce rules to deduce new GCIs from $\mathcal{T}$. In the rules below, $A$, the $A_i$'s and $B$ are concept names in $\text{voc}(\mathcal{T})$, the $C_i$'s use only concept names and role names from $\text{voc}(\mathcal{T})$ too. We write $\mathcal{T} \vdash C \sqsubseteq D$ when $C \sqsubseteq D$ can be derived from $\mathcal{T}$ by applying the rules below.

- $\frac{C \sqsubseteq D \in \mathcal{T}}{\mathcal{T} \vdash C \sqsubseteq D}$ (\text{-rule})
- $\frac{}{\mathcal{T} \vdash A \sqsubseteq A}$ (\text{id-rule})
- $\frac{}{\mathcal{T} \vdash A \sqsubseteq \top}$ (\text{T-rule})

- $\frac{\mathcal{T} \vdash A \sqsubseteq A_1, \quad \mathcal{T} \vdash A \sqsubseteq A_2, \quad \mathcal{T} \vdash A_1 \sqcap A_2 \sqsubseteq B}{\mathcal{T} \vdash A \sqsubseteq B}$ (\text{\&-rule})

- $\frac{\mathcal{T} \vdash A \sqsubseteq \exists r. A_1, \quad \mathcal{T} \vdash A_1 \sqsubseteq B}{\mathcal{T} \vdash A \sqsubseteq \exists r. B}$ (\text{\exists-rule})

- $\frac{\mathcal{T} \vdash C_1 \sqsubseteq C_2, \quad \mathcal{T} \vdash C_2 \sqsubseteq C_3}{\mathcal{T} \vdash C_1 \sqsubseteq C_3}$ (\text{trans-rule})

under the condition: $\{C_1 \sqsubseteq C_3\}$ is simple.

Show that

$$\{\exists r. B \sqsubseteq B_1 , A_1 \sqsubseteq B_2 , B_1 \sqcap B_2 \sqsubseteq A_2 , A_1 \sqsubseteq \exists r. A_1 , \top \sqsubseteq B\} \vdash A_1 \sqsubseteq A_2.$$
3. Prove that if \( T \vdash C \sqsubseteq D \), then \( T \) and \( T \cup \{ C \sqsubseteq D \} \) are satisfied by exactly the same interpretations. Conclude \( C \sqsubseteq D \in T^c \) (with \( T^c \) from Question 2.) implies \( T \models C \sqsubseteq D \).

4. Let \( \mathcal{I} \) be the interpretation defined as follows (depending on \( T \) via \( T^c \)).
   - \( \Delta^\mathcal{I} \) is the set of concept names from \( \text{voc}(T) \) (including \( \top \)).
   - \( A^\mathcal{I} \overset{\text{def}}{=} \{ B \in \Delta^\mathcal{I} \mid B \sqsubseteq A \in T^c \} \) for all concept names \( A \) in \( \text{voc}(T) \).
   - \( r^\mathcal{I} \overset{\text{def}}{=} \{(A, B) \mid A \sqsubseteq \exists r.B \in T^c \} \) for all role names \( r \) in \( \text{voc}(T) \).

   Verify that \( \top^\mathcal{I} = \Delta^\mathcal{I} \). Show that \( \mathcal{I} \models T^c \).

5. Conclude that for all \( A, B \in \text{voc}(T) \), we have \( A \sqsubseteq B \notin T^c \) implies \( T \not\models A \sqsubseteq B \).

6. Show that given an arbitrary simple TBox \( T \) and \( A, B \in \text{voc}(T) \), checking whether \( T \models A \sqsubseteq B \) can be done in polynomial time in the size of \( T \).

** Exercises related to today session**

**Exercise 4.** By using the tableaux calculus for \( \mathcal{ALC} \) and its properties, show that the concept below is not satisfiable.

\[
(\forall r.((\neg A) \sqcup B)) \sqcap (\forall r.A) \sqcap \exists r.\neg B
\]

(uses the property that \( A \) is consistent iff there is a complete and clash-free ABox \( A' \) derivable from \( A \) with the tableaux rules)

**Exercise 5.** Show that

\[
\{(a, b) : s, (a, c) : r\} \cup \{a : A_1 \cap \exists s.A_5, a : \forall s.\neg A_5 \sqcup \neg A_2, b : A_2, c : A_3 \cap \exists s.A_4\}
\]

is a consistent ABox.
Exercise 6. Using the tableaux calculus for $\mathcal{ALC}$ with blocking, show that the knowledge base $(\mathcal{T}, \mathcal{A})$ below is consistent.

$$\mathcal{T} = \{ \top \sqsubseteq (\neg A) \sqcup \exists r.B \} \quad \mathcal{A} = \{ a : A \sqcap B, a : \forall r.\forall r.C \}$$

Exercise 7. Complete the case for the $\forall$-rule in the soundness proof of the tableau proof system for $\mathcal{ALC}$ without TBoxes.

Exercise 8. Complete the case for the $\forall$-rule to show that $\mathcal{A}''$ is complete in the soundness proof for the tableau proof system for $\mathcal{ALC}$ with TBoxes (and with blocking technique).

Exercise 9. Complete in the soundness proof for $\mathcal{ALC}$ with TBoxes the property: for all $a : C \in \mathcal{A}''$, we have $a^\mathcal{I} \in C^\mathcal{I}$.

Exercise 10. Let us consider the extension of $\mathcal{ALC}$ TBoxes by allowing role axioms of the form $\text{Disj}(r, s)$ such that $\mathcal{I} \models \text{Disj}(r, s)$ iff $r^\mathcal{I} \cap s^\mathcal{I} = \emptyset$. Extend the tableau-style proof system for $\mathcal{ALC}$ knowledge base consistency and prove termination, soundness and completeness.

Exercise 11. (Tableaux with RIAs) Let us consider the extension of $\mathcal{ALC}$ TBoxes by allowing role inclusion axioms $r \sqsubseteq s$. Extend the tableau-style proof system for $\mathcal{ALC}$ knowledge base consistency and prove termination, soundness and completeness.

Exercise 12. Let us propose an alternative definition for blocking that does not rely on the ancestor-relation but rather on the notion of age. Whenever the $\exists$-rule is applied, the age of the fresh individual name $c$ is equal to the number of previous applications of the $\exists$-rule plus one. The age of the root individuals is equal to zero. An individual name $b$ in some ABox $\mathcal{A}'$ derived from $\mathcal{A}$ is blocked by $a$ if

- the age of $a$ is strictly less that the age of $b$,
- $\text{con}_{\mathcal{A}'}(b) \subseteq \text{con}_{\mathcal{A}'}(a)$,
- $a$ is not blocked.

Show that the tableau-style proof system for $\mathcal{ALC}$ knowledge base consistency with this notion of blocking provides also a decision procedure.
Exercise 13. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base with

$$\mathcal{T} = \{ A \sqsubseteq \neg \forall r. \neg A \} \quad \mathcal{A} = \{ a : (A \cap (\exists r. B)) \cap (\neg \forall r. B) \}$$

1. Using the tableaux calculus for $\mathcal{ALC}$, show that $\mathcal{K}$ is consistent.

2. Based on the derivation from Question 1., define an interpretation satisfying $\mathcal{K}$.

Exercise 14. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base for $\mathcal{ALC}$ with role axioms, with $\mathcal{T} = \mathcal{T}_{GC} \cup \{ r \circ s \equiv s \circ r \}$. $\mathcal{T}_{GC}$ is made of GCIs. Show that $\mathcal{K}$ is consistent iff $\mathcal{K}' = (\mathcal{T}_{GC} \cup \mathcal{T}_{RA}, \mathcal{A})$ is consistent with $\mathcal{T}_{RA}$ equal to $\{ r \circ s \sqsubseteq q, q \sqsubseteq r \circ s, s \circ r \sqsubseteq q, q \sqsubseteq s \circ r \}$ for some new role name $q$.

Exercise 15. (from exam 2022/2023) In this exercise, we consider the tableaux rules for $\mathcal{ALC}$ with the blocking technique. We recall that given a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ such that all the concepts are in negation normal form and the GCIs in $\mathcal{T}$ are of the form $\top \sqsubseteq C$, we have $\mathcal{K}$ is consistent iff $\mathcal{A} \models A'$ for some complete and clash-free ABox derived from the rules for $\mathcal{ALC}$ and using the GCIs from $\mathcal{T}$ to apply the $\sqsubseteq$-rule.

1. Given a finite set $\mathcal{T}$ of GCIs for $\mathcal{ALC}$ and $C^* \sqsubseteq D^*$ be a GCI, we write $\mathcal{T} \models C^* \sqsubseteq D^*$ for all interpretations $\mathcal{I}$, we have (for all $C \sqsubseteq D \in \mathcal{T}$, we have $\mathcal{I} \models C \sqsubseteq D$) implies $\mathcal{I} \models C^* \sqsubseteq D^*$. Show that $\mathcal{T} \models C^* \sqsubseteq D^*$ iff $(\mathcal{T}, \{ a : C^*, a : \neg D^* \})$ is not consistent.

2. Using the tableaux calculus for $\mathcal{ALC}$, show that

$$\{ A_0 \sqsubseteq \exists r. A_0 \} \cup \{ A_0 \sqsubseteq \forall r. A_i \mid i \in [1, 5] \} \models A_0 \sqsubseteq \exists r. (A_4 \sqcap A_2 \sqcap A_0),$$

where $A_0, \ldots, A_5$ are distinct concept names.