# Reversal-Bounded Counter Machines (part 2)

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# Slides and lecture notes

http://www.lsv.fr/~demri/notes-de-cours.html

https://wikimpri.dptinfo.ens-cachan.fr/doku.php?id=cours:c-2-9-1

# Plan of the lecture

- Previous lecture:
  - The Presburger sets and the semilinear sets coincide.
  - Application: Parikh image of regular languages.
  - Introduction to reversal-bounded counter machines.
  - Runs in normal form.
- Reachability sets are computable Presburger sets.
- Decidable and undecidable extensions.
- Repeated reachability problems.

# The previous lecture in 4 slides (1/4)

- A linear set $X$ is defined by a basis $\mathbf{b} \in \mathbb{N}^d$ and by $\mathfrak{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_m\} \subseteq \mathbb{N}^d$:

$$X = \{\mathbf{b} + \sum_{i=1}^{m} \lambda_i \mathbf{p}_i : \lambda_1, \ldots, \lambda_m \in \mathbb{N}\}$$

- Semilinear sets are finite unions of linear sets.
- Semilinear sets and Presburger sets coincide.
- $\{n^2 \mid n \in \mathbb{N}\}$ and $\{2^n \mid n \in \mathbb{N}\}$ are not Presburger sets.
- Simple vector addition systems with states (VASS) have reachability sets that are not Presburger sets.

# The previous lecture in 4 slides (2/4)

- Parikh image of $a\ b\ a\ a\ b$ is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

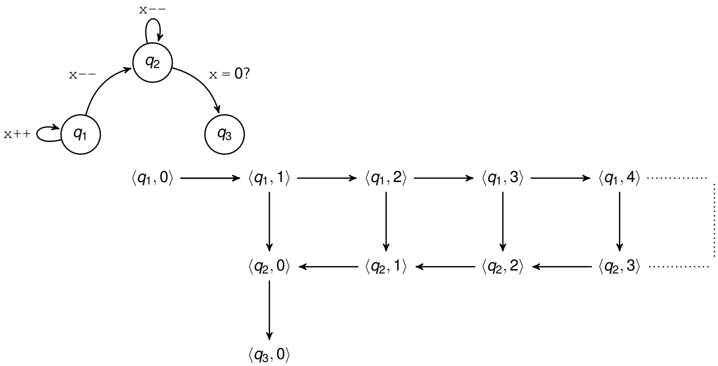
- $\mathrm{L} \subseteq \Sigma^*$ is bounded and regular iff it is a finite union of languages of the form

$$u_0 v_1^* u_1 \cdots v_k^* u_k$$

- The Parikh images of bounded and regular languages are Presburger sets.

- For every regular language $\mathrm{L}$, there is a bounded and regular language $\mathrm{L}'$ such that
  1. $\mathrm{L}' \subseteq \mathrm{L}$,
  2. $\Pi(\mathrm{L}') = \Pi(\mathrm{L})$.

# The previous lecture in 4 slides (3/4)




- Reversal: Alternation from nonincreasing mode to nondecreasing mode and vice-versa.
- A run is $r$-reversal-bounded whenever the number of reversals of each counter is less or equal to $r$.

# The previous lecture in 4 slides (4/4)

- Notion of extended path for which no reversal occurs and satisfaction of the guards remains constant.

$$\pi_0 \; S_1 \; \pi_1 \; \cdots \; S_\alpha \; \pi_\alpha$$

- Runs in normal form.
- I.e., any finite $r$-reversal-bounded run can be generated by a small sequence of small such extended paths.

# Guards and intervals

- Transition labelled by $\langle g, \mathbf{a} \rangle$ with $\mathbf{a} \in \mathbb{Z}^d$ and $g$ is a guard:

$$g ::= \top \mid \bot \mid \mathrm{x} \sim k \mid g \wedge g \mid g \vee g \mid \neg g$$

where $\sim \in \{\leq, \geq, =\}$ and $k \in \mathbb{N}$.

- Linear ordering on $\mathcal{I}$ (for non-empty intervals):

$$[k_1, k_1] \leq [k_1+1, k_2-1] \leq [k_2, k_2] \leq [k_2+1, k_3-1] \leq [k_2, k_2] \leq \dots$$

$$\dots \leq [k_K, k_K] \leq [k_K + 1, +\infty)\}$$

- Interval map $\mathfrak{im} : C \to \mathcal{I}$ and symbolic satisfaction relation $\mathfrak{im} \vdash g$.

- Guarded mode $\mathfrak{gmd} = \langle \mathfrak{im}, \mathfrak{md} \rangle$ where $\mathfrak{im}$ is an interval map and $\mathfrak{md} \in \{\mathrm{INC}, \mathrm{DEC}\}^d$.

# Small extended path compatible with $\mathfrak{gmd}$

- Extended path **P**:

$$\pi_0 \; S_1 \; \pi_1 \; \cdots \; S_\alpha \; \pi_\alpha$$

- Small extended path:
  1. $\pi_0$ and $\pi_\alpha$ have at most $2 \times \mathrm{card}(Q)$ transitions,
  2. $\pi_1, \ldots, \pi_{\alpha-1}$ have at most $\mathrm{card}(Q)$ transitions,
  3. for each $q \in Q$, there is at most one set $S$ containing simple loops on $q$.

- For every transition $t = q \xrightarrow{\langle g, \mathbf{a} \rangle} q'$:
  1. $\mathfrak{im} \vdash g$,
  2. for every $i \in [1, d]$,
     - $\mathfrak{md}(i) = \mathrm{INC}$ implies $\mathbf{a}(i) \geq 0$,
     - $\mathfrak{md}(i) = \mathrm{DEC}$ implies $\mathbf{a}(i) \leq 0$.

# Normal forms

- $r$-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0 \rangle \cdots \langle q_\ell, \mathbf{x}_\ell \rangle$.
- $\rho$ can be divided as a sequence $\rho = \rho_1 \cdot \rho_2 \cdots \rho_{L'}$ such that
  - each $\rho_i$ respects a small extended path $\mathbf{P}_i$ compatible with some guarded mode $\mathfrak{gmd}_i$.
  - $L' \leq ((d \times r) + 1) \times 2Kd$.

## Reachability Sets are Presburger Sets

- Small extended path **P** compatible with $\mathfrak{gmd} = \langle \mathfrak{im}, \mathfrak{md} \rangle$

$$\pi_0 \ \{sl_1^1, \ldots, sl_1^{n_1}\} \ \pi_1 \ \cdots \ \{sl_\alpha^1, \ldots, sl_\alpha^{n_\alpha}\} \ \pi_\alpha$$

  where $q_0$ is the first control state in $\pi_0$ and $q_f$ is the last control state in $\pi_\alpha$ ($= \pi'_\alpha \cdot t$).

- There is $\varphi(\bar{x}, \bar{y})$ of exponential size in $|\mathcal{M}|$ such that

$$[\![\varphi]\!] = \{\langle \mathbf{x}_0, \mathbf{y} \rangle : \text{ there is a run } \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{*} \langle q_f, \mathbf{y} \rangle \text{ respecting } \mathbf{P}\}$$

- $\varphi$ states the following properties:
  1. $\mathbf{x}_0$ belong to the right intervals induced by $\mathfrak{im}$,
  2. the counter values for the penultimate configuration $\langle q'_f, \mathbf{y}' \rangle$ belong to the right intervals induced by $\mathfrak{im}$,
  3. the values for $\bar{y}$ are obtained from $\bar{x}$ by considering the effects of the paths $\pi_i$ plus a finite amount of times the effects of each simple loop occurring in **P**.

# Arghhh !!!!!

$$\exists\, z_1^1, \ldots, z_1^{n_1}, \ldots, z_\alpha^1, \ldots, z_\alpha^{n_\alpha}$$

$$(z_1^1 \geq 1) \wedge \cdots \wedge (z_1^{n_1} \geq 1) \wedge \cdots \wedge (z_\alpha^1 \geq 1) \wedge \cdots \wedge (z_\alpha^{n_\alpha} \geq 1) \wedge$$

$$(\bar{y} = \bar{x} + \mathfrak{ef}(\pi_0) + \cdots + \mathfrak{ef}(\pi_\alpha) + \sum_{i,j} z_i^j \mathfrak{ef}(sl_i^j)) \wedge$$

$$(\bigwedge_{\mathfrak{im} \vdash \mathrm{x}_c \sim k} x_c \sim k) \wedge (\bigwedge_{\text{not } \mathfrak{im} \vdash \mathrm{x}_c \sim k} \neg(x_c \sim k)) \wedge$$

$$(\bigwedge_{j \in [1,d]} (x_j \in \mathfrak{im}(\mathrm{x}_j) \wedge (y_j \in \mathfrak{im}(\mathrm{x}_j))) \wedge$$

$$(\bigwedge_{\mathfrak{im} \vdash \mathrm{x}_c \sim k} (x_c + \mathfrak{ef}(\pi_0)(c) + \cdots + \mathfrak{ef}(\pi_{\alpha-1})(c) + \mathfrak{ef}(\pi'_\alpha)(c) + \sum_{i,j} z_i^j \mathfrak{ef}(sl_i^j)(c)) \sim k) \wedge$$

$$(\bigwedge_{\text{not } \mathfrak{im} \vdash \mathrm{x}_c \sim k} \neg(x_c + \mathfrak{ef}(\pi_0)(c) + \cdots + \mathfrak{ef}(\pi_{\alpha-1})(c) + \mathfrak{ef}(\pi'_\alpha)(c) + \sum_{i,j} z_i^j \mathfrak{ef}(sl_i^j)(c) \sim k)$$

'$z_j \in [l, l']$' stands for $l \leq z_j \wedge z_j \leq l'$ and $z_j \in [k_K + 1, +\infty)$ stands for $k_K + 1 \leq z_j$.

# One more step

- Sequence of small extended paths $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$.
- There is $\varphi(\bar{x}, \bar{y})$ such that

$$[\![\varphi]\!] = \{\langle \mathbf{x}, \mathbf{y} \rangle : \text{ there is a run } \langle q_0, \mathbf{x} \rangle \xrightarrow{*} \langle q_f, \mathbf{y} \rangle \text{ respecting } \mathbf{P}_1 \cdots \mathbf{P}_{L'}\}$$

- $\varphi_i(\bar{x}, \bar{y})$ for each $\mathbf{P}_i$.

$$\exists\, \bar{z_0}, \ldots, \bar{z_{L'}}\ (\bar{x} = \bar{z_0}) \wedge (\bar{y} = \bar{z_{L'}}) \wedge$$

$$\varphi_1(\bar{z_0}, \bar{z_1}) \wedge \varphi_2(\bar{z_1}, \bar{z_2}) \wedge \cdots\ \varphi_{L'-1}(z_{\overline{L'}-2}, z_{\overline{L'}-1}) \wedge \varphi_{L'}(z_{\overline{L'}-1}, \bar{z_{L'}}).$$

- $r$-reversal-bounded $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ that is for some $r \geq 0$.
- For each $q' \in Q$, the set
$$\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$$
is a computable Presburger set.
- Formula $\varphi(\bar{y})$:
$$\exists\, \bar{x}\ (\bigwedge_{i \in [1,d]} \mathbf{x}(i) = x_i) \wedge \bigvee_{\text{small seq. } \sigma = \mathbf{P}_1 \cdots \mathbf{P}_{L'} \text{ ending by } q'} \varphi_\sigma(\bar{x}, \bar{y})$$
- Assuming that $\mathcal{M}$ is uniformly $r$-reversal-bounded for some $r \geq 0$. For all $q$, $q'$, one can compute $\varphi(\bar{x}, \bar{y})$ such that
$$[\![\varphi]\!] = \{\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{N}^{2d} : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$$

# Time to reap the rewards!

- Reachability problem with bounded number of reversals.

  Input: a CM $\mathcal{M}$, $r \in \mathbb{N}$, $\langle q_0, \mathbf{x}_0 \rangle$ and $\langle q_f, \mathbf{x}_f \rangle$.

  Question: Is there a run from $\langle q_0, \mathbf{x}_0 \rangle$ to $\langle q_f, \mathbf{x}_f \rangle$ such that each counter has at most $r$ reversals?

- When $\langle \mathcal{M}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ is $r'$-reversal-bounded for some $r' \leq r$, we get an instance of the reachability problem with initial configuration $\langle q_0, \mathbf{x_0} \rangle$.

- The reachability problem with bounded number of reversals is decidable.

- Next, a proof that abstracts away from small sequences of small extended paths (but still these are implicitly used).

# Proof (1/3)

- $\mathcal{M} = \langle Q, T, C \rangle$, $r \in \mathbb{N}$, $\langle q_0, \mathbf{x}_0 \rangle$ and $\langle q_f, \mathbf{x}_f \rangle$.
- $\mathcal{M}' = \langle Q', T', C \rangle$ with

$$Q' = Q \times \{\mathrm{DEC}, \mathrm{INC}\}^d \times [0, r]^d$$

- New control states record the type of phase and the current number of reversals (with a bound on $r$).
- By construction, $\langle \mathcal{M}', \langle \langle q_0, \mathrm{INC}, \mathbf{0} \rangle, \mathbf{x}_0 \rangle \rangle$ is $r$-reversal-bounded.

# Proof (2/3)

- $\langle q, \mathfrak{md}, \sharp\mathfrak{alt}\rangle \xrightarrow{\langle g, \mathbf{a}\rangle} \langle q', \mathfrak{md}', \sharp\mathfrak{alt}'\rangle \in T' \overset{\text{def}}{\Leftrightarrow} q \xrightarrow{\langle g, \mathbf{a}\rangle} q' \in T$ and

| $\mathbf{a}$ | $\mathfrak{md}(i)$ | $\mathfrak{md}'(i)$ | $\sharp\mathfrak{alt}'(i)$ |
|---|---|---|---|
| $\mathbf{a}(i) < 0$ | DEC | DEC | $\sharp\mathfrak{alt}(i)$ |
| $\mathbf{a}(i) < 0$ | INC | DEC | $\sharp\mathfrak{alt}(i) + 1$ and $\sharp\mathfrak{alt}(i) < r$ |
| $\mathbf{a}(i) > 0$ | INC | INC | $\sharp\mathfrak{alt}(i)$ |
| $\mathbf{a}(i) > 0$ | DEC | INC | $\sharp\mathfrak{alt}(i) + 1$ and $\sharp\mathfrak{alt}(i) < r$ |
| $\mathbf{a}(i) = 0$ | DEC | DEC | $\sharp\mathfrak{alt}(i)$ |
| $\mathbf{a}(i) = 0$ | INC | INC | $\sharp\mathfrak{alt}(i)$ |

- Equivalence between:
  - there is a run of $\mathcal{M}$ from $\langle q_0, \mathbf{x}_0\rangle$ to $\langle q_f, \mathbf{x}_f\rangle$ such that each counter has at most $r$ reversals,
  - $\langle\langle q_f, \mathfrak{md}, \sharp\mathfrak{alt}\rangle, \mathbf{x}_f\rangle$ is reachable from $\langle\langle q_0, \mathbf{INC}, \mathbf{0}\rangle, \mathbf{x_0}\rangle$ in $\mathcal{M}'$ for some $\mathfrak{md}$, $\sharp\mathfrak{alt}$.

# Proof (3/3)

- The number of distinct pairs $\langle \mathfrak{md}, \sharp\mathfrak{alt} \rangle$ is bounded by $2^d \times (r+1)^d$.

- We have seen that

  $$X_{\langle \mathfrak{md}, \sharp\mathfrak{alt} \rangle} = \{ \mathbf{x}' \in \mathbb{N}^d : \langle \langle q_0, \mathrm{INC}, \mathbf{0} \rangle, \mathbf{x}_0 \rangle \xrightarrow{*} \langle \langle q_f, \mathfrak{md}, \sharp\mathfrak{alt} \rangle, \mathbf{x}' \rangle \}$$

  is a computable Presburger set.

- $\mathbf{x} \in X_{\langle \mathfrak{md}, \sharp\mathfrak{alt} \rangle}$ amounts to check the satisfiability status of

  $$(\bigwedge_{i=1}^{d} \mathrm{x}_i = \mathbf{x}(i)) \wedge \varphi_{\langle \mathfrak{md}, \sharp\mathfrak{alt} \rangle}.$$

- It amounts to checking satisfiability of a a disjunctive formula with at most $2^d (r+1)^d$ disjuncts.

# Complexity

- The reachability problem with bounded number of reversals is NP-complete, assuming that all the natural numbers are encoded in binary except the number of reversals.
- The problem is NEXPTIME-complete assuming that all the natural numbers are encoded in binary.

  [Gurari & Ibarra, ICALP'81; Howell & Rosier, JCSS 87]
- NEXPTIME-hardness as a consequence of the standard simulation of Turing machines. [Minsky, 67]

# Doubly-exponential number of times loops are visited

- If $\langle q_0, \mathbf{x}_0 \rangle \xrightarrow{*} \langle q_f, \mathbf{x}_f \rangle$ is $r$-reversal-bounded, then there is an $r$-reversal-bounded run between these configurations
  1. respecting a small sequence of small extended paths,
  2. each simple loop is visited at most a doubly-exponential number of times in $log(r) + |\mathbf{x}_0| + |\mathbf{x}_f| + |\mathcal{M}|$.
- We only need to prove the constraints on the number of times the loops are visited.
- So, there is an $r$-reversal-bounded run $\rho'$ that respects a small sequence of small extended paths $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$.

# Back to quantifier-free formulae

- Formula $\varphi(\bar{\mathrm{x}}, \bar{\mathrm{y}})$ for that sequence is equivalent to an existential formula of exponential size in $log(r) + |\mathcal{M}|$.

$$(\bigwedge_{j \in [1,d]} (\mathrm{x}_j = \mathbf{x}_0(j) \wedge \mathrm{y}_j = \mathbf{x}_f(j)) \wedge \varphi(\bar{\mathrm{x}}, \bar{\mathrm{y}})$$

- We get the doubly-exponential bound thanks to:
  - $\varphi$ quantifier-free formula with variables $\mathrm{x}_1, \ldots, \mathrm{x}_n$ is satisfiable iff there is a valuation

$$\mathfrak{v} : \{\mathrm{x}_1, \ldots, \mathrm{x}_n\} \to [0, 2^{p(|\varphi|)}] \text{ such that } \mathfrak{v} \models \varphi$$

$p(\cdot)$ is a polynomial independent of $\varphi$ and $\mathrm{x}_1, \ldots, \mathrm{x}_n$.

(see the lecture on Presburger arithmetic on Oct. 9th)

# EXPSPACE upper bound

- NEXPTIME$\subseteq$ EXPSPACE.
- A small sequence of small extended paths has at most $((d \times r) + 1) \times 2Kd$ extended paths.
- Each extended path has at most $\mathrm{card}(T)^{\mathrm{card}(Q)}$ simple loops and at most $\mathrm{card}(Q)(3 + \mathrm{card}(Q))$ transitions, that do not occur in simple loops
- A nondeterministic exponential space algorithm can guess such a run.

# Nondeterministic algorithm with bound $\mathfrak{B}$

- Algorithm for $\mathcal{M} = \langle Q, T, C \rangle$, $r \in \mathbb{N}$, $\langle q_0, \mathbf{x}_0 \rangle$ and $\langle q_f, \mathbf{x}_f \rangle$.

1. $i := 0$; $\mathbf{x_c} := \mathbf{x}_0$; $q_c := q_0$ (current configuration);
2. While $(\mathbf{x}' \neq \mathbf{x}_f$ or $q_c \neq q_f)$ and $i < \mathfrak{B}$ do
   2.1 Guess a transition $\langle q, \langle g, \mathbf{a} \rangle, q' \rangle \in T$;
   (nondeterministic step !)

   2.2 If $q \neq q_c$ or $\mathbf{x}_c$ does not satisfy $g$ or $\mathbf{x}_c + \mathbf{a} \notin \mathbb{N}^d$ then abort;

   2.3 $i := i + 1$; $\mathbf{x_c} := \mathbf{x_c} + \mathbf{a}$; $q_c := q'$;
3. If $\langle \mathbf{x}_c, q_c \rangle \neq \langle q_f, \mathbf{x}_f \rangle$ then abort else accept;

+ need to counter the number of reversals per counter.

# Why in EXPSPACE?

- A counter with an exponential amount of bits can count until a doubly-exponential value.
- Only two configurations need to be stored thanks to nondeterminism.
- Comparing or adding two natural numbers requires logarithmic space only.
- Taking an exponential amount of loops and doubly-exponential amount of times, is still of doubly-exponential magnitude.
- [Savitch, JCSS 70]: a nondeterministic procedure for a given problem using space $\mathfrak{f}(N) \geq log(N)$ can be turned into a deterministic procedure using $\mathfrak{f}(N) \times \mathfrak{f}(N)$ space.
- Exponential functions are closed under multiplication.

# NExpTime upper bound

- Instance $\mathcal{M}$, $r$, $\langle q_0, \mathbf{x}_0 \rangle$ and $\langle q_f, \mathbf{x}_f \rangle$ of size $N$.
- $r$-reversal-bounded run from $\langle q_0, \mathbf{x}_0 \rangle$ to $\langle q_f, \mathbf{x}_f \rangle$ with
  - a sequence of small extended paths of length at most
  $$((d \times r) + 1) \times 2Kd$$
  - each extended path has at most $\mathrm{card}(T)^{\mathrm{card}(Q)}$ simple loops and at most $1 + \mathrm{card}(Q)$ paths of length at most $3 \times \mathrm{card}(Q)$,
- Algorithm guesses on-the-fly the small sequence and computes the effects of taking loops a doubly-exponential number of times, or of taking non-loop paths.
- All the computations can be performed in exponential time (but the values involved in the computations can be of doubly-exponential magnitude).

$$\langle q_0, \mathbf{x_0} \rangle \xrightarrow{\pi_1} \langle q_1, \mathbf{x_1} \rangle \xrightarrow{\gamma_2 \text{ times } sl_2} \langle q_2, \mathbf{x_2} \rangle \xrightarrow{\gamma_3 \text{ times } sl_3} \langle q_3, \mathbf{x_3} \rangle \ \ldots \xrightarrow{\pi_\alpha} \langle q_\alpha, \mathbf{x}_\alpha \rangle$$

- $\gamma_i \leq 2^{2^{p'(N)}}$.
- Number of paths of length at most $3 \times \mathrm{card}(Q)$ or the number of simple loops visited is bounded by:

$$G = ((d \times r) + 1) \times 2Kd \times (\mathrm{card}(T)^{\mathrm{card}(Q)} + \mathrm{card}(Q) + 1)$$

- $\alpha \leq G$.

# Algorithm

1. $\langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle := \langle q_0, \mathbf{x}_0 \rangle$; Guess $\alpha \leq G$; $\beta := 1$;

2. While $\beta \leq \alpha$ do
   - 2.1 Guess either a path $\pi$ of length at most $3 \times \mathrm{card}(Q)$ or, a simple loop $sl$ and a guarded mode $\mathfrak{gmd} = \langle \mathfrak{im}, \mathfrak{md} \rangle$ and $\gamma$ of double exponential value in $N$ such that $sl$ is compatible with $\mathfrak{gmd}$;
   - 2.2 If a simple loop is guessed in (a), then check that $\mathbf{x}_{\mathrm{cur}}$ and $\mathbf{x}_{\mathrm{cur}} + (\gamma - 1)\mathfrak{ef}(sl) + sl^{\backslash \mathrm{last}}$ are in the right intervals: for every $i \in [1, d]$, $\mathbf{x}_{\mathrm{cur}}(i)$ and
   $$(\mathbf{x}_{\mathrm{cur}} + (\gamma - 1)\mathfrak{ef}(sl) + \mathfrak{ef}(sl^{\backslash \mathrm{last}}))(i)$$
   belong to $\mathfrak{im}(\mathtt{x}_i)$.
   - 2.3 If a path $\pi$ is guessed in (a), then check that the sequence of transitions in $\pi$ can be fired from $\langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle$ and set $\langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle := \langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle + \mathfrak{ef}(\pi)$.
   - 2.4 $\beta$++;

3. Return $(\langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle = \langle q_f, \mathbf{x}_f \rangle)$.

+ need to counter the number of reversals per counter.

# NEXPTIME-hardness

- Nondeterministic Turing machine $M = \langle Q, q_0, \Sigma, \delta, q_a \rangle$:
  - $Q$: set of control states.
  - $q_0$: initial state; $q_a$: accepting state.
  - $\Sigma$: tape symbols (including a blank symbol or an end symbol).
  - Transition relation $\delta : Q \times \Sigma \to \mathcal{P}(Q \times \overbrace{\{-1, 0, 1\}}^{\text{moves}} \times \Sigma)$.
- We can assume that the Turing machine starts with an "empty" tape.

# Simulating a Turing machine (ideas only)

- A Turing machine can be simulated by two stacks (the tape is cut in half).
  - E.g., moving the head left or right is equivalent to popping a bit from one stack and pushing it onto the other
- A stack over a binary alphabet can be simulated by two counters. One counter contains the binary representation of the bits on the stack.
  - E.g., pushing a one is equivalent to doubling and adding 1, assuming that in the binary representation the least significant bit is on the top.
- Each step in the Turing machine is simulated by an exponential amount of steps in the counter machines.

# Two or Three Extensions

# Adding equality constraints

- Guards so far:

$$g ::= \top \mid \bot \mid \mathrm{x} \sim k \mid g \wedge g \mid g \vee g \mid \neg g$$

  where $\sim \in \{\leq, \geq, =\}$ and $k \in \mathbb{N}$.

- Adding equalities $\mathrm{x} = \mathrm{x}'$ and inequalities $\mathrm{x} \neq \mathrm{x}'$.
- Updates are still equal to $\mathbf{a} \in \mathbb{Z}^d$.

# Deterministic Minsky machines

- A counter stores a single natural number.
- A Minsky machine can be viewed as a finite-state machine with two counters.
- Operations on counters:
  - Check whether the counter is zero.
  - Increment the counter by one.
  - Decrement the counter by one if nonzero.

# 2-counter Minsky machines

- Set of $n$ instructions.
- The $l$th instruction has one of the forms below ($i \in \{1, 2\}$, $l' \in \{1, \ldots, n\}$):

  l: $\mathrm{x}_i := \mathrm{x}_i + 1$; goto $l'$
  l: if $\mathrm{x}_i = 0$ then goto $l'$ else $\mathrm{x}_i := \mathrm{x}_i - 1$; goto $l''$
  n: halt

- Configurations are elements of $[1, n] \times \mathbb{N} \times \mathbb{N}$.
- Initial configuration: $\langle 1, 0, 0 \rangle$.

# Computations

- A computation is a sequence of configurations starting from the initial configuration and such that two successive configurations respect the instructions.
- The Minsky machine

  1: $\mathtt{x}_1 := \mathtt{x}_1 + 1$; goto 2
  2: $\mathtt{x}_2 := \mathtt{x}_2 + 1$; goto 1
  3: halt

  has unique computation

  $\langle 1, 0, 0\rangle \to \langle 2, 1, 0\rangle \to \langle 1, 1, 1\rangle \to \langle 2, 2, 1\rangle \to \langle 1, 2, 2\rangle \to \langle 2, 3, 2\rangle \ldots$

# Halting problem

- Halting problem:

  input: a 2-counter Minsky machine $\mathcal{M}$;
  question: is there a finite computation that ends with location equal to $n$?

  ($n$ is understood as a special instruction that halts the machine)

- Theorem: The halting problem is undecidable. [Minsky,67]

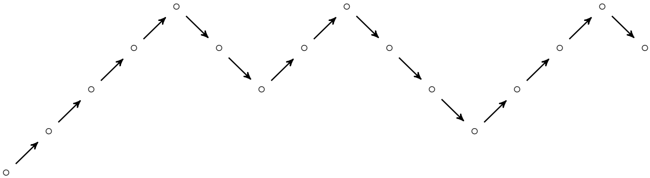
- Minsky machines are Turing-complete.

# Undecidability

- Minsky machine $\mathcal{M}$ with *n* instructions and 2 counters.
- Each counter $\mathrm{x}$ in $\mathcal{M}$ is given two counters $\mathrm{x}^{inc}$ and $\mathrm{x}^{dec}$.
- Zero-test on $\mathrm{x}$ is simulated by the guard $\mathrm{x}^{inc} = \mathrm{x}^{dec}$.
- A decrement on $\mathrm{x}$ first check that $\mathrm{x}^{inc} \neq \mathrm{x}^{dec}$ and then increment $\mathrm{x}^{dec}$.
- $\mathcal{M}$ can be simulated by a 0-reversal-bounded counter machine with four counters.
- $\mathcal{M}$ halts iff the set of counter values for reaching the state *n* in the 0-reversal-bounded counter machine is not empty.

# Safely enriching the set of guards

- Atomic formulae in guards are of the form $t \leq k$ or $t \geq k$ with $k \in \mathbb{Z}$ and $t$ is of the form $\sum_i a_i \mathsf{x}_i$ with the $a_i$'s in $\mathbb{Z}$.
- $\mathtt{T}$: a finite set of terms including $\{\mathsf{x}_1, \ldots, \mathsf{x}_d\}$.
- A run is $r$-$\mathtt{T}$-reversal-bounded $\stackrel{\text{def}}{\Leftrightarrow}$ the number of reversals of each term in $\mathtt{T} \leq r$ times.



# Reversal-boundedness leads to semilinearity

- Given a counter machine $\mathcal{M}$, $\mathtt{T}_{\mathcal{M}} \stackrel{\text{def}}{=}$ the set of terms $t$ occurring in $t \sim k$ with $\sim \in \{\leq, \geq\}$ + counters in $\{\mathsf{x}_1, \ldots, \mathsf{x}_d\}$.

- $\langle \mathcal{M}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ is reversal-bounded $\stackrel{\text{def}}{\Leftrightarrow}$ there is $r \geq 0$ such that every run from $\langle q_0, \mathbf{x}_0 \rangle$ is $r$-$\mathtt{T}_{\mathcal{M}}$-reversal-bounded.

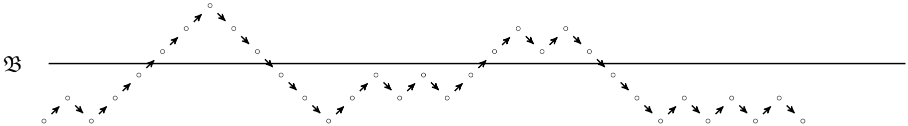
- When $\mathtt{T} = \{\mathsf{x}_1, \ldots, \mathsf{x}_d\}$, $\mathtt{T}$-reversal-boundedness is equivalent to reversal-boundedness from [Ibarra, JACM 78].

- Given a counter machine $\mathcal{M}$, $r \geq 0$ and $q, q' \in Q$, one can effectively compute a Presburger formula $\varphi_{q,q'}(\overline{\mathsf{x}}, \overline{\mathsf{y}})$ such that for all $\mathfrak{v}$, propositions below are equivalent:
  - $\mathfrak{v} \models \varphi_{q,q'}(\overline{\mathsf{x}}, \overline{\mathsf{y}})$,
  - there is an $r$-$\mathtt{T}_{\mathcal{M}}$-reversal-bounded run from $\langle q, \langle \mathfrak{v}(\mathsf{x}_1), \ldots, \mathfrak{v}(\mathsf{x}_d) \rangle \rangle$ to $\langle q', \langle \mathfrak{v}(\mathsf{y}_1), \ldots, \mathfrak{v}(\mathsf{y}_d) \rangle \rangle$.

[Ibarra, JACM 78; Demri & Bersani, FROCOS'11]

# Weak reversal-boundedness

- Reversals are recorded only above a bound $\mathfrak{B}$:




- Effective semilinearity of the reachability sets.

[Finkel & Sangnier, MFCS'08]

# Formal definition

- Counter machine $\mathcal{M} = \langle Q, T, C \rangle$ and bound $\mathfrak{B} \in \mathbb{N}$.
- From $\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle, \ldots$, we defined a sequence of mode vectors $\mathfrak{md}_0, \mathfrak{md}_1, \ldots$ with each $\mathfrak{md}_i \in \{\mathrm{INC}, \mathrm{DEC}\}^d$.
- Set of positions $Rev_i^{\mathfrak{B}}$:

$$\{j \in [0, |\rho| - 1] : \mathfrak{md}_j(i) \neq \mathfrak{md}_{j+1}(i), \{\mathbf{x}_j(i), \mathbf{x}_{j+1}(i)\} \not\subseteq [0, \mathfrak{B}]\}$$

- $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is $r$-reversal-$\mathfrak{B}$-bounded $\overset{\text{def}}{\Leftrightarrow}$ for every finite run $\rho$ starting at $\langle q, \mathbf{x} \rangle$, $\mathrm{card}(Rev_i^{\mathfrak{B}}) \leq r$ for every $i \in [1, d]$.
- $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is weakly reversal-bounded $\overset{\text{def}}{\Leftrightarrow}$ there are $r, \mathfrak{B} \geq 0$ such that $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is $r$-reversal-$\mathfrak{B}$-bounded.
- $r$-reversal-boundedness = $r$-reversal-0-boundedness.

# Reachability sets are Presburger sets too!

- $r$-reversal-$\mathfrak{B}$-bounded counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$.
- For each $q' \in Q$,

$$\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$$

  is a computable Presburger set.
- This extends the results for $r$-reversal-boundedness.
- ... but the proof uses simply those results.

# Proof (1/10)

- $\mathcal{M} = \langle Q, T, C \rangle$ and $r, \mathfrak{B} \geq 0$.
- W.l.o.g.,
  - $\mathfrak{B} \geq k$ for any atomic guard $\mathrm{x} \sim k$,
  - $\mathfrak{B} \geq \mid \mathbf{a}(i) \mid$ for any update $\mathbf{a}$.
- $r$-reversal-$\mathfrak{B}$-boundedness $\rightarrow$ $r$-reversal-$\mathfrak{B}'$-boundedness (when $\mathfrak{B}' \geq \mathfrak{B}$)
- Counter machine $\mathcal{M}' = \langle Q', T', C \rangle$ with $Q' = Q \times [0, \mathfrak{B}]^d$.
- In $\mathcal{M}'$, we encode in the control states the fact that a counter value is below $\mathfrak{B}$.

# The general principle (2/10)

- $\langle q, \mathbf{v}, \mathbf{x} \rangle$ for $\mathcal{M}'$ with $\mathbf{v}(i) = \alpha < \mathfrak{B}$, we require $\mathbf{x}(i) = 0$.
- The intended counter value for $\mathrm{x}_i$ from $\mathcal{M}$ is precisely $\alpha$.
- Updating the counters below $\mathfrak{B}$ does not create any reversal (the counter values remains equal to zero).
- $\langle q, \mathbf{v}, \mathbf{x} \rangle$ with $\mathbf{v}(i) = \mathfrak{B}$, $\mathbf{x}(i)$ can take any value.
- The intended counter value for $\mathrm{x}_i$ from $\mathcal{M}$ is precisely $\mathfrak{B} + \mathbf{x}(i)$.
- Let us implement that principle for transitions in $T$.

# The map $\mathfrak{f}$ between configurations (3/10)

$$\mathfrak{f} : (Q \times \mathbb{N}^d) \rightarrow ((Q \times [0, \mathfrak{B}]^d) \times \mathbb{N}^d)$$

$\mathfrak{f}(\langle q, \mathbf{x} \rangle) \stackrel{\text{def}}{=} \langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle$ with

1. $q = q'$,
2. for every $i \in [1, d]$,
   - if $\mathbf{x}(i) < \mathfrak{B}$ then $\mathbf{v}(i) \stackrel{\text{def}}{=} \mathbf{x}(i)$ and $\mathbf{x}'(i) \stackrel{\text{def}}{=} 0$,
   - otherwise $\mathbf{x}'(i) \stackrel{\text{def}}{=} \mathbf{x}(i) - \mathfrak{B}$ and $\mathbf{v}(i) \stackrel{\text{def}}{=} \mathfrak{B}$.

   (so $\mathbf{x}' + \mathbf{v} = \mathbf{x}$)

- With $\mathfrak{B} = 3$, $\mathfrak{f}(\langle q, 7 \rangle) = \langle q, 3, 4 \rangle$ and $\mathfrak{f}(\langle q, 2 \rangle) = \langle q, 2, 0 \rangle$.
- $\mathfrak{f}^{-1}(\langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle)$ defined if ($\mathbf{x}'(i) > 0$ implies $\mathbf{v}(i) = \mathfrak{B}$).

# Working on the guards (4/10)

- Guard $g$ in $\mathcal{M}$ and $\mathbf{v} \in [0, \mathfrak{B}]^d$.
- $[g]_{\mathbf{v}}$ in $\mathcal{M}'$ is defined as:
  - $[\mathrm{x}_i \sim k]_{\mathbf{v}} \stackrel{\text{def}}{=} \mathbf{v}(i) \sim k$,
  - $[\cdot]_{\mathbf{v}}$ is homomorphic for Boolean connectives.
- $[g]_{\mathbf{v}}$ is equivalent either to $\top$ or to $\perp$.
- It is easy to determine whether $[g]_{\mathbf{v}}$ is equivalent to $\top$.

# The property on guards (5/10)

- $\mathfrak{f}(\langle q, \mathbf{x} \rangle) = \langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle$.
- For all guards $g$ in $\mathcal{M}$,

  we have $\mathbf{x} \models g$ iff $[g]_{\mathbf{v}}$ is equivalent to $\top$.

  ($\mathbf{v}$ is the truncation of $\mathbf{x}$ w.r.t. $\mathfrak{B}$)
- We use that $\mathfrak{B} \geq k$ for any atomic guard $\mathrm{x} \sim k$.

# Transitions (6/10)

- For each $q \xrightarrow{\langle g, \mathbf{a} \rangle} q'$ in $T$, we consider all

$$\langle q, \mathbf{v} \rangle \xrightarrow{\langle g', \mathbf{a}' \rangle} \langle q', \mathbf{v}' \rangle$$

with $g' \stackrel{\text{def}}{=} [g]_{\mathbf{v}} \wedge \ldots$ and for all $i \in [1, d]$:

- $\mathbf{v}(i) < \mathfrak{B}$ and $\mathbf{v}(i) + \mathbf{a}(i) < \mathfrak{B}$:

$$\mathbf{v}'(i) \stackrel{\text{def}}{=} \mathbf{v}(i) + \mathbf{a}(i) \quad \text{and} \quad \mathbf{a}'(i) \stackrel{\text{def}}{=} 0$$

- $\mathbf{v}(i) < \mathfrak{B}$ and $\mathbf{v}(i) + \mathbf{a}(i) \geq \mathfrak{B}$:

$$\mathbf{v}'(i) \stackrel{\text{def}}{=} \mathfrak{B} \quad \text{and} \quad \mathbf{a}'(i) \stackrel{\text{def}}{=} \mathbf{v}(i) + \mathbf{a}(i) - \mathfrak{B}$$

# Transitions (7/10)

- $\mathbf{v}(i) = \mathfrak{B}$ and $\mathbf{a}(i) \geq 0$:

$$\mathbf{v}'(i) \stackrel{\text{def}}{=} \mathfrak{B} \quad \text{and} \quad \mathbf{a}'(i) \stackrel{\text{def}}{=} \mathbf{a}(i)$$

- $\mathbf{v}(i) = \mathfrak{B}$ and $\mathbf{a}(i) < 0$:
  (the value for $\mathrm{x}_i$ is $\geq -\mathbf{a}(i)$)

$$\mathbf{v}'(i) \stackrel{\text{def}}{=} \mathfrak{B} \quad \text{and} \quad \mathbf{a}'(i) \stackrel{\text{def}}{=} \mathbf{a}(i)$$

- These two cases can be merged !!

# Transitions (8/10)

- Remaining case: a counter is decremented in $\mathcal{M}$ from a value above the bound $\mathfrak{B}$ to a value below the bound $\mathfrak{B}$.

- $\mathbf{v}(i) = \mathfrak{B}$ and $\mathbf{a}(i) < 0$:
(the value for $\mathrm{x}_i$ equal to $\alpha$ in $[0, -\mathbf{a}(i) - 1]$)

$$\mathbf{v}'(i) \stackrel{\text{def}}{=} \underbrace{\mathfrak{B} + \overbrace{\alpha + \mathbf{a}(i)}^{<0}}_{\in[0,\mathfrak{B}]} \quad \text{and} \quad \mathbf{a}'(i) \stackrel{\text{def}}{=} -\alpha$$

- We add the conjunct $\mathrm{x}_i = \alpha$ to the guard $[g]_{\mathbf{v}}$ (or to its extensions).

- We use the assumption that $-\mathbf{a}(i) \leq \mathfrak{B}$.

# Time to wrap-up (9/10)

- $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is weakly reversal-bounded with respect to $r$ and $\mathfrak{B}$ iff $\langle \mathcal{M}', \mathfrak{f}(\langle q, \mathbf{x} \rangle) \rangle$ is $r$-reversal-bounded.

- For every run in $\mathcal{M}'$

$$\langle \langle q_0, \mathbf{v}_0 \rangle, \mathbf{y}_0 \rangle \quad \rightarrow \cdots \rightarrow \quad \langle \langle q_n, \mathbf{v}_n \rangle, \mathbf{y}_n \rangle$$

  with $\mathfrak{f}(\langle q, \mathbf{x} \rangle) = \langle \langle q_0, \mathbf{v}_0 \rangle, \mathbf{y}_0 \rangle$,

$$\mathfrak{f}^{-1}(\langle \langle q_0, \mathbf{v}_0 \rangle, \mathbf{y}_0 \rangle) \quad \rightarrow \cdots \rightarrow \quad \mathfrak{f}^{-1}(\langle \langle q_n, \mathbf{v}_n \rangle, \mathbf{y}_n \rangle)$$

  is a run in $\mathcal{M}$.

- For every $q' \in Q$, $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$ is equal to

$$\bigcup_{\mathbf{v} \in [0, \mathfrak{B}]^d} \{\pi_2(\mathfrak{f}^{-1}(\langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle)) : \mathfrak{f}(\langle q, \mathbf{x} \rangle) \xrightarrow{*} \langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle\}$$

# Presburger set $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x}\rangle \xrightarrow{*} \langle q', \mathbf{y}\rangle\}$ (10/10)

- $\langle \mathcal{M}', \mathfrak{f}(\langle q, \mathbf{x}\rangle)\rangle$ is $r$-reversal-bounded.

- For every $\mathbf{v}$, there is $\varphi_{\mathbf{v}}(\mathsf{y}_1, \ldots, \mathsf{y}_d)$ such that

$$[\![\varphi_{\mathbf{v}}]\!] = \{\mathbf{y} \in \mathbb{N}^d : \mathfrak{f}(\langle q, \mathbf{x}\rangle) \xrightarrow{*} \langle\langle q', \mathbf{v}\rangle, \mathbf{y}\rangle\}$$

- $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x}\rangle \xrightarrow{*} \langle q', \mathbf{y}\rangle\}$ characterised by $\varphi(\mathsf{z}_1, \ldots, \mathsf{z}_d)$

$$\bigvee_{\mathbf{v}} \exists\, \mathsf{y}_1, \ldots, \mathsf{y}_d \; (\varphi_{\mathbf{v}}(\mathsf{y}_1, \ldots, \mathsf{y}_d) \wedge$$

$$\bigwedge_{i \in [1,d]} ((\mathbf{v}(i) = \mathfrak{B} \Rightarrow \mathsf{z}_i = \mathsf{y}_i + \mathfrak{B}) \wedge (\mathbf{v}(i) < \mathfrak{B} \Rightarrow \mathsf{z}_i = \mathbf{v}(i))))$$

# The Reversal-Boundedness Detection Problem

# The reversal-boundedness detection problem

- The reversal-boundedness detection problem:

  Input: Counter machine $\mathcal{M}$ of dimension $d$, configuration $\langle \mathcal{M}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ and $i \in [1, d]$.

  Question: Is $\langle \mathcal{M}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ reversal-bounded with respect to the counter $\mathtt{x}_i$?

- Undecidability due to [Ibarra, JACM 78].
- Restriction to VASS is decidable [Finkel & Sangnier, MFCS'08].

# Undecidability proof

- Minsky machine $\mathcal{M}$ with halting state $q_H$ (2 counters).
- Either $\mathcal{M}$ has a unique infinite run (and never visits $q_H$) or $\mathcal{M}$ has a finite run (and halts at $q_H$).
- Counter machine $\mathcal{M}'$: replace $t = q_i \xrightarrow{\varphi} q_j$ by

$$q_i \xrightarrow{\texttt{++x}_1} q_{1,t}^{new} \xrightarrow{\texttt{--x}_1} q_{2,t}^{new} \xrightarrow{\varphi} q_j$$

- We have the following equivalences:
  - $\mathcal{M}$ halts.
  - For $\mathcal{M}'$, $q_H$ is reached from $\langle q_0, \mathbf{0} \rangle$.
  - Unique run of $\mathcal{M}'$ starting by $\langle q_0, \mathbf{0} \rangle$ is finite.
  - $\mathcal{M}'$ is reversal-bounded from $\langle q_0, \mathbf{0} \rangle$.

# EXPSPACE-hardness of VASS decision problems

- Covering and boundedness problems are EXPSPACE-complete [Lipton, TR 76; Rackoff, TCS 78].
- Control state reachability is EXPSPACE-complete too.
- Reachability problem for VAS is decidable

  [Mayr, STOC 81; Kosaraju, STOC 82; Reutenauer, 89]

  See also [Leroux, LICS 09]
  - No primitive recursive algorithm is known.
  - EXPSPACE-hardness [Lipton, TR 76].
- Checking whether two VASS produce the same set of configurations is undecidable [Hack, TCS 76].

# ExpSpace-hardness

- Reduction from the control state reachability problem for VASS.
- Instance: $\mathcal{M} = \langle Q, T, C \rangle$, $\langle q_0, \mathbf{x}_0 \rangle$ and $q_f$.
- We build the VASS $\mathcal{M}' = \langle Q', T', C \cup \{\mathtt{x}_{d+1}\} \rangle$ and $\langle q_0', \mathbf{x}_0' \rangle$ such that

$$\langle q_0, \mathbf{x}_0 \rangle \xrightarrow{*} \langle q_f, \mathbf{x}_f \rangle \text{ for some } \mathbf{x}_f \in \mathbb{N}^d$$

iff

$\langle \mathcal{M}', \langle q_0', \mathbf{x}_0' \rangle \rangle$ is not reversal-bounded with respect to $\mathtt{x}_{d+1}$.

- ExpSpace-hardness and coExpSpace= ExpSpace imply that the reversal-boundedness detection problem restricted to VASS is ExpSpace-hard too.

# Definition of $\mathcal{M}' = \langle Q', T', C \cup \{\mathtt{x}_{d+1}\}\rangle$

- $T'$ contains all the transitions of $T$, but with no update on $\mathtt{x}_{d+1}$.
- Two new transitions:

$$q_f \xrightarrow{\mathtt{x}_{d+1}{}^{++}} q_f \quad \text{and} \quad q_f \xrightarrow{\mathtt{x}_{d+1}{}^{--}} q_f$$

- $q_0' \stackrel{\text{def}}{=} q_0$.
- $\mathbf{x}_0'$ equal to $\mathbf{x}_0$ on the $d$ first counters and $\mathbf{x}_0'(d+1) \stackrel{\text{def}}{=} 0$.

$\langle q_0, \mathbf{x}_0\rangle \xrightarrow{*} \langle q_f, \mathbf{x}_f\rangle$ for some $\mathbf{x}_f \in \mathbb{N}^d$
iff
$\langle \mathcal{M}', \langle q_0', \mathbf{x}_0'\rangle\rangle$ is not reversal-bounded with respect to $\mathtt{x}_{d+1}$.

# ExpSpace upper bound

- ExpSpace upper bound by reduction into the place-boundedness problem for VASS. [Demri, JCSS 13]

- Place boundedness problem for VASS:

  Input: A VASS $\mathcal{M} = \langle Q, T, C \rangle$, $\langle q_0, \mathbf{x}_0 \rangle$ and $\mathtt{x}_j \in C$.

  Question: Is there a bound $\mathfrak{B} \in \mathbb{N}$ such that $\langle q_0, \mathbf{x}_0 \rangle \xrightarrow{*} \langle q', \mathbf{x}' \rangle$ implies $\mathbf{x}'(j) \leq \mathfrak{B}$?

- Proof idea: add a new counter that counts the number of reversals for the distinguished counter $\mathtt{x}_j$.

# EXPSPACE upper bound

- Instance: $\mathcal{M} = \langle Q, T, C \rangle$, $\langle q_0, \mathbf{x}_0 \rangle$ and $\mathrm{x}_j \in C$.
- $\mathcal{M}' = \langle Q', T', C \cup \{\mathrm{x}_{d+1}\} \rangle$ with $Q' = Q \times \{\mathrm{DEC}, \mathrm{INC}\}$.
- In $\mathcal{M}'$, the number of reversals for $\mathrm{x}_j$ is recorded in $\mathrm{x}_{d+1}$.
- $\langle \mathcal{M}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ is reversal-bounded with respect to $\mathrm{x}_j$ iff $\langle \mathcal{M}', \langle q_0', \mathbf{x}_0' \rangle \rangle$ is bounded with respect to $\mathrm{x}_{d+1}$.
- $q_0' \stackrel{\text{def}}{=} \langle q_0, \mathrm{INC} \rangle$.
- $\mathbf{x}_0'$ restricted to the $d$ first counters is $\mathbf{x_0}$ and $\mathbf{x}_0'(d+1) \stackrel{\text{def}}{=} 0$.

# Decidable Repeated Reachability Problems

# The problems

- Control state repeated reachability problem with bounded number of reversals:

  Input: CM $\mathcal{M}$, $\langle q_0, \mathbf{x}_0 \rangle$, $r \geq 0$, state $q_f$.

  Question: is there an infinite $r$-reversal-bounded run starting from $\langle q_0, \mathbf{x}_0 \rangle$ such that $q_f$ is repeated infinitely often?

- Control state reachability reachability problem with bounded number of reversals:

  Input: CM $\mathcal{M}$, $\langle q_0, \mathbf{x}_0 \rangle$, $r \geq 0$, state $q_f$.

  Question: is there a finite $r$-reversal-bounded run starting from $\langle q_0, \mathbf{x}_0 \rangle$ such that $q_f$ is reached?

- Control state reachability reachability problem with bounded number of reversals is decidable.

- Control state repeated reachability problem with bounded number of reversals is decidable. (proof follows).

[Dang & Ibarra & San Pietro, FSTTCS'01]

# A variant

- $\exists$-Presburger infinitely often problem:

  Input: Initialized CM $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ that is $r$-reversal-bounded and $\psi = \mathsf{GF}\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_d)$ where $\varphi$ is a Presburger formula on counters.

  Question: Is there an infinite run from $\langle q, \mathbf{x} \rangle$ satisfying $\psi$?

- $\exists$-Presburger infinitely often problem is decidable.

  [Dang & San Pietro & Kemmerer, TCS 03]

# Idea of the proof
# (for control state repeated reachability problem)

- Initialized CM $\langle \mathcal{M}, \langle q_0, \mathbf{x_0} \rangle \rangle$, $q_f \in Q$ and $r \geq 0$.
- Reduction to an instance of the control state reachability problem with a bounded number of reversals (decidable).
- $k_{max} \in \mathbb{N}$: maximal constant $k$ occurring in an atomic guard of the form $\mathtt{x} \sim k$.
- Property $(\star)$: there is an $r$-reversal-bounded infinite run from $\langle q_0, \mathbf{x_0} \rangle$ such that $q_f$ is repeated infinitely often.
- We reduce $(\star)$ to a reachability question for a new reversal-bounded counter machine $\mathcal{M}'$.

# Property ($\star\star$)

There exist an $r$-reversal-bounded run

$$\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle \cdots \xrightarrow{t_\ell} \langle q_\ell, \mathbf{x}_\ell \rangle$$

$\ell' \in [0, \ell - 1]$ and $C_= \subseteq C$ such that

(a) $q_\ell = q_{\ell'} = q_f$,

(b) for all $\mathtt{x}_i \in C_=$ and $j \in [\ell' + 1, \ell]$, $\mathbf{x}_{j-1}(i) = \mathbf{x}_j(i)$,

(c) for all $\mathtt{x}_i \in (C \setminus C_=)$ and $j \in [\ell' + 1, \ell]$, $\mathbf{x}_{j-1}(i) \leq \mathbf{x}_j(i)$,

(d) for all $\mathtt{x}_i \in (C \setminus C_=)$, we have $k_{max} < \mathbf{x}_{l'}(i)$,

(e) for all $\mathtt{x}_i \in C_=$, have $\mathbf{x}_{\ell'}(i) \leq k_{max}$.

# Equivalence

- By showing ($\star$) and ($\star\star$) are equivalent, we can then reduce control state repeated reachability to control state reachability.
- Checking ($\star\star$) amounts to introduce $2^d$ copies of $\mathcal{M}$, one for each subset of $C$.
- Proof in two steps:
  1. Equivalence between ($\star$) and ($\star\star$).
  2. ($\star\star$) reduces to an instance of control state reachability with a bounded number of reversals.

# $(\star)$ implies $(\star\star)$

- Infinite $r$-reversal-bounded run

$$\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle \xrightarrow{t_2} \langle q_2, \mathbf{x}_2 \rangle \cdots$$

  such that $q_f$ is repeated infinitely often.

- $C^{\rho}_{=} \subseteq C$: counters whose values are less or equal to $k_{max}$, apart from a finite prefix.

- Since $\rho$ is $r$-reversal-bounded, there exists $I \geq 0$ such that for some $n \geq I$, no counters in $C \setminus C^{\rho}_{=}$ is decremented and their values are strictly greater than $k_{max}$.

- Since $q_f$ is repeated infinitely often, there are $I \leq \ell' < \ell$ such that $q_{\ell} = q_{\ell'} = q_f$ and (b)-(e) hold.

# $(\star\star)$ implies $(\star)$

- $r$-reversal-bounded run

$$\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle \cdots \xrightarrow{t_\ell} \langle q_\ell, \mathbf{x}_\ell \rangle,$$

  $\ell' \in [0, \ell - 1]$ and $C_= \subseteq C$ witnessing the satisfaction of $(\star\star)$.

- $\omega$-sequence of transitions

$$t_1 \cdots t_{\ell'} (t_{\ell'+1} \cdots t_\ell)^\omega$$

  allows us to define an infinite $r$-reversal-bounded run $\rho'$ that extends $\rho$.

- $q_f$ is repeated infinitely often.

- Guards on transitions are satisfied by the counter values.

- Indeed, the conditions (c),(d) and (e) and the values for counters in $(C \setminus C_=)$ are non-negative thanks to (c) and (d).

# Reduction to a reachability question

- Reversal-bounded $\mathcal{M}' = \langle Q', T', C \rangle$ such that $(\star\star)$ iff there is a $r$-reversal-bounded run from $\langle q_0, \mathbf{x}_0 \rangle$ that reaches $q_{new}$.
- $\mathcal{M}' = \mathcal{M} \uplus 2^d$ "copies" of $\mathcal{M}$.
  (one copy per subset of $\{\mathtt{x}_1, \ldots, \mathtt{x}_d\}$.)
- $C_=$-copy of $\mathcal{M}$:
  - no transition in the $C_=$-copy modifies $\mathtt{x}$ in $C_=$,
  - no transition in the $C_=$-copy decrements $\mathtt{x}$ in $(C \setminus C_=)$.
  - Control states are pairs in $Q \times \{C_=\}$.

# Principles for constructing $\mathcal{M}'$

- To simulate $\langle q_{\ell'}, \mathbf{x}_{\ell'}\rangle \cdots \langle q_{\ell}, \mathbf{x}_{\ell}\rangle$ for the satisfaction of $(\star\star)$ in $\mathcal{M}$, we nondeterministically move from the original copy to some $C_=$-copy in $\mathcal{M}'$.

- For every $C_=$, we consider in $\mathcal{M}'$ a transition from $q_f$ to $\langle q_f, C_=\rangle$ that checks:
  1. all counters in $C_=$ have values $\leq k_{max}$,
  2. all counters in $(C \setminus C_=)$ have values $> k_{max}$.

  $$(\bigwedge_{\mathtt{x} \in (C \setminus C_=)} \mathtt{x} \geq (k_{max} + 1)) \wedge (\bigwedge_{\mathtt{x} \in C_=} \mathtt{x} \leq k_{max})$$

  (and the transition has no effect)

- As soon as in the $C_=$-copy, we reach again a control state whose first component is $q_f$, we may jump to the final control state $q_{new}$.
- In $\mathcal{M}'$, it is sufficient to look for a $r$-reversal-bounded run.

# Next lecture on November 13th

- Lecturer: Philippe Schnoebelen (phs@lsv.fr).

# Exercise (1/5)

- Goal: Show decidability of the problem:

  Input: $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ and semilinear set $X \subseteq \mathbb{N}^d$ defined by $\langle \mathbf{b}_1, \mathfrak{P}_1 \rangle, \ldots, \langle \mathbf{b}_\alpha, \mathfrak{P}_\alpha \rangle$.

  Question: Is there an infinite $r$-reversal-bounded run from $\langle q, \mathbf{x} \rangle$ such that infinitely often the counter values are in $X$?

A) Show that we can restrict ourselves to $\alpha = 1$ and infinitely often the counter values belong to the linear set $\langle \mathbf{b}_1, \mathfrak{P}_1 \rangle$ and simulaneously the location is some fixed $q'$.

# Exercise (2/5)

B) Linear set $X$ characterised by $\mathbf{b}$ and $\mathbf{p}_1, \ldots, \mathbf{p}_N$.
Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be an infinite sequence of elements in $X$.
Show that there are $\ell' < \ell$ and $\mathbf{a}, \mathbf{c} \in \mathbb{N}^N$ such that

(I) $\mathbf{x}_{\ell'} \preceq \mathbf{x}_\ell$,

(II) $\mathbf{x}_{\ell'} = \mathbf{b} + \sum_{k \in [1,N]} \mathbf{a}(k)\mathbf{p}_k$,

(III) $\mathbf{x}_\ell = \mathbf{b} + \sum_{k \in [1,N]} \mathbf{c}(k)\mathbf{p}_k$,

(IV) $\mathbf{a} \preceq \mathbf{c}$.

C) Design a 0-reversal-bounded counter machine with $d$ counters such that for some state $q_0, q_f \in Q$, for all $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{x} \in X$ iff there is a run from $\langle q_0, \mathbf{x} \rangle$ to $\langle q_f, \mathbf{0} \rangle$.

# Exercise (3/5)

D) Design a 1-reversal-bounded CM with $2d$ counters such that for some state $q_0, q_f \in Q$, for all $\mathbf{x} \in \mathbb{N}^{2d}$ such that the restriction to $\mathbf{x}$ to the $d$ last counters equal to $\mathbf{0}$,

the restriction of $\mathbf{x}$ to the $d$ first counters belongs to $X$
iff
there is a run from $\langle q_0, \mathbf{x} \rangle$ to $\langle q_f, \mathbf{x} \rangle$.

E) Design a 1-reversal-bounded CM with $4d$ counters such that for some state $q_0, q_f \in Q$, for all $\mathbf{x} \in \mathbb{N}^{4d}$ such that the restriction to $\mathbf{x}$ to the $2d$ last counters equal to $\mathbf{0}$,

there are $\lambda_1, \ldots, \lambda_N \in \mathbb{N}$ such that for all $i \in [1, d]$,
$$\mathbf{x}(d+i) - \mathbf{x}(i) = \lambda_1 \mathbf{p}_1(i) + \cdots \lambda_N \mathbf{p}_N(i)$$
iff
there is a run from $\langle q_0, \mathbf{x} \rangle$ to $\langle q_f, \mathbf{x} \rangle$.

# Exercise (4/5)

Show that the conditions below are equivalent:

($\star$) There is an infinite $r$-reversal-bounded run from $\langle q_0, \mathbf{x}_0 \rangle$ such that counter values belong to $X$ and the state is $q'$ infinitely often.

($\star\star$) There exist a finite $r$-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle \cdots \xrightarrow{t_l} \langle q_\ell, \mathbf{x}_\ell \rangle$, $\ell' \in [0, \ell - 1]$ and $C_= \subseteq C$ such that

(a) $q_\ell = q_{\ell'} = q'$,
(b) $\mathbf{x}_{\ell'}, \mathbf{x}_\ell \in X$,
(c) (I)–(IV) above,
(d) for $\mathrm{x}_i \in C_=$ and $j \in [\ell' + 1, \ell]$, $\mathbf{x}_j(i) - \mathbf{x}_{j-1}(i) = 0$,
(e) for $\mathrm{x}_i \in (C \setminus C_=)$ and $j \in [\ell' + 1, \ell]$, $\mathbf{x}_{j-1}(i) \leq \mathbf{x}_j(i)$,
(f) for $\mathrm{x}_i \in (C \setminus C_=)$, we have $k_{max} < \mathbf{x}_{\ell'}(i)$.
(g) for all $\mathrm{x}_i \in C_=$, have $\mathbf{x}_{\ell'}(i) \leq k_{max}$.

$k_{max}$: maximal constant $k$ occurring in guards

# Exercise (5/5)

- Design a reduction from ($\star\star$) to an instance of the reachability problem with bounded number of reversals.
- Conclude that checking whether an initialized counter machine has an infinite $r$-reversal-bounded run visiting infinitely often a semilinear set can be decided in NEXPTIME.