Dynamic Axioms in Description Logics

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Description logics and updates

• Description logics are well-known logical formalisms for knowledge representation. [Baader et al., Book 2017]

• Interpretations are labeled directed graphs satisfying inclusions $C \sqsubseteq C'$, assertions $C(a)$, role inclusion axioms $r_1 \circ \cdots \circ r_n \sqsubseteq s$, ...

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- How to specify the evolution of the satisfaction of inclusions or assertions when the current interpretation is updated?
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- How to specify the evolution of the satisfaction of inclusions or assertions when the current interpretation is updated?

    This talk: proposal for a framework based on separating connectives from separation logics.

- Separation logics designed to verify heap-manipulating programs. Use of connectives $\ast$, $\neg\ast$, $\otimes$, ...
**A LC in a nutshell**

- Complex concepts.

\[ C ::= \top \mid \bot \mid A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.C \mid \forall r.C, \]

where \( A \in \mathbb{N}_C \) and \( r \in \mathbb{N}_R \).
\textbf{ALC in a nutshell}

- Complex concepts.

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where \( A \in N_C \) and \( r \in N_R \).

- Interpretation \( \mathcal{I} \stackrel{\text{def}}{=} (\Delta^\mathcal{I}, \cdot^\mathcal{I}) \)
  
  - \( \Delta^\mathcal{I} \): non-empty set (the \textit{domain}).
  
  - \( \cdot^\mathcal{I} \): \textit{interpretation function} such that
    \[ A^\mathcal{I} \subseteq \Delta^\mathcal{I} \quad r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I} \quad a^\mathcal{I} \in \Delta^\mathcal{I} \]
**A LC in a nutshell**

- Complex concepts.

\[ C ::= \top \ | \bot \ | \ A \ | \neg C \ | \ C \cap C \ | \ C \cup C \ | \exists r.C \ | \forall r.C, \]

where \( A \in \mathbb{N}_C \) and \( r \in \mathbb{N}_R \).

- Interpretation \( I \) \text{ def } (\Delta^I, \cdot^I)

  - \( \Delta^I \): non-empty set (the domain).
  
  - \( \cdot^I \): interpretation function such that

\[ A^I \subseteq \Delta^I \quad r^I \subseteq \Delta^I \times \Delta^I \quad a^I \in \Delta^I \]

Concept name \( A \) / role name \( r \) / individual name \( a \)

\( \approx \)

unary predicate / binary predicate / constant
Semantics for complex concepts

\[ \top^I \overset{\text{def}}{=} \Delta^I \]

\[ \bot^I \overset{\text{def}}{=} \emptyset \]

\[ (\neg C)^I \overset{\text{def}}{=} \Delta^I \setminus C^I \]

\[ (C_1 \sqcup C_2)^I \overset{\text{def}}{=} C_1^I \cup C_2^I \]

\[ (C_1 \sqcap C_2)^I \overset{\text{def}}{=} C_1^I \cap C_2^I \]

\[ (\exists r. C)^I \overset{\text{def}}{=} \{ d \in \Delta^I \mid r^I(d) \cap C^I \neq \emptyset \} \]

\[ (\forall r. C)^I \overset{\text{def}}{=} \{ d \in \Delta^I \mid r^I(d) \subseteq C^I \} \]

\[ \mathcal{R}(d) \overset{\text{def}}{=} \{ e \mid (d, e) \in \mathcal{R} \} \]
Inclusions and assertions

• Expressions of the form $C \sqsubseteq D$ are called general concept inclusions (GCIs).

\[ \text{Employee} \sqsubseteq \exists \text{WorksFor}. \top \]

\[ \mathcal{I} \models C \sqsubseteq D \quad \overset{\text{def}}{=} \quad C^\mathcal{I} \subseteq D^\mathcal{I} \]

\[ \text{Concept assertion:} \quad C(a) (\text{Student} \sqcap \neg \exists \text{Pays}. \text{Tax})(Alice) \]

\[ \mathcal{I} \models C(a) \quad \overset{\text{def}}{=} \quad a^\mathcal{I} \in C^\mathcal{I} \]

\[ \text{Role assertion:} \quad r(a, b) (\text{WorksFor}(Laura, \text{CNRS})) \]

\[ \mathcal{I} \models r(a, b) \quad \overset{\text{def}}{=} \quad (a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I} \]
Inclusions and assertions

- Expressions of the form $C \sqsubseteq D$ are called general concept inclusions (GCIs).

- Concept assertion: $C(a)$ \quad (Student $\sqcap \neg \exists$Pays.Tax)(Alice)

\[
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Inclusions and assertions

- Expressions of the form $C \sqsubseteq D$ are called general concept inclusions (GCIs).

- **Concept assertion:** $C(a)$ \quad (Student $\sqcap \neg \exists \text{Pays.Tax})(\text{Alice})$

  $\mathcal{I} \models C(a) \iff a^\mathcal{I} \in C^\mathcal{I}$

- **Role assertion:** $r(a, b)$ \quad \text{WorksFor}(\text{Laura, CNRS})

  $\mathcal{I} \models r(a, b) \iff (a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$
Knowledge bases (a.k.a. ontologies)

- Terminological Box (TBox) \( \mathcal{T} \): finite collection of GCIs.
- Assertional Box (ABox) \( \mathcal{A} \): finite collection of assertions.
- Knowledge base \( \mathcal{K} \) is a pair \( (\mathcal{T}, \mathcal{A}) \).
Knowledge bases (a.k.a. ontologies)

- Terminological Box (TBox) $T$: finite collection of GCIs.
- Assertional Box (ABox) $A$: finite collection of assertions.
- Knowledge base $K$ is a pair $(T, A)$.
- Interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, knowledge base $K = (T, A)$.

\[ \mathcal{I} \models K \iff \text{for all } \alpha \in A \cup T, \mathcal{I} \models \alpha \]
Knowledge bases (a.k.a. ontologies)

- Terminological Box (TBox) $\mathcal{T}$: finite collection of GCIs.
- Assertional Box (ABox) $\mathcal{A}$: finite collection of assertions.
- Knowledge base $\mathcal{K}$ is a pair $(\mathcal{T}, \mathcal{A})$.

- Interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I}^\mathcal{I})$, knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$.
  \[ \mathcal{I} \models \mathcal{K} \iff \text{for all } \alpha \in \mathcal{A} \cup \mathcal{T}, \mathcal{I} \models \alpha \]

- $\mathcal{K}$ is consistent $\iff$ there is some $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$.
- Consistency problem for $\mathcal{ALC}$ is $\text{ExpTime}$-complete.
Adding dynamic axioms in KBs

- Problems from [Liu et al., KR’06] are related to the construction of $\mathcal{A}'$ equivalent to $\mathcal{A}$ after (deterministic) update $\mathcal{U}$.

$$\mathcal{A}' = \mathcal{A} \ast \mathcal{U} \quad \text{Inter}(\mathcal{A}') = \{ \mathcal{I}^{\mathcal{U}} \mid \mathcal{I} \in \text{Inter}(\mathcal{A}) \}$$

(symbol $\ast$ overloaded)
Adding dynamic axioms in KBs

• Problems from [Liu et al., KR’06] are related to the construction of $A'$ equivalent to $A$ after (deterministic) update $U$.

$$A' = A \ast U \quad \text{Inter}(A') = \{ \mathcal{I}^U \mid \mathcal{I} \in \text{Inter}(A) \}$$

(s symbol * overloaded)

• In this talk, dynamic axioms update the interpretations using the separating connectives $\ast$ and $\ominus$.

• Partial composition operator $\oplus : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I}$ with AC $\oplus$. 
Adding dynamic axioms in KBs

- Problems from [Liu et al., KR’06] are related to the construction of $A'$ equivalent to $A$ after (deterministic) update $U$.

$$A' = A \ast U \quad \text{Inter}(A') = \{ I^U | I \in \text{Inter}(A) \}$$

(symmetric * overloaded)

- In this talk, dynamic axioms update the interpretations using the separating connectives $\ast$ and $\ominus$.

- Partial composition operator $\oplus : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I}$ with AC $\oplus$.

- KB $\mathcal{K} = (\mathcal{I}, \mathcal{A}, \mathcal{D})$ with dynamic box $\mathcal{D}$. 
Connective $\bigotimes$ in dynamic axioms

- $\mathcal{I} \not\models (\exists r. T \sqsubseteq \exists s. T)$ but there is $\mathcal{J}$ such that $\mathcal{I} \oplus \mathcal{J}$ is defined and satisfies $\mathcal{I} \oplus \mathcal{J} \models \exists r. T \sqsubseteq \exists s. T$. 

Connective $\bigotimes$ in separation logics introduced in [Vafeiadis & Parkinson, CONCUR’07].
Connective $\ominus$ in dynamic axioms

- $I \not\models (\exists r. T \subseteq \exists s. T)$ but there is $J$ such that $I \oplus J$ is defined and satisfies $I \oplus J \models \exists r. T \subseteq \exists s. T$.

- Property specified by
  \[ \neg (\exists r. T \subseteq \exists s. T) \cap (T \ominus (\exists r. T \subseteq \exists s. T)) \]

- Connective $\ominus$ in separation logics introduced in
  [Vafeiadis & Parkinson, CONCUR’07]
Positive dynamic axioms

\[ U, V := \top \mid C(a) \mid r(a, b) \mid C \sqsubseteq D \mid \]

\[
\begin{align*}
U \cup V \mid U \cap V & \mid U \ast V \mid U \leftarrow \ast V \\
& \text{positive Boolean part} \quad \text{compositional part}
\end{align*}
\]

(existential separating connectives)
Positive dynamic axioms

\[ U, V := \top | C(a) | r(a, b) | C \sqsubseteq D | \]

\[
\begin{array}{l}
U \sqcup V \mid U \sqcap V \mid U \ast V \mid U \smallcircleast V \\
positive \ Boolean \ part \mid \ \text{compositional part}
\end{array}
\]

(existental separating connectives)

\[ \mathcal{I} \models U_1 \ast U_2 \ \text{iff} \ \text{there are } \mathcal{I}_1, \mathcal{I}_2 \ \text{s.t. } \mathcal{I} = \mathcal{I}_1 \oplus \mathcal{I}_2, \]
\[ \mathcal{I}_1 \models U_1 \text{ and } \mathcal{I}_2 \models U_2 \]

\[ \mathcal{I} \models U_1 \smallcircleast U_2 \ \text{iff} \ \text{there is } \mathcal{I}' \ \text{s.t. } \mathcal{I} \oplus \mathcal{I}' \text{ is defined,} \]
\[ \mathcal{I}' \models U_1 \text{ and } \mathcal{I} \oplus \mathcal{I}' \models U_2. \]
Interpretation composition (for this talk)

- \( \mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 \) whenever,
  - \( \mathcal{I}, \mathcal{I}_1 \) and \( \mathcal{I}_2 \) share the same domain,
  - they agree on the interpretation of the individual names \( a \) and concept names \( A \),
  - for all \( r \in \mathbb{N}_R \), we have \( r^\mathcal{I} = r^\mathcal{I}_1 \cup r^\mathcal{I}_2 \).

\[\begin{array}{ccc}
\begin{array}{ccc}
A, B & A, B & A, B \\
& A & A, B \\
é & e & f
\end{array}
\end{array}\]

\[\begin{array}{ccc}
\begin{array}{ccc}
A, B & A, B & A, B \\
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\end{array}\]

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A, B & A, B & A, B \\
& A & A, B \\
é & e & f
\end{array}
\end{array}\]
Consistency problem with dynamic axioms

- Dynamic axioms (closure under Boolean operators)

\[ U, V ::= \top \mid \neg U \mid U \sqcup V \mid U \sqcap V \]

- EL concepts:
  \[ C ::= \top \mid A \mid C \sqcap D \mid \exists r.C \]
  (EL fragment of ALC, no \( \neg \), \( \sqcup \))

<table>
<thead>
<tr>
<th>Logic \ Dynamic axioms</th>
<th>Positive DAs</th>
<th>DAs</th>
</tr>
</thead>
<tbody>
<tr>
<td>EL</td>
<td>in PTIME</td>
<td>undecidable</td>
</tr>
<tr>
<td>ALC</td>
<td>EXPTIME-complete</td>
<td>undecidable</td>
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</tbody>
</table>
Fragment $\mathcal{EL}$

- $\mathcal{EL}$ concepts: $C ::= \top | A | C \sqcap D | \exists r.C$

- Consistency problem for PDA with $\mathcal{EL}$ (no $\sqcup, \neg$; $C, D \in \mathcal{EL}$)

\[ U, V ::= \top | C(a) \mid r(a, b) \mid C \sqsubseteq D \mid U \sqcap V \mid U \ast V \mid U \ominus V \]
Fragment $\mathcal{EL}$

- **$\mathcal{EL}$ concepts:** $C ::= \top \mid A \mid C \cap D \mid \exists r.C$

- Consistency problem for PDA with $\mathcal{EL}$ (no $\sqcup, \neg$; $C, D \in \mathcal{EL}$)
  \[
  U, V ::= \top \mid C(a) \mid r(a, b) \mid C \sqsubseteq D \mid U \cap V \mid U \star V \mid U \ominus V
  \]

- A KB with PDAs for $\mathcal{EL}$ is not always consistent.
  \[
  r(a, b) \sqcap (r(a, b) \ominus \top)
  \]
Interpretable positions

- PDA $\mathbb{U}$ represented by a labelled finite tree.

$$(\mathbb{U}_1 \oplus \mathbb{U}_2) \cap (\mathbb{U}'_1 \oplus \mathbb{U}'_2)$$
Interpretable positions

- PDA $\mathcal{U}$ represented by a labelled finite tree.

$$\big((\mathcal{U}_1 \ominus \mathcal{U}_2) \cap (\mathcal{U}_1' \ominus \mathcal{U}_2')\big)$$

- $\text{Int}_{\mathcal{U}}$: set of interpretable positions, smallest subset of positions such that
  - $\varepsilon \in \text{Int}_{\mathcal{U}}$ and,
  - $n \cdot 1, n \cdot 2 \in \text{Int}_{\mathcal{U}}$, for $n$ labelled by a PDA of the form either $\mathcal{U}_1 \ominus \mathcal{U}_2$ or $\mathcal{U}_1 \ast \mathcal{U}_2$. 

...
PTime upper bound

- Derivation of statements of the form \((n \in \text{Int}_U)\)

\[ n : r(a, b), \quad n : \neg r(a, b), \quad \bot \]

(see paper for the definition of the calculus)
PTime upper bound

- Derivation of statements of the form \((n \in \text{Int}_U)\)

\[
\begin{align*}
\varepsilon : r(a, b), & \quad \varepsilon : \neg r(a, b), \quad \bot \\
\text{(see paper for the definition of the calculus)}
\end{align*}
\]

\[
\begin{align*}
\varepsilon : r(a, b) & \quad \varepsilon + 2.1 = 2.2 \\
\frac{2.1 : r(a, b)}{\varepsilon : \neg r(a, b)} \quad 2.1 : r(a, b) \\
\bot
\end{align*}
\]
PTime upper bound

- Derivation of statements of the form \((n \in \text{Int}_U)\)

\[ n : r(a, b), \quad n : \neg r(a, b), \quad \bot \]

(see paper for the definition of the calculus)

\[ r(a, b) \sqcap (r(a, b) \neg \top) \]

\[ \varepsilon : r(a, b) \quad \varepsilon + 2.1 = 2.2 \quad 2.1 : r(a, b) \]

\[ \bot \]

- Equivalence between:
  - \( \mathcal{X} = (T, A, D) \) is consistent,
  - \( \bot \) cannot be derived from \( U = \bigcap_{\alpha \in T \cup A \cup D} \alpha \).

- Consistency problem for \( \mathcal{EL} \) with PDAs is in \( \text{PTime} \).
Designing disjointness axioms from PDAs

- \(i(n) \overset{\text{def}}{=} \) maximal prefix of \(n\) that is in \(\text{Int}_U\).
Designing disjointness axioms from PDAs

• $i(n) \overset{\text{def}}{=} \text{maximal prefix of } n \text{ that is in } \text{Int}_U$.

\begin{center}
\begin{tikzpicture}
  \node (U_1) at (0,0) {$U_1$};
  \node (U_2) at (1.5,0) {$U_2$};
  \node (U_1') at (0,1.5) {$U_1'$};
  \node (U_2') at (1.5,1.5) {$U_2'$};
  \draw (U_1) edge (U_1') edge (U_2) edge (U_2');
  \draw (U_2) edge (U_1') edge (U_2');
\end{tikzpicture}
\end{center}

• $\text{Disj}_U$: smallest set of disjointness axioms of the form $n = n_1 + n_2$ with $n, n_1, n_2 \in \text{Int}_U$ such that
  • if $n$ labelled by $U_1 \ast U_2$ then $i(n) = (n \cdot 1) + (n \cdot 2) \in \text{Disj}_U$,
  • if $n$ labelled by $U_1 \ominus U_2$ then $(n \cdot 2) = (n \cdot 1) + i(n) \in \text{Disj}_U$.

• With $U^* = (U_1 \ominus U_2) \cap (U_1' \ominus U_2')$,
  \[
  \text{Disj}_{U^*} \supseteq \{ 1 \cdot 2 = \varepsilon + 1 \cdot 1, \ 2 \cdot 2 = \varepsilon + 2 \cdot 1 \} \]
Designing disjointness axioms from PDAs (II)

- With $U^* = (U_1 \oplus U_2) \cap (U'_1 \ominus U'_2)$,

\[
\text{Disj}_{U^*} \supseteq \{1 \cdot 2 = \varepsilon + 1 \cdot 1, \ 2 \cdot 2 = \varepsilon + 2 \cdot 1\}
\]
Designing disjointness axioms from PDAs (II)

- With $U^* = (U_1 ∪ U_2) \cap (U'_1 ∪ U'_2)$,

  $$\text{Disj}_{U^*} \supseteq \{1 \cdot 2 = \varepsilon + 1 \cdot 1, \ 2 \cdot 2 = \varepsilon + 2 \cdot 1\}$$

- A map $g: \text{Int}_U \rightarrow \mathbb{I}$ is said to be a complete witness for $U$ iff for all $n \in \text{Int}_U$,
  - $g(n) \models U_n$ with $n$ labelled by $U_n$ and,
  - if $n = n_1 + n_2$ is in $\text{Disj}_U$, then $g(n) = g(n_1) + g(n_2)$. 

Consistency of $U$ amounts to find interpretations for the interpretable positions that can be composed according to disjointness axioms.

$U$ is consistent iff there is a complete witness for $U$. Useful to show that consistency problem of PDAs with A LC concepts in ExpTime.
Designing disjointness axioms from PDAs (II)

- With $\mathbb{U}^* = (\mathbb{U}_1 \oplus \mathbb{U}_2) \cap (\mathbb{U}_1' \oplus \mathbb{U}_2')$,

  $$\text{Disj}_{\mathbb{U}^*} \supseteq \{1 \cdot 2 = \varepsilon + 1 \cdot 1, \ 2 \cdot 2 = \varepsilon + 2 \cdot 1\}$$

- A map $g : \text{Int}_\mathbb{U} \to \mathbb{I}$ is said to be a complete witness for $\mathbb{U}$ iff for all $n \in \text{Int}_\mathbb{U}$,
  - $g(n) \models \mathbb{U}_n$ with $n$ labelled by $\mathbb{U}_n$ and,
  - if $n = n_1 + n_2$ is in Disj$_\mathbb{U}$, then $g(n) = g(n_1) + g(n_2)$.

- Consistency of $\mathbb{U}$ amounts to find interpretations for the interpretable positions that can be composed according to disjointness axioms.
Designing disjointness axioms from PDAs (II)

- With $U^* = (U_1 \oplus U_2) \cap (U_1' \ominus U_2')$,
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- Consistency of $U$ amounts to find interpretations for the interpretable positions that can be composed according to disjointness axioms.

- $U$ is consistent iff there is a complete witness for $U$. 
Designing disjointness axioms from PDAs (II)

- With $\mathbb{U}^* = (\mathbb{U}_1 \oplus \mathbb{U}_2) \cap (\mathbb{U}_1' \ominus \mathbb{U}_2')$,
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  - $g(n) \models \mathbb{U}_n$ with $n$ labelled by $\mathbb{U}_n$ and,
  - if $n = n_1 + n_2$ is in Disj$_\mathbb{U}$, then $g(n) = g(n_1) + g(n_2)$.

- Consistency of $\mathbb{U}$ amounts to find interpretations for the interpretable positions that can be composed according to disjointness axioms.

- $\mathbb{U}$ is consistent iff there is a complete witness for $\mathbb{U}$.

- Useful to show that consistency problem of PDAs with $A\mathcal{LC}$ concepts in $\text{ExpTime}$. 
A simple proof system (parameterised by $\mathbb{U}$)

Derivation of statements of the form $(n \in \text{Int}_\mathbb{U})$

\[ n : r(a, b), \quad n : \neg r(a, b), \quad \bot \]

\[ \frac{n : r(a, b) \quad n : \neg r(a, b)}{\bot} \quad \frac{n \text{ labelled by } r(a, b)}{i(n) : r(a, b)} \]

\[ n = n_1 + n_2 \in \text{Disj}_\mathbb{U} \quad n_i : r(a, b) \]

\[ n : r(a, b) \]

\[ n = n_1 + n_2 \in \text{Disj}_\mathbb{U} \quad n_i : r(a, b) \]

\[ n_{3-i} : \neg r(a, b) \]

\text{etc.}
Correctness

- $\mathbb{U} \vdash n : r(a, b)$ implies for all complete witnesses $g$ for $\mathbb{U}$, we have $g(n) \models r(a, b)$.

- $\mathbb{U} \vdash n : \neg r(a, b)$ implies for all complete witnesses $g$ for $\mathbb{U}$, we have $g(n) \not\models r(a, b)$.

- So, $\mathcal{H}$ consistent implies $\bot$ is not derivable.
Sketch of the proof (other direction)

• $\mathcal{RA}(U)$: set of role assertions occurring in $U$.

• Complete snapshot $f: \text{Int}_U \times \mathcal{RA}(U) \rightarrow \{0, 1\}$ such that for all $r(a, b) \in \mathcal{RA}(U)$,
  
  • for all $n \in \text{Int}_U$, $U \vdash n: r(a, b)$ implies $f(n, r(a, b)) = 1$, and $U \vdash n: \neg r(a, b)$ implies $f(n, r(a, b)) = 0$,
  
  • $f(n, r(a, b)) = f(n_1, r(a, b)) + f(n_2, r(a, b)) \leq 1$, whenever $n = n_1 + n_2 \in \text{Disj}_U$. 

The other direction is much more involved and requires to use the following properties.

• Non-derivability of $\bot$ implies the existence of a complete snapshot.

• $(T, A)$ is $\text{EL}$ consistent, say $I | = (T, A)$.

• The interpretation for $U$ is made of copies of $I$ taking care of disjointness axioms and the complete snapshot.
Sketch of the proof (other direction)

- $RA(U)$: set of role assertions occurring in $U$.

- Complete snapshot $f: \text{Int}_U \times RA(U) \to \{0, 1\}$ such that for all $r(a, b) \in RA(U)$,
  - for all $n \in \text{Int}_U$, $U \vdash n: r(a, b)$ implies $f(n, r(a, b))=1$, and
  - $U \vdash n: \neg r(a, b)$ implies $f(n, r(a, b)) = 0$,

- $f(n, r(a, b)) = f(n_1, r(a, b)) + f(n_2, r(a, b)) \leq 1$, whenever $n = n_1 + n_2 \in \text{Disj}_U$.

- The other direction is much more involved and requires to use the following properties.
  - Non-derivability of $\bot$ implies the existence of a complete snapshot.
  - $(\mathcal{I}, \mathcal{A})$ is $\mathcal{EL}$ consistent, say $\mathcal{I} \models (\mathcal{I}, \mathcal{A})$.
  - The interpretation for $U$ is made of copies of $\mathcal{I}$ taking care of disjointness axioms and the complete snapshot.
**$\mathcal{ALC}$ with dynamic axioms**

- Consistency of DA with $\mathcal{ALC}$ concepts is undecidable.
- Reduction from $\mathcal{ALC}$ concept satisfiability w.r.t. RBoxes $R$. 

---

$\mathcal{I}_1 \sqsubseteq \cdots \sqsubseteq \mathcal{I}_n \sqsubseteq s \iff \mathcal{I}_1 \sqsubseteq \cdots \sqsubseteq \mathcal{I}_n \subseteq s_{\text{def}}$
**ALC with dynamic axioms**

- Consistency of DA with \( ALC \) concepts is undecidable.

- Reduction from \( ALC \) concept satisfiability w.r.t. RBoxes \( \mathcal{R} \).

\[
\mathcal{I} \models r_1 \circ \cdots \circ r_n \sqsubseteq s \; \text{def} \; r_1^\mathcal{I} \circ \cdots \circ r_n^\mathcal{I} \subseteq s^\mathcal{I}.
\]

- \( ALC \) concept satisfiability w.r.t. RBoxes is undecidable.

[Baldoni et al., TABLEAUX’98]
**ALC with dynamic axioms**

- Consistency of DA with ALC concepts is undecidable.

- Reduction from ALC concept satisfiability w.r.t. RBoxes $R$.

- $I \models r_1 \circ \cdots \circ r_n \sqsubseteq s \iff r_1^I \circ \cdots \circ r_n^I \subseteq s^I$.

- ALC concept satisfiability w.r.t. RBoxes is undecidable. [Baldoni et al., TABLEAUX’98]

- $C$ satisfiable w.r.t. $R$ iff DA below consistent ($V$’s are PDAs):

$$C(a) \cap ((\exists t_1.T) \sqsubseteq \bot) \cap \prod_{r \sqsubseteq s \in R} \neg V(r \sqsubseteq s, t_1, t_2)$$
Expressing $t^\mathcal{I} = \emptyset$ and $C^\mathcal{I} \neq \emptyset$

- Fresh role names $t$, $t_1$, $t_2$ interpreted by the empty relation.
- We use $t \equiv \emptyset$ for $(\exists t. \top) \sqsubseteq \bot$. 
Expressing \( t^\mathcal{I} = \emptyset \) and \( C^\mathcal{I} \neq \emptyset \)

- Fresh role names \( t, t_1, t_2 \) interpreted by the empty relation.

- We use \( t \equiv \emptyset \) for \( (\exists t. \top) \sqsubseteq \bot \).

- \( C \) built over the role names in \( \{s_1, \ldots, s_m\} \).

\[
\langle C \not\equiv t \perp \rangle \overset{\text{def}}{=} (\prod_{r \in \{s_1, \ldots, s_m\}} (r \equiv \emptyset)) \oslash (\top \sqsubseteq \exists t. C)
\]

- \( C^\mathcal{I} \) is non-empty iff \( \mathcal{I} \models \langle C \not\equiv t \perp \rangle \) holds.
Encoding \( r_1 \circ \cdots \circ r_n \sqsubseteq s \)

- When \( t_1^\mathcal{I} = \emptyset \),
  \( \mathcal{I} \models \neg V((r_1, \ldots, r_n), s, t_1, t_2) \) iff \( \mathcal{I} \models r_1 \circ \cdots \circ r_n \sqsubseteq s \).

\[ V((r_1, \ldots, r_n), s, t_1, t_2) \overset{\text{def}}{=} (\prod_{r \in \{r_1, \ldots, r_n, s\}} (r \equiv \emptyset) \odot (\langle \exists r_1 \ldots \exists r_n \exists t_1. \top \cap \neg \exists s. \exists t_1. \top \rangle \neq t_2 \bot)) \]
**Encoding** $r_1 \circ \cdots \circ r_n \sqsubseteq s$

- When $t_1^\mathcal{I} = \emptyset$,
  
  $\mathcal{I} \models \neg \mathcal{V}((r_1, \ldots, r_n), s, t_1, t_2)$ iff $\mathcal{I} \models r_1 \circ \cdots \circ r_n \sqsubseteq s$.

\[
\mathcal{V}((r_1, \ldots, r_n), s, t_1, t_2) \overset{\text{def}}{=} \left( \bigcap_{r \in \{r_1, \ldots, r_n, s\}} (r \equiv \emptyset) \right) \text{ and } \left( \langle \exists r_1 \cdots \exists r_n \exists t_1. \top \land \neg \exists s. \exists t_1. \top \rangle \neq t_2 \bot \right)
\]

- $C$ satisfiable wrt $\mathcal{R}$ iff DA below is consistent:

\[
C(a) \sqcap (t_1 \equiv \emptyset) \sqcap \prod_{\overline{r} \subseteq s} \neg \mathcal{V}((r_1, \ldots, r_n), s, t_1, t_2)
\]

- Consistency problem for $\mathcal{ALC}$ with dynamic axioms is undecidable. (a refinement leads to undec. for $\mathcal{EL}$ too)
Concluding remarks

- New framework for updates in DL interpretations.
- Complexity/decidability results for some specific DLs and composition.
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Alternative (non-aggregative) compositions.
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- New framework for updates in DL interpretations.
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- Subsumption problem with PDA for $\text{EL}$ (no $\neg$, $\sqcup$): $\mathcal{U} \models \mathcal{V}$?
  Study of subproblems/variants with student Pranay Agrawal.

- Alternative (non-aggregative) compositions.