Logical Aspects of Artificial Intelligence
Tableaux for DLs & Undecidability

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Plan of the lecture

- Tableaux calculus for checking $ALC$ concept satisfiability.
- Tableaux calculus for checking $ALC$ knowledge base consistency.
- Undecidability result with role axioms.
- Exercises session.
Recapitulation of the Previous Lecture(s)
$\mathcal{ALC}$ in a nutshell

$C ::= \top \mid \bot \mid A \mid \neg C \mid C \cap C \mid C \cup C \mid \exists r.C \mid \forall r.C$

- Interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$.

- $\text{TBox } \mathcal{T} = \{ C \sqsubseteq D, \ldots \}$.  

- $\text{ABox } \mathcal{A} = \{ a : C, (b, b') : r, \ldots \}$.  

- Knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. (a.k.a. ontology)

- Decision problems include concept satisfiability, knowledge base consistency, and other problems for classification.
$\top^{\mathcal{I}} \quad \text{def} \quad \Delta^{\mathcal{I}}$

$\bot^{\mathcal{I}} \quad \text{def} \quad \emptyset$

$(\neg C)^{\mathcal{I}} \quad \text{def} \quad \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$

$(C_1 \sqcup C_2)^{\mathcal{I}} \quad \text{def} \quad C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$

$(C_1 \sqcap C_2)^{\mathcal{I}} \quad \text{def} \quad C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$

$(\exists r. C)^{\mathcal{I}} \quad \text{def} \quad \{ a \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(a) \cap C^{\mathcal{I}} \neq \emptyset \}$

$(\forall r. C)^{\mathcal{I}} \quad \text{def} \quad \{ a \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(a) \subseteq C^{\mathcal{I}} \}$
A few properties about \textit{ALC}

\begin{itemize}
\item Concept satisfiability problem is PSPACE-complete.
\item Knowledge base consistency problem is EXPTIME-complete.
\item \textit{ALC} has many well-known fragments and extensions, some of them to deal with
  \begin{itemize}
  \item inverse roles,
  \item number restrictions,
  \item properties on the role interpretations,
  \item inclusions between the composition of roles,
  \item etc..
  \end{itemize}
\item Reduction of decision problems for DLs to first-order logic.
  \begin{footnotesize}(to modal logics too, but not presented herein)\end{footnotesize}
\item Filtration construction leading to an NEXPTIME upper bound for the \textit{ALC} knowledge base consistency problem.
\end{itemize}
Expansion rules for $\mathcal{ALC}$ ABox consistency

$\sqcap$-rule: If $a : C \sqcap D \in \mathcal{A}$ and $\{a : C, a : D\} \not\subseteq \mathcal{A}$ then

\[ \mathcal{A} \longrightarrow \mathcal{A} \cup \{a : C, a : D\} \]

$\sqcup$-rule: If $a : C \sqcup D \in \mathcal{A}$ and $\{a : C, a : D\} \cap \mathcal{A} = \emptyset$ then

\[ \mathcal{A} \longrightarrow \mathcal{A} \cup \{a : E\} \quad \text{for some } E \in \{C, D\} \]
Expansion rules for $\mathcal{ALC}$ ABox consistency

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\textbf{⊔-rule:} If $a : C \cup D \in \mathcal{A}$ and $\{a : C, a : D\} \cap \mathcal{A} = \emptyset$ then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{a : E\} \quad \text{for some } E \in \{C, D\}$$

\textbf{∃-rule:} If $a : \exists r. C \in \mathcal{A}$ and there is no $b$ such that $\{(a, b) : r, b : C\} \subseteq \mathcal{A}$ then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{(a, c) : r, c : C\} \quad \text{where } c \text{ is fresh}$$
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$\exists$-rule: If $a : \exists r.C \in \mathcal{A}$ and there is no $b$ such that $\{(a, b) : r, b : C\} \subseteq \mathcal{A}$ then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{(a, c) : r, c : C\} \quad \text{where } c \text{ is fresh}$$

$\forall$-rule: If $\{(a, b) : r, a : \forall r.C\} \subseteq \mathcal{A}$ and $b : C \notin \mathcal{A}$, then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{b : C\}$$
Today’s objectives

- Termination, soundness, completeness, blocking technique.

- Equivalences between
  - \((\mathcal{T}, \mathcal{A})\) is consistent (for \(\mathcal{ALC}\))
  - \(\mathcal{A} \rightarrow \mathcal{A}'\) for some complete and clash-free ABox \(\mathcal{A}'\) \((\rightarrow \text{ depends on } \mathcal{T})\)
  - \(\mathcal{A} \rightarrow \mathcal{A}'\) for some complete and clash-free ABox \(\mathcal{A}'\) derivable in at most \(f(\text{size}(\mathcal{T}, \mathcal{A}))\) steps.
Example

\[ A = \{(a, b) : s, (a, c) : r\}\cup \{a : A_1 \cap \exists s. A_5, a : \forall s. (\neg A_5 \cup \neg A_2), b : A_2, c : A_3 \cap \exists s. A_4\} \]
Example

\[ \mathcal{A} = \{(a, b) : s, (a, c) : r\} \cup \{a : A_1 \land \exists s.A_5, a : \forall s.(-A_5 \sqcup -A_2), b : A_2, c : A_3 \land \exists s.A_4\} \]

\[ \mathcal{A} \xrightarrow{*} \mathcal{A} \cup \{a : A_1, a : \exists s.A_5, a_{\text{new}} : A_5, (a, a_{\text{new}}) : s, b : -A_5 \sqcup -A_2, a_{\text{new}} : (-A_5 \sqcup -A_2), b : -A_5, a_{\text{new}} : -A_2, c : A_3, c : \exists s.A_4, c_{\text{new}} : A_4, (c, c_{\text{new}}) : s\} \]

(is it complete?)
Example

\[ \mathcal{A} = \{ (a, b) : s, (a, c) : r \} \cup \{ a : A_1 \cap \exists s. A_5, a : \forall s. (\neg A_5 \sqcup \neg A_2), b : A_2, c : A_3 \cap \exists s. A_4 \} \]

\[ \mathcal{A} \xrightarrow{*} \mathcal{A} \cup \{ a : A_1, a : \exists s. A_5, a_{\text{new}} : A_5, (a, a_{\text{new}}) : s, b : \neg A_5 \sqcup \neg A_2, a_{\text{new}} : (\neg A_5 \sqcup \neg A_2), b : \neg A_5, a_{\text{new}} : \neg A_2, c : A_3, c : \exists s. A_4, c_{\text{new}} : A_4, (c, c_{\text{new}}) : s \} \]

(is it complete?)
Terminology: root vs. tree individuals

- **Tree individuals** are generated by application of the \(\exists\)-rule.
- If \((a, b) : r\) is added by application of the \(\exists\)-rule, \(b\) is an \(r\)-successor of \(a\).
- Root individuals have no predecessors or ancestors.
Why “Tableaux”?

\[(a, b): s\]
\[(a, c): r\]

\[a: \forall s. \neg A_5 \lor \neg A_2\]
\[b: A_2\]
\[c: A_3 \land \exists s. A_4\]

\[a: \forall s\]
\[a: \exists s. A_5\]
\[a_{\text{new}}: A_5\]
\[(a, a_{\text{new}}): s\]
\[b: \forall A_5 \lor \forall A_2\]
\[a_{\text{new}}: \forall A_5 \lor \forall A_2\]

\[b: \forall A_2\]

\[a_{\text{new}}: \forall A_5\]
\[c: A_3\]
\[c: \exists s. A_4\]
\[c_{\text{new}}: A_4\]
\[(c, c_{\text{new}}): s\]
Termination

- The **∃-weight** of $C$ is the number of its subconcepts of the form $\exists r. D$.

$$w_\exists(C) \overset{\text{def}}{=} \text{card}(\{\exists r. D \mid \exists r. D \in \text{sub}(C)\})$$

⚠️ The definition assumes that $C$ is in NNF.

- $w_\exists(\mathcal{A}) \overset{\text{def}}{=} \sum_{a: C \in \mathcal{A}} w_\exists(C)$.
Termination

- The **∃-weight** of $C$ is the number of its subconcepts of the form $\exists r. D$.

\[
  w_\exists(C) \overset{\text{def}}{=} \text{card}(\{\exists r. D \mid \exists r. D \in \text{sub}(C)\})
\]

⚠️ The definition assumes that $C$ is in NNF.

- $w_\exists(A) \overset{\text{def}}{=} \sum_{a: C \in A} w_\exists(C)$.

- The **∀∃-depth** of $C$, written $d_{\forall \exists}(C)$, is the maximal number of imbrications of $\exists$ and $\forall$s in $C$.

  \[(\text{a.k.a. quantifier depth, modal depth})\]

- $d_{\forall \exists}(\exists r. \top \sqcup \forall r. \exists s. A) = 2$

- $d_{\forall \exists}(A) = \max\{d_{\forall \exists}(C) \mid a : C \in A\}$. 

Decorating individual names

Let $\mathcal{A}$ be an ABox with $W = w_\exists(\mathcal{A})$, $D = d_\forall(\mathcal{A})$ and $N$ is the number of distinct individual names in $\mathcal{A}$.

Let $\mathcal{A}^0$ be the variant of $\mathcal{A}$ where $a : C$ is replaced by $a^0 : C$. 
Decorating individual names

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Let $\mathcal{A}^0$ be the variant of $\mathcal{A}$ where $a : C$ is replaced by $a^0 : C$.

\(\square\)-rule: If $a^i : C \cap D \in \mathcal{A}$ and $\{a^i : C, a^i : D\} \not\subseteq \mathcal{A}$ then
\[
\mathcal{A} \to \mathcal{A} \cup \{a^i : C, a^i : D\}
\]

\(\Box\)-rule: If $a^i : C \cup D \in \mathcal{A}$ and $\{a^i : C, a^i : D\} \cap \mathcal{A} = \emptyset$ then
\[
\mathcal{A} \to \mathcal{A} \cup \{a^i : E\} \quad \text{for some } E \in \{C, D\}
\]

\(\exists\)-rule: If $a^i : \exists r.C \in \mathcal{A}$ and there is no $b^j$ such that $\{(a^i, b^j) : r, b^j : C\} \subseteq \mathcal{A}$ then
\[
\mathcal{A} \to \mathcal{A} \cup \{(a^i, c^{i+1}) : r, c^{i+1} : C\} \quad \text{where } c \text{ is fresh}
\]

\(\forall\)-rule: If $\{(a^i, b^j) : r, a^i : \forall r.C\} \subseteq \mathcal{A}$ and $b^j : C \not\in \mathcal{A}$, then
\[
\mathcal{A} \to \mathcal{A} \cup \{b^j : C\}
\]
Quantities about $\mathcal{A}^0 \rightarrow \mathcal{A}'$

- If $a^i : C \in \mathcal{A}'$, then $i + d_{\forall \exists}(C) \leq D$.

Trees from individual names labelled by zero have depth at most $D$. 
Quantities about $A^0 \rightarrow A'$

- If $a^i : C \in A'$, then $i + d_{\forall\exists}(C) \leq D$. Trees from individual names labelled by zero have depth at most $D$.

- $a^i : C \in A'$ implies $\text{card}((a^i, b^j) \mid (a^i, b^j) : r \in A') \leq N + W$ (necessarily either $i = j = 0$ or $j = i + 1$)

The maximum branching degree of nodes in the trees is at most $N + W$. (rough overapproximation)
Quantities about $\mathcal{A}^0 \rightarrow \mathcal{A}'$

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The maximum branching degree of nodes in the trees is at most $N + W$. (rough overapproximation)

- $a^i : C \in \mathcal{A}'$ implies $C \in \text{sub}(\mathcal{A})$. 

Quantities about $A^0 \to A'$

- If $a^i : C \in A'$, then $i + d_{\forall \exists}(C) \leq D$.
  Trees from individual names labelled by zero have depth at most $D$.

- $a^i : C \in A'$ implies $\text{card}((a^i, b^i) \mid (a^i, b^i) : r \in A') \leq N + W$
  (necessarily either $i = j = 0$ or $j = i + 1$)

  The maximum branching degree of nodes in the trees is at most $N + W$. (rough overapproximation)

- $a^i : C \in A'$ implies $C \in \text{sub}(A)$.

- The length of the derivation $A^0 \to A'$ is at most
  $$N \times (N + W)^{D+1} \times \text{card}(\text{sub}(A))$$
  (why?)
Main algorithm

- We shall show that $\mathcal{A}$ is consistent iff $\mathcal{A} \rightarrow^* \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$. 
Main algorithm

We shall show that $\mathcal{A}$ is consistent iff $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

Existence of $\mathcal{A}'$ amounts to explore a finite tree of bounded depth and bounded degree.
The auxiliary function \( \exp \)

- **Expansion function** \( \exp(\mathcal{A}, \mathcal{R}, X) \) taking as arguments
  - an ABox \( \mathcal{A} \),
  - an expansion rule \( \mathcal{R} \),
  - a subset \( X \) of \( \mathcal{A} \) (with one or two elements) allowing the application of \( \mathcal{R} \)

- ... and returning the set of ABoxes obtained from \( \mathcal{A} \) by applying the rule \( \mathcal{R} \) with main assertions in \( X \).

- \( \exp(\{a : E, a : C \sqcup D\}, \sqcup\text{-rule}, \{a : C \sqcup D\}) \) is equal to

  \[
  \{\{a : E, a : C \sqcup D, a : C\}, \{a : E, a : C \sqcup D, a : D\}\}
  \]
Algorithm for depth-first search

1: **procedure** EXPAND(\(\mathcal{A}\))
2: if \(\mathcal{A}\) has a clash then return \(\emptyset\)
3: end if
4: if \(\mathcal{A}\) is clash-free and complete then return \(\mathcal{A}\)
5: end if
6: for applicable \(R, X\) on \(\mathcal{A}\) and \(\mathcal{A}' \in \exp(\mathcal{A}, R, X)\) do
7: if EXPAND(\(\mathcal{A}'\)) \(\neq \emptyset\) then return EXPAND(\(\mathcal{A}'\))
8: end if
9: end for
10: return \(\emptyset\)
11: end procedure
Soundness

Let $\mathcal{A}$ be a finite ABox with at least one concept assertion, complete, clash-free and all the concepts in NNF. Then, $\mathcal{A}$ is consistent.
Soundness

- Let $\mathcal{A}$ be a finite ABox with at least one concept assertion, complete, clash-free and all the concepts in NNF. Then, $\mathcal{A}$ is consistent.

- For each individual name $a$ occurring in $\mathcal{A}$, we write $\text{con}_{\mathcal{A}}(a)$ to denote the set $\{C \mid a : C \in \mathcal{A}\}$. 
Soundness

Let $\mathcal{A}$ be a finite ABox with at least one concept assertion, complete, clash-free and all the concepts in NNF. Then, $\mathcal{A}$ is consistent.

For each individual name $a$ occurring in $\mathcal{A}$, we write $\text{con}_{\mathcal{A}}(a)$ to denote the set $\{ C \mid a : C \in \mathcal{A} \}$.

Let us define $\mathcal{I} \overset{\text{def}}{=} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ as follows.

\[
\begin{align*}
\Delta^\mathcal{I} & \overset{\text{def}}{=} \{ a \mid a : C \in \mathcal{A} \}, \\
a^\mathcal{I} & \overset{\text{def}}{=} a \text{ for all individual names } a \text{ in } \mathcal{A}, \\
A^\mathcal{I} & \overset{\text{def}}{=} \{ a \mid A \in \text{con}_{\mathcal{A}}(a) \} \text{ for all concept names } A \in \text{sub}(\mathcal{A}), \\
r^\mathcal{I} & \overset{\text{def}}{=} \{ (a, b) \mid (a, b) : r \in \mathcal{A} \}.
\end{align*}
\]

Let us show that for all $a : C \in \mathcal{A}$, we have $a^\mathcal{I} \in C^\mathcal{I}$. 

Proof by structural induction

▶ The base case with concept assertions $a : A$ is immediate by definition of $A^I$. 

▶ Case $a : C \sqcup D$ in the induction step.

▶ As $A$ is complete, $a : C \in A$ or $a : D \in A$.

▶ W.l.o.g., suppose $a : C \in A$. By (IH), $a I \in C^I$.

▶ By definition of $\cdot^I$, we conclude $a I \in (C \sqcup D)^I$.

▶ Case $a : \exists r . C$ in the induction step.

▶ As $A$ is complete, $\{ (a, b) : r, b : C \} \subseteq A$ for some $b$.

▶ By definition of $r^I$, $(a, b) \in r^I$.

▶ By (IH), $b I \in C^I$.

▶ By definition of $\cdot^I$, we conclude $a I \in (\exists r . C)^I$. 
Proof by structural induction

- The base case with concept assertions $a : A$ is immediate by definition of $A^\mathcal{I}$.

- The base case with concept assertions $a : \neg A$ is immediate by definition of $A^\mathcal{I}$ as $\mathcal{A}$ is clash-free.
Proof by structural induction

- The base case with concept assertions $a : A$ is immediate by definition of $A^\mathcal{I}$.

- The base case with concept assertions $a : \neg A$ is immediate by definition of $A^\mathcal{I}$ as $\mathcal{A}$ is clash-free.

- Case $a : C \sqcup D$ in the induction step.
  - As $\mathcal{A}$ is complete, $a : C \in \mathcal{A}$ or $a : D \in \mathcal{A}$.
  - W.l.o.g., suppose $a : C \in \mathcal{A}$. By (IH), $a^\mathcal{I} \in C^\mathcal{I}$.
  - By definition of $\cdot^\mathcal{I}$, we conclude $a^\mathcal{I} \in (C \sqcup D)^\mathcal{I}$. 
Proof by structural induction

- The base case with concept assertions $a : A$ is immediate by definition of $A^\mathcal{I}$.

- The base case with concept assertions $a : \neg A$ is immediate by definition of $A^\mathcal{I}$ as $A$ is clash-free.

- Case $a : C \sqcup D$ in the induction step.
  - As $A$ is complete, $a : C \in A$ or $a : D \in A$.
  - W.l.o.g., suppose $a : C \in A$. By (IH), $a^\mathcal{I} \in C^\mathcal{I}$.
  - By definition of $\cdot^\mathcal{I}$, we conclude $a^\mathcal{I} \in (C \sqcup D)^\mathcal{I}$.

- Case $a : \exists r. C$ in the induction step.
  - As $A$ is complete, $\{(a, b) : r, b : C\} \subseteq A$ for some $b$.
  - By definition of $r^\mathcal{I}$, $(a, b) \in r^\mathcal{I}$.
  - By (IH), $b^\mathcal{I} \in C^\mathcal{I}$.
  - By definition of $\cdot^\mathcal{I}$, we conclude $a^\mathcal{I} \in (\exists r. C)^\mathcal{I}$.
Concluding the soundness

- The cases in the induction step for \( \sqcap \)-concept assertions and \( \forall \)-concept assertions are similar.
Concluding the soundness

- The cases in the induction step for $\land$-concept assertions and $\forall$-concept assertions are similar.

- If $\text{expand}(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is consistent.
Concluding the soundness

- The cases in the induction step for $\cap$-concept assertions and $\forall$-concept assertions are similar.

- If $\text{expand}(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is consistent.

- Indeed, $\text{expand}(\mathcal{A}) \neq \emptyset$ if there is some $\mathcal{A}'$ with $\mathcal{A} \subseteq \mathcal{A}'$ such that $\mathcal{A}'$ is complete and clash-free.

- Consistency of $\mathcal{A}'$ leads to the consistency of $\mathcal{A}$.
Moving towards completeness

- If $\mathcal{A}$ is consistent, then $\mathcal{A} \rightarrow^* \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$. 
Moving towards completeness

- If $\mathcal{A}$ is consistent, then $\mathcal{A} \rightarrow \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

- Let $\mathcal{I} \overset{\text{def}}{=} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be such that $\mathcal{I} \models \mathcal{A}$.

- If $\mathcal{A}$ is complete, we are done.

- Otherwise, if $\mathcal{A}$ is not complete, we show that there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.
Moving towards completeness

- If $\mathcal{A}$ is consistent, then $\mathcal{A} \rightarrow \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

- Let $\mathcal{I} \overset{\text{def}}{=} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be such that $\mathcal{I} \models \mathcal{A}$.

- If $\mathcal{A}$ is complete, we are done.

- Otherwise, if $\mathcal{A}$ is not complete, we show that there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.

- As the length of a derivation from $\mathcal{A}$ is bounded by an exponential in the size of $\mathcal{A}$, there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow^* \mathcal{A}'$ and $\mathcal{A}'$ is complete, clash-free (and consistent).
It remains to prove that non-completeness implies the existence of one expansion preserving consistency.

Guidance from the interpretations to choose disjuncts and tree individuals.

If the $\sqcup$-rule is applicable on $a : C \sqcup D$, then there is $E \in \{ C, D \}$ such that $\mathcal{I} \models \mathcal{A} \cup \{ a : E \}$.

$\mathcal{A} \rightarrow \mathcal{A} \cup \{ a : E \}$ and $\mathcal{I} \models \mathcal{A} \cup \{ a : E \}$.
Single steps in the completeness proof (II)

- If the $\exists$-rule is applicable on $a : \exists r . C$, then we use the fact that $a^I \in (\exists r . C)^I$.

- There is $a \in \Delta^I$ such that $a \in C^I$ and $(a^I, a) \in r^I$.

- Let $I'$ be equal to $I$ except that $I'(c) = a$ for some fresh $c$.

- Then, $A \rightarrow A \cup \{c : C, (a, c) : r\}$ and

  $I' \models A \cup \{c : C, (a, c) : r\}$

  (freshness is required here)
Decision procedure of ABox consistency

- $\mathcal{A}$ is consistent iff $\mathcal{A} \overset{*}{\rightarrow} \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

- Derivations $\mathcal{A} \overset{*}{\rightarrow} \mathcal{A}'$ have length bounded by an exponential in $\text{size}(\mathcal{A})$.

- Existence of $\mathcal{A}'$ amounts to explore a tree of bounded depth and bounded degree.
Adding a TBox – First properties

- $\mathcal{I} \models C \subseteq D$ iff $\mathcal{I} \models \top \subseteq \neg C \sqcup D$.

- $\mathcal{I} \models C \equiv D$ iff $\mathcal{I} \models \top \subseteq (\neg C \sqcup D) \cap (\neg D \sqcup C)$.

- In the sequel, GCIs are of the form $\top \subseteq E$ with $E$ in NNF.
Adding a TBox – First properties

- $\models I \models C \subseteq D$ iff $\models I \models \top \subseteq \neg C \sqcup D$.

- $\models I \models C \equiv D$ iff $\models I \models \top \subseteq (\neg C \sqcup D) \cap (\neg D \sqcup C)$.

- In the sequel, GCIs are of the form $\top \subseteq E$ with $E$ in NNF.

\hline
\hline
\textbf{⊑-rule}: If $a$ in $\mathcal{A}$, $\top \subseteq D \in \mathcal{T}$ and $a : D \notin \mathcal{A}$, then

$\mathcal{A} \rightarrow \mathcal{A} \cup \{a : D\}$

\hline
\hline
Adding a TBox – First properties

\[ \mathcal{I} \models C \subseteq D \text{ iff } \mathcal{I} \models \top \subseteq \neg C \cup D. \]

\[ \mathcal{I} \models C \equiv D \text{ iff } \mathcal{I} \models \top \subseteq (\neg C \cup D) \cap (\neg D \cup C). \]

\[ \text{In the sequel, GCIs are of the form } \top \subseteq E \text{ with } E \text{ in NNF.} \]

\[ \sqsubseteq \text{-rule: If } a \text{ in } \mathcal{A}, \top \subseteq D \in \mathcal{T} \text{ and } a : D \notin \mathcal{A}, \text{ then} \]

\[ \mathcal{A} \longrightarrow \mathcal{A} \cup \{a : D\} \]

\[ \text{The termination argument for ABox consistency does not work anymore. \quad (Why?)} \]
Termination with the blocking technique

Given $A \xrightarrow{*} A'$, $a$ is an ancestor of $b$ in $A'$ iff

$$\{(a_1, a_2) : r_1, \ldots, (a_k, a_{k+1}) : r_k\} \subseteq A'$$

with $a_1 = a$, $a_{k+1} = b$ and $b$ is a tree individual.

⚠️ The notion of ancestor assumes that one can distinguish the root individuals (individual names from $A$) from the tree individuals (those introduced by applying the $\exists$-rule).
Termination with the blocking technique

- Given $\mathcal{A} \xrightarrow{\ast} \mathcal{A}'$, $a$ is an **ancestor** of $b$ in $\mathcal{A}'$ iff

$$\{(a_1, a_2) : r_1, \ldots, (a_k, a_{k+1}) : r_k\} \subseteq \mathcal{A}'$$

with $a_1 = a$, $a_{k+1} = b$ and $b$ is a tree individual.

⚠️ The notion of ancestor assumes that one can distinguish the **root individuals** (individual names from $\mathcal{A}$) from the **tree individuals** (those introduced by applying the $\exists$-rule).

- An individual name $b$ in $\mathcal{A}'$ is **blocked by** $a$ if
  - $a$ is an ancestor of $b$,
  - $\text{con}_{\mathcal{A}'}(b) \subseteq \text{con}_{\mathcal{A}'}(a)$.

- An individual name $b$ is **blocked in** $\mathcal{A}'$ iff it is blocked by some individual name or, one or more of its ancestors is blocked in $\mathcal{A}'$. 
$b$ blocked by $a$

$A_0 \rightarrow A_2 \rightarrow \ldots \rightarrow A_k = \{ \}

\{ D_0, \ldots, D_m \} \subseteq \{ c_0, \ldots, c_m \}$

root individuals

\( a: c_1, \ldots, a: c_n \)

\( b: D_{d_1}, \ldots, b: D_{d_m} \)

\( \ldots \)
Expansion rules with blocking

□-rule: If \( a : C \cap D \in A \), \( a \) is not blocked and \( \{ a : C, a : D \} \not\subseteq A \) then \( A \rightarrow A \cup \{ a : C, a : D \} \).

⊔-rule: If \( a : C \cup D \in A \), \( a \) is not blocked and \( \{ a : C, a : D \} \cap A = \emptyset \) then \( A \rightarrow A \cup \{ a : E \} \) for some \( E \in \{ C, D \} \).

∃-rule: If \( a : \exists r.C \in A \), \( a \) is not blocked and there is no \( b \) such that \( \{(a, b) : r, b : C\} \subseteq A \) then
\[
A \rightarrow A \cup \{(a, c) : r, c : C\} \quad \text{where } c \text{ is fresh}
\]

∀-rule: If \( \{(a, b) : r, a : \forall r.C\} \subseteq A \), \( a \) is not blocked and \( b : C \not\subseteq A \), then \( A \rightarrow A \cup \{b : C\} \).

⊑-rule: If \( a \) in \( A \), \( \top \subseteq D \in T \), \( a \) is not blocked and \( a : D \not\subseteq A \), then
\[
A \rightarrow A \cup \{a : D\}.
\]
Termination

- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCIs of the form $\top \sqsubseteq D$.

- $\mathbf{N}$: number of root individuals in $\mathcal{A}$, $\mathbf{M} = \text{card}(\text{sub}(\mathcal{K}))$, $\mathbf{W} = w_\exists(\mathcal{K})$. 
Termination

- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCIs of the form $\top \sqsubseteq D$.

- $\mathbf{N}$: number of root individuals in $\mathcal{A}$, $\mathbf{M} = \text{card}(\text{sub}(\mathcal{K}))$, $\mathbf{W} = w_{\exists}(\mathcal{K})$.

- $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ and $a : C \in \mathcal{A}'$ imply
  \[ \text{card}(\{(a, b) \mid (a, b) : r \in \mathcal{A}'\}) \leq \mathbf{N} + \mathbf{W} \]

- $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ and $a : C \in \mathcal{A}'$ imply $C \in \text{sub}(\mathcal{K})$.  
  ("subconcept property")
Termination

- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCIIs of the form $\top \subseteq D$.

- $N$: number of root individuals in $\mathcal{A}$, $M = \text{card}(\text{sub}(\mathcal{K}))$, $W = w_3(\mathcal{K})$.

- $\mathcal{A} \rightarrow^* \mathcal{A}'$ and $a : C \in \mathcal{A}'$ imply
  \[ \text{card}(\{(a, b) | (a, b) : r \in \mathcal{A}'\}) \leq N + W \]

- $\mathcal{A} \rightarrow^* \mathcal{A}'$ and $a : C \in \mathcal{A}'$ imply $C \in \text{sub}(\mathcal{K})$.  
  \[ \text{("subconcept property")} \]

- $\{(a_1, a_2) : r_1, \ldots, (a_k, a_{k+1}) : r_k\} \subseteq \mathcal{A}'$ and $a_2$ is a tree individual imply $k \leq 2^M$. 

Termination

- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCIs of the form $\top \subseteq D$.

- $N$: number of root individuals in $\mathcal{A}$, $M = \text{card}(\text{sub}(\mathcal{K}))$, $W = w_\exists(\mathcal{K})$.

- $\mathcal{A} \xrightarrow{\ast} \mathcal{A}'$ and $a : C \in \mathcal{A}'$ imply
  \[
  \text{card}(\{(a, b) \mid (a, b) : r \in \mathcal{A}'\}) \leq N + W
  \]

- $\mathcal{A} \xrightarrow{\ast} \mathcal{A}'$ and $a : C \in \mathcal{A}'$ imply $C \in \text{sub}(\mathcal{K})$.
  ("subconcept property")

- $\{(a_1, a_2) : r_1, \ldots, (a_k, a_{k+1}) : r_k\} \subseteq \mathcal{A}'$ and $a_2$ is a tree individual imply $k \leq 2^M$.

- The length of the derivation $\mathcal{A} \xrightarrow{\ast} \mathcal{A}'$ is at most
  \[
  N \times (N + W)^{(2^M + 1)} \times M
  \]
Soundness

- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCIs of the form $\top \sqsubseteq D$.

- $\mathcal{A} \overset{*}{\to} \mathcal{A}'$ with $\mathcal{A}'$ complete and clash-free.

- We construct $\mathcal{A}''$ as the ABox made of the following assertions

\[
\{a : C \mid a : C \in \mathcal{A}', \ a \text{ is not blocked}\} \cup
\{(a, b) : r \mid (a, b) : r \in \mathcal{A}', \ b \text{ is not blocked}\} \cup
\{(a, b') : r \mid (a, b) : r \in \mathcal{A}', \ a \text{ is not blocked and } b \text{ is blocked by } b'\}
\]
Construction of $A''$

Diagram showing the construction process with nodes labeled $a_0$, $b_0$, and $a_{10}$, among others, with arrows indicating relationships and blocking conditions.
Properties of $\mathcal{A}''$

- $\mathcal{A} \subseteq \mathcal{A}''$ as root individuals cannot be blocked and $\mathcal{A} \subseteq \mathcal{A}'$. 
Properties of $A''$

- $A \subseteq A''$ as root individuals cannot be blocked and $A \subseteq A'$.
- None of the individual names occurring in $A''$ is blocked.
Properties of $\mathcal{A}''$

- $\mathcal{A} \subseteq \mathcal{A}''$ as root individuals cannot be blocked and $\mathcal{A} \subseteq \mathcal{A}'$.
- None of the individual names occurring in $\mathcal{A}''$ is blocked.
- For all $a$ in $\mathcal{A}''$, we have $\text{con}_{\mathcal{A}''}(a) = \text{con}_{\mathcal{A}'}(a)$.

(Left as an exercise)
Properties of $\mathcal{A}''$

- $\mathcal{A} \subseteq \mathcal{A}''$ as root individuals cannot be blocked and $\mathcal{A} \subseteq \mathcal{A}'$.
- None of the individual names occurring in $\mathcal{A}''$ is blocked.
- For all $a$ in $\mathcal{A}''$, we have $\text{con}_{\mathcal{A}''}(a) = \text{con}_{\mathcal{A}'}(a)$.
  
  \begin{center}
  \textit{(left as an exercise)}
  \end{center}

- $\mathcal{A}''$ is complete and clash-free.
Proof: $\mathcal{A}''$ is complete

\[(\star) \; \text{con}_{\mathcal{A}''}(a) = \text{con}_{\mathcal{A}'}(a) \quad \text{for all} \; a \in \mathcal{A}''\]
Proof: $\mathcal{A}''$ is complete

\[(\star) \text{ } \text{con}_{\mathcal{A}''}(a) = \text{con}_{\mathcal{A}'}(a) \text{ for all } a \in \mathcal{A}''\]

- Suppose $a : C \cap D \in \mathcal{A}''$.
  - By $(\star)$, $a : C \cap D \in \mathcal{A}'$.
  - As $\mathcal{A}'$ is complete, $\{a : C, a : D\} \subseteq \mathcal{A}'$.
  - By $(\star)$, $\{a : C, a : D\} \subseteq \mathcal{A}''$. 
Proof: $\mathcal{A}''$ is complete

$(\star) \ con_{\mathcal{A}''}(a) = con_{\mathcal{A}'}(a)$ for all $a \in \mathcal{A}''$

- Suppose $a : C \cap D \in \mathcal{A}''$.
  
  By $(\star)$, $a : C \cap D \in \mathcal{A}'$.
  
  As $\mathcal{A}'$ is complete, $\{a : C, a : D\} \subseteq \mathcal{A}'$.
  
  By $(\star)$, $\{a : C, a : D\} \subseteq \mathcal{A}''$.

- Suppose that $a : C \in \mathcal{A}''$ and $\top \subseteq D \in \mathcal{T}$.
  
  By $(\star)$, $a : C \in \mathcal{A}'$.
  
  As $\mathcal{A}'$ is complete, $a : D \in \mathcal{A}'$.
  
  By $(\star)$, $a : D \in \mathcal{A}''$. 
Case with the $\exists$-rule

- Suppose that $a : \exists r. C \in A''$.
  By $(\star)$, $a : \exists r. C \in A'$ and $a$ not blocked.
  By completeness of $A'$, there is $b$ such that
  $\{(a, b) : r, b : C\} \subseteq A'$.

- If $b$ is not blocked, then $\{(a, b) : r, b : C\} \subseteq A''$. 
Case with the $\exists$-rule

- Suppose that $a : \exists r. C \in A''$.
  By (⋆), $a : \exists r. C \in A'$ and $a$ not blocked.
  By completeness of $A'$, there is $b$ such that
  $\{(a, b) : r, b : C\} \subseteq A'$.

- If $b$ is not blocked, then $\{(a, b) : r, b : C\} \subseteq A''$.

- As $a$ is not blocked, if $b$ is blocked, then $b$ is blocked by $b'$ in $A'$ and $b'$ is not blocked.
Case with the $\exists$-rule

- Suppose that $a : \exists r.C \in A''$.
  By $(\star)$, $a : \exists r.C \in A'$ and $a$ not blocked.
  By completeness of $A'$, there is $b$ such that
  $\{(a, b) : r, b : C\} \subseteq A'$.

- If $b$ is not blocked, then $\{(a, b) : r, b : C\} \subseteq A''$.

- As $a$ is not blocked, if $b$ is blocked, then $b$ is blocked by $b'$ in $A'$ and $b'$ is not blocked.

- By definition of $A''$, $(a, b') : r \in A''$. 
Case with the \( \exists \)-rule

> Suppose that \( a : \exists r. C \in A'' \).

By (\( \star \)), \( a : \exists r. C \in A' \) and \( a \) not blocked.
By completeness of \( A' \), there is \( b \) such that
\[
\{(a, b) : r, b : C\} \subseteq A'.
\]

> If \( b \) is not blocked, then \( \{(a, b) : r, b : C\} \subseteq A'' \).

> As \( a \) is not blocked, if \( b \) is blocked, then \( b \) is blocked by \( b' \)
in \( A' \) and \( b' \) is not blocked.

> By definition of \( A'' \), \( (a, b') : r \in A'' \).

> \( \text{con}_{A'}(b) \subseteq \text{con}_{A'}(b') \) (blocking). By (\( \star \)),
\[
C \in \text{con}_{A'}(b) \subseteq \text{con}_{A'}(b') = \text{con}_{A''}(b')
\]
So, \( b' : C \in A'' \).
Case with the $\exists$-rule

- Suppose that $a : \exists r. C \in A''$.
  
  By $(\star)$, $a : \exists r. C \in A'$ and $a$ not blocked.
  
  By completeness of $A'$, there is $b$ such that
  
  $(a, b) \cup r, b : C) \subseteq A'$.

- If $b$ is not blocked, then $(a, b) \cup r, b : C) \subseteq A''$.

- As $a$ is not blocked, if $b$ is blocked, then $b$ is blocked by $b'$
  in $A'$ and $b'$ is not blocked.

- By definition of $A''$, $(a, b') : r \in A''$.

- $\text{con}_{A'}(b) \subseteq \text{con}_{A'}(b')$ (blocking). By $(\star)$,
  
  $C \in \text{con}_{A'}(b) \subseteq \text{con}_{A'}(b') = \text{con}_{A''}(b')$

  So, $b' : C \in A''$.

- Case with the $\forall$-rule left as an exercise.
More about the soundness proof

- $A \xrightarrow{*} A'$ with $A'$ complete and clash-free and $A''$ computed as above.

- Let us define $\mathcal{I} \overset{\text{def}}{=} (\Delta \mathcal{I}, a \mathcal{I})$ from $A''$ as follows.
  - $\Delta \mathcal{I} \overset{\text{def}}{=} \{ a \mid a : C \in A'' \}$.
  - $a \mathcal{I} \overset{\text{def}}{=} a$ for all individual names $a$ in $A''$.
  - $A \mathcal{I} \overset{\text{def}}{=} \{ a \mid A \in \text{con}_{A''}(a) \}$ for all concept names $A \in \text{sub}(A'')$.
  - $r \mathcal{I} \overset{\text{def}}{=} \{ (a, b) \mid (a, b) : r \in A'' \}$.

(Previous construction with $A''$ instead)

- One can show that for all $a : C \in A''$, we have $a \mathcal{I} \in C \mathcal{I}$.
  (left as an exercise.)
The final step about soundness

► It remains to check that $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$. 

► One can show that for all $a : C \in \mathcal{A}^{''}$, we have $a \mathcal{I} \in C$.

► Consequently, $\mathcal{I} \models \mathcal{A}$ as $\mathcal{A} \subseteq \mathcal{A}^{''}$.

► Moreover, $\mathcal{I} \models \mathcal{T}$ for all $\mathcal{T} \subseteq \mathcal{C}$.

► $a \in \Delta \mathcal{I} \rightarrow a : C \in \mathcal{A}^{''}$ (see above)
The final step about soundness

- It remains to check that $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$.

- One can show that for all $a : C \in \mathcal{A}''$, we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

- Consequently, $\mathcal{I} \models \mathcal{A}$ as $\mathcal{A} \subseteq \mathcal{A}''$. 
The final step about soundness

- It remains to check that $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$.

- One can show that for all $a : C \in \mathcal{A}''$, we have $a^\mathcal{I} \in C^\mathcal{I}$.

- Consequently, $\mathcal{I} \models \mathcal{A}$ as $\mathcal{A} \subseteq \mathcal{A}''$.

- Moreover, $\mathcal{I} \models \top \subseteq C$ for all $\top \subseteq C \in \mathcal{T}$.

- $a \in \Delta^\mathcal{I}$
  $\rightarrow a : C \in \mathcal{A}''$ ($\mathcal{A}''$ is complete)
  $\rightarrow a \in C^\mathcal{I}$ (see above)
Completeness (bis)

If $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is consistent, then $\mathcal{A} \rightarrow \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$. 

Let $I \overset{\text{def}}{=} (\Delta_I, \cdot_I)$ be such that $I \models (\mathcal{T}, \mathcal{A})$.

If $\mathcal{A}$ is complete, we are done.

Otherwise ($\mathcal{A}$ is not complete), we show there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.

As the length of a derivation from $\mathcal{A}$ is bounded by a double-exponential in the size of $\mathcal{A}$, there is $\mathcal{A}'$ such that $\mathcal{A}^* \rightarrow \mathcal{A}'$ and $\mathcal{A}'$ is complete, clash-free (and consistent).

One can prove that non-completeness implies the existence of one expansion preserving consistency.
Completeness (bis)

- If $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is consistent, then $\mathcal{A} \rightarrow^\ast \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

- Let $\mathcal{I} \overset{\text{def}}{=} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be such that $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$.

- If $\mathcal{A}$ is complete, we are done.

- Otherwise ($\mathcal{A}$ is not complete), we show there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.
Completeness (bis)

- If $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is consistent, then $\mathcal{A} \rightarrow^* \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

- Let $\mathcal{I} \overset{\text{def}}{=} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be such that $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$.

- If $\mathcal{A}$ is complete, we are done.

- Otherwise ($\mathcal{A}$ is not complete), we show there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.

- As the length of a derivation from $\mathcal{A}$ is bounded by a double-exponential in the size of $\mathcal{A}$, there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow^* \mathcal{A}'$ and $\mathcal{A}'$ is complete, clash-free (and consistent).
Completeness (bis)

- If $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is consistent, then $\mathcal{A} \xrightarrow{\ast} \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

- Let $\mathcal{I} \overset{\text{def}}{=} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be such that $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$.

- If $\mathcal{A}$ is complete, we are done.

- Otherwise ($\mathcal{A}$ is not complete), we show there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.

- As the length of a derivation from $\mathcal{A}$ is bounded by a double-exponential in the size of $\mathcal{A}$, there is $\mathcal{A}'$ such that $\mathcal{A} \xrightarrow{\ast} \mathcal{A}'$ and $\mathcal{A}'$ is complete, clash-free (and consistent).

- One can prove that non-completeness implies the existence of one expansion preserving consistency.
Complexity issues

- $\mathcal{ALC}$ concept satisfiability in $\text{PSPACE}$, knowledge base consistency in $\text{EXPTIME}$.

- The algorithm for ABox consistency runs in exponential space:
  - Because of the nondeterministic $\sqcup$-rule, exponentially many ABoxes may be generated.
  - Complete ABoxes may be exponentially large.

- $\text{PSPACE}$ bound for ABox consistency can be regained by exploring the tree-like interpretations in a depth-first manner having only one path at a time.
Recapitulation: Tableaux for $\mathcal{ALC}$ knowledge base consistency

- Tableaux-based algorithm to decide $\mathcal{ALC}$ knowledge base consistency.

- All other standard decision problems can be handled too.

- Termination is guaranteed thanks to the blocking technique.

- In the worst-case, exponential space is used but optimisations exist to meet the optimal upper bound $\text{EXPTime}$.

- Tableaux can be extended to richer variants of $\mathcal{ALC}$ (with inverses, nominals, number restrictions, etc.)
Undecidability with Role Inclusion Axioms
DLs: a playground to study extensions and fragments

- Many developments to extend $\mathcal{ALC}$ while preserving the decidability status / complexity of the main decision problems.

- Many developments to study fragments of $\mathcal{ALC}$ (or variants) to identify tractable fragments.

- It is also important to identify undecidable extensions.
Tiling system

- **Tiling system**: \((T, H, V, t_0)\) where
  - \(T\) is a finite set of **tile types** and \(t_0 \in T\),
  - \(H, V \subseteq T \times T\) are two relations referred to as the **horizontal**, resp. **vertical matching relation**.
Tiling system

- **Tiling system**: \(( T, H, V, t_0 )\) where
  - \( T \) is a finite set of **tile types** and \( t_0 \in T \),
  - \( H, V \subseteq T \times T \) are two relations referred to as the horizontal, resp. vertical matching relation.

- **A set of tile types** (a.k.a. **tiles**)

  \[
  t_1 = \begin{array}{c}
  2 \\
  1 \\
  0 \\
  2 \\
  \end{array}
  \quad t_2 = \begin{array}{c}
  1 \\
  2 \\
  1 \\
  2 \\
  \end{array}
  \quad t_3 = \begin{array}{c}
  0 \\
  1 \\
  2 \\
  \end{array}
  \quad t_4 = \begin{array}{c}
  2 \\
  2 \\
  2 \\
  \end{array}
  \]

- **...with its matching relations**
  - \( H = \{(t_1, t_3), (t_1, t_4), (t_2, t_1), (t_3, t_2), (t_4, t_1)\} \)
  - \( V = \{(t_1, t_2), (t_1, t_4), (t_2, t_3), (t_4, t_1), (t_4, t_2)\} \)
A tiling for the \(([0, 3] \times [0, 2])\)-arena

\[
\text{tiling } \tau : [0, 3] \times [0, 2] \rightarrow T
\]
An undecidable tiling problem

▶ The ($\infty \times \infty$)-tiling problem.

Input: A tiling system $(T, H, V, t_0)$.

Question: Is there a tiling $\tau : \mathbb{N} \times \mathbb{N} \rightarrow T$ such that for all $i, j \in \mathbb{N}$,

(hori) if $\tau(i, j) = t$ and $\tau(i + 1, j) = t'$, then $(t, t') \in H$,

(verti) if $\tau(i, j) = t$ and $\tau(i, j + 1) = t'$, then $(t, t') \in V$

▶ The ($\infty \times \infty$)-tiling problem is undecidable.
Complexity about $\mathcal{ALC}$ problems

▶ Concept satisfiability problem is PSPACE-complete.

▶ PSPACE-hardness by reduction from $(n \times n)$-tiling game problem.
Complexity about $\mathcal{ALC}$ problems

- Concept satisfiability problem is $\text{PSpace}$-complete.

- $\text{PSpace}$-hardness by reduction from $(n \times n)$-tiling game problem.

- $\text{ExpTime}$-complete knowledge base consistency problem. $\text{ExpTime}$-hardness from $(n \times \infty)$-tiling game problem.
A standard undecidability result

- $\mathcal{ALC}$ + role axioms $r \circ s \sqsubseteq q$ and $q \sqsubseteq r \circ s$ has undecidable knowledge base consistency problem.

  (actually CBox consistency is undecidable)

- Reduction from $(\infty \times \infty)$-tiling problem.
A standard undecidability result

- $\mathcal{ALC} + \text{role axioms } r \circ s \sqsubseteq q \text{ and } q \sqsubseteq r \circ s \text{ has undecidable knowledge base consistency problem.}$

  (actually CBox consistency is undecidable)

- Reduction from $(\infty \times \infty)$-tiling problem.

- $\mathcal{ALC} + \text{local role value maps } r \circ s \sqsubseteq q \text{ and } q \sqsubseteq r \circ s \text{ has undecidable concept satisfiability problem.}$

  (not presented herein)
An undecidable extension of \( \text{ALC} \)

Let us consider the extension of \( \text{ALC} \) in which we allow role axioms of the form

\[ r \circ s \sqsubseteq q \quad q \sqsubseteq r \circ s, \]

\[ \mathcal{I} \models r \circ s \sqsubseteq q \quad \text{def} \quad r^{\mathcal{I}} \circ s^{\mathcal{I}} \sqsubseteq q^{\mathcal{I}} \]

\[ \mathcal{I} \models q \sqsubseteq r \circ s \quad \text{def} \quad q^{\mathcal{I}} \subseteq r^{\mathcal{I}} \circ s^{\mathcal{I}} \]
An undecidable extension of ALC

Let us consider the extension of ALC in which we allow role axioms of the form

\[ r \circ s \sqsubseteq q \quad q \sqsubseteq r \circ s, \]

\[ \mathcal{I} \models r \circ s \sqsubseteq q \iff r^\mathcal{I} \circ s^\mathcal{I} \subseteq q^\mathcal{I} \quad \mathcal{I} \models q \sqsubseteq r \circ s \iff q^\mathcal{I} \subseteq r^\mathcal{I} \circ s^\mathcal{I} \]

Role axioms \( r \circ s \equiv s \circ r \) can be encoded by introducing a fresh role name \( q \):

\[ \{ r \circ s \sqsubseteq q, q \sqsubseteq r \circ s, s \circ r \sqsubseteq q, q \sqsubseteq s \circ r \} \]

(correctness left as an exercise)
An undecidable extension of \( \text{ALC} \)

- Let us consider the extension of \( \text{ALC} \) in which we allow role axioms of the form

\[
\begin{align*}
    r \circ s & \subseteq q \quad q \subseteq r \circ s, \\
    \mathcal{I} \models r \circ s \subseteq q & \iff r^\mathcal{I} \circ s^\mathcal{I} \subseteq q^\mathcal{I} \quad \mathcal{I} \models q \subseteq r \circ s & \iff q^\mathcal{I} \subseteq r^\mathcal{I} \circ s^\mathcal{I}
\end{align*}
\]

- Role axioms \( r \circ s \equiv s \circ r \) can be encoded by introducing a fresh role name \( q \):

\[
\{ r \circ s \subseteq q, q \subseteq r \circ s, s \circ r \subseteq q, q \subseteq s \circ r \} 
\]

(correctness left as an exercise)

- Reduction from the \( (\infty \times \infty) \)-tiling problem to knowledge base consistency for such an \( \text{ALC} \) extension.
The reduction

Given a tiling system $\mathcal{T} = (T, H, V, t_0)$, we introduce two role names $r_x$ and $r_y$.

We build a TBox $\mathcal{T}_\mathcal{T}$ such that $\mathcal{T}$ is a positive instance of the $(\infty \times \infty)$-tiling problem iff $\mathcal{T}_\mathcal{T}$ is consistent.
The reduction

- Given a tiling system $\mathcal{T} = (T, H, V, t_0)$, we introduce two role names $r_x$ and $r_y$.

- We build a TBox $\mathcal{T}_T$ such that $\mathcal{T}$ is a positive instance of the $(\infty \times \infty)$-tiling problem iff $\mathcal{T}_T$ is consistent.

- Every individual has a horizontal and a vertical successor:
  
  $T \subseteq \exists r_x. T \land \exists r_y. T$
The reduction

Given a tiling system \( T = (T, H, V, t_0) \), we introduce two role names \( r_x \) and \( r_y \).

We build a TBox \( \mathcal{T}_T \) such that \( T \) is a positive instance of the \((\infty \times \infty)\)-tiling problem iff \( \mathcal{T}_T \) is consistent.

Every individual has a horizontal and a vertical successor:
\[
\top \subseteq \exists r_x. T \sqcap \exists r_y. T
\]

Every individual belongs to a unique tile type.
\[
\top \subseteq \bigcup_{t \in T} (t \sqcap \bigsqcap_{t' \neq t} \neg t')
\]
The reduction

Given a tiling system $\mathcal{T} = (T, H, V, t_0)$, we introduce two role names $r_x$ and $r_y$.

We build a TBox $\mathcal{T}_T$ such that $\mathcal{T}$ is a positive instance of the $(\infty \times \infty)$-tiling problem iff $\mathcal{T}_T$ is consistent.

Every individual has a horizontal and a vertical successor:

$$\top \sqsubseteq \exists r_x. T \cap \exists r_y. T$$

Every individual belongs to a unique tile type.

$$\top \sqsubseteq \bigsqcup_{t \in T} (t \cap \bigsqcap_{t' \neq t} \neg t')$$

Tile types of adjacent individuals satisfy the matching relations:

$$\top \sqsubseteq \bigsqcup_{(t, t') \in H} (t \cap \forall r_x. t') \cap \bigsqcup_{(t, t') \in V} (t \cap \forall r_y. t')$$
The properties

- The set of $r_x r_y$-successors is equal to the set of $r_y r_x$-successors.

\[ r_x \circ r_y \equiv r_y \circ r_x \]
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- The set of $r_xr_y$-successors is equal to the set of $r_yr_x$-successors.
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- $\mathcal{T}_T$ is made of the above GCIs and role axioms.
The properties

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  \[ r_x \circ r_y \equiv r_y \circ r_x \]

- $\mathcal{T}_T$ is made of the above GCIs and role axioms.

- $\mathcal{T}_T$ is consistent iff $\mathcal{T}$ is a positive instance.

- TBox consistency problem for $\mathcal{ALC}$ augmented with role axioms of the form $r \circ s \sqsubseteq q$ and $q \sqsubseteq r \circ s$ is undecidable.
Correctness proof (or how to extract a grid)

Let $\mathcal{I}$ be an interpretation satisfying the TBox $\mathcal{T}_T$.

We define a map $f : \mathbb{N} \times \mathbb{N} \rightarrow \Delta^\mathcal{I}$ such that for all $i, j$

- $(f(i, j), f(i + 1, j)) \in r_x^\mathcal{I}$
- $(f(i, j), f(i, j + 1)) \in r_y^\mathcal{I}$

Unicity of $t$ guaranteed by $I |= \top \sqsubseteq F_t \in T (t \sqcap dt \neq t \neg t \neg t')$.

Afterwards, easy to check $\tau$ is a tiling as $I |= \mathcal{T}_T$.
Correctness proof (or how to extract a grid)

- Let $\mathcal{I}$ be an interpretation satisfying the TBox $\mathcal{T}_{\mathcal{I}}$.

- We define a map $f : \mathbb{N} \times \mathbb{N} \rightarrow \Delta^\mathcal{I}$ such that for all $i, j$
  - $(f(i, j), f(i + 1, j)) \in r^\mathcal{I}_x$
  - $(f(i, j), f(i, j + 1)) \in r^\mathcal{I}_y$

- Then, we define $\tau : \mathbb{N} \times \mathbb{N} \rightarrow T$ from $f$ as follows:
  \[
  \tau(i, j) \overset{\text{def}}{=} \text{unique } t \text{ such that } f(i, j) \in t^\mathcal{I}
  \]
Correctness proof (or how to extract a grid)

▶ Let $\mathcal{I}$ be an interpretation satisfying the TBox $\mathcal{T}_T$.

▶ We define a map $\emptyset: \mathbb{N} \times \mathbb{N} \rightarrow \Delta^\mathcal{I}$ such that for all $i, j$
  
  ▶ $(\emptyset(i, j), \emptyset(i + 1, j)) \in r^\mathcal{I}_x$

  ▶ $(\emptyset(i, j), \emptyset(i, j + 1)) \in r^\mathcal{I}_y$

▶ Then, we define $\tau: \mathbb{N} \times \mathbb{N} \rightarrow T$ from $\emptyset$ as follows:

$$\tau(i, j) \overset{\text{def}}{=} \text{unique } t \text{ such that } \emptyset(i, j) \in t^\mathcal{I}$$

▶ Unicity of $t$ guaranteed by $\mathcal{I} \models T \subseteq \bigcup_{t \in T} (t \cap \bigcap_{t' \neq t} t')$.

▶ Afterwards, easy to check $\tau$ is a tiling as $\mathcal{I} \models \mathcal{T}_T$. 
How to define $f$ while maintaining properties?

- $f(0,0)$ is chosen arbitrarily in $\Delta^I$ (non-empty).
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- $f(0, 0)$ is chosen arbitrarily in $\Delta^I$ (non-empty).

- As $\mathcal{I} \models \top \subseteq \exists r_x. \top$, when $f(i, i)$ is already defined, pick $a \in \Delta^I$ such that
  - $(f(i, i), a) \in r_x^I$,
  - $f(i + 1, i) \overset{\text{def}}{=} a$
How to define $f$ while maintaining properties?

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- As $\mathcal{I} \models \top \sqsubseteq \exists r_y. \top$, when $f(i + 1, i)$ is already defined, pick $b \in \Delta^I$ such that
  - $(f(i + 1, i), b) \in r_y^I$,
  - $f(i + 1, i + 1) \overset{\text{def}}{=} b$
More cases for defining $f$

- As $\mathcal{I} \models r_x \circ r_y \equiv r_y \circ r_x$, when

$$f(i,j), f(i+1,j), f(i+1,j+1)$$

are defined and $f(i,j+1)$ undefined, pick $a \in \Delta^\mathcal{I}$ such that

- $(f(i,j), a) \in r_y^\mathcal{I}$
- $(a, f(i+1,j+1)) \in r_x^\mathcal{I}$
- $f(i,j+1) \overset{\text{def}}{=} a$
More cases for defining \( f \)

- As \( \mathcal{I} \models r_x \circ r_y \equiv r_y \circ r_x \), when
  \[
  f(i, j), f(i + 1, j), f(i + 1, j + 1)
  \]
  are defined and \( f(i, j + 1) \) undefined, pick \( \alpha \in \Delta^\mathcal{I} \) such that
  \[
  (f(i, j), \alpha) \in r_y^\mathcal{I}, \quad \alpha \xrightarrow{r_x} f(i + 1, j + 1)
  \]
  \[
  (\alpha, f(i + 1, j + 1)) \in r_x^\mathcal{I}, \quad \xrightarrow{r_y} f(i, j)
  \]
  \[
  f(i, j + 1) \overset{\text{def}}{=} \alpha
  \]

- When \( f(i, j), f(i, j + 1), f(i + 1, j + 1) \) are defined and \( f(i + 1, j) \) undefined, pick \( \alpha \in \Delta^\mathcal{I} \) such that
  \[
  (f(i, j), \alpha) \in r_x^\mathcal{I}, \quad \xrightarrow{r_y} f(i + 1, j + 1)
  \]
  \[
  (\alpha, f(i + 1, j + 1)) \in r_y^\mathcal{I}, \quad \xrightarrow{r_x} f(i, j)
  \]
  \[
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More cases for defining $f$

- As $\mathcal{I} \models r_x \circ r_y \equiv r_y \circ r_x$, when

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- $(f(i, j), a) \in r_y^{\mathcal{I}}$
- $(a, f(i + 1, j + 1)) \in r_x^{\mathcal{I}}$
- $f(i, j + 1) \overset{\text{def}}{=} a$

- When $f(i, j), f(i, j + 1), f(i + 1, j + 1)$ are defined and $f(i + 1, j)$ undefined, pick $a \in \Delta^{\mathcal{I}}$ such that

  - $(f(i, j), a) \in r_x^{\mathcal{I}}$
  - $(a, f(i + 1, j + 1)) \in r_y^{\mathcal{I}}$
  - $f(i + 1, j) \overset{\text{def}}{=} a$

- With these four cases, how to build $f$ on $\mathbb{N} \times \mathbb{N}$?
Construction of the map $f$: a bit of organisation

Ordering to define $f$:
$f : \mathbb{N} \times \mathbb{N} \to \Delta$
The other direction (easy)

Let $\mathcal{T} = (T, H, V, t_0)$ be a tiling system and $\tau : \mathbb{N} \times \mathbb{N} \rightarrow T$ be a tiling.

Interpretation $\mathcal{I} \overset{\text{def}}{=} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

- $\Delta^\mathcal{I} \overset{\text{def}}{=} \mathbb{N} \times \mathbb{N}$
- $r_x^\mathcal{I} \overset{\text{def}}{=} \{(((i, j), (i + 1, j)) \mid i, j \in \mathbb{N}\}$
- $r_y^\mathcal{I} \overset{\text{def}}{=} \{(((i, j), (i, j + 1)) \mid i, j \in \mathbb{N}\}$
- $t^\mathcal{I} \overset{\text{def}}{=} \{(n, m) \mid \tau(n, m) = t\}$ for every $t \in T$

It is easy to check $\mathcal{I}$ satisfies all the GCIs and the role axioms from $\mathcal{T}_\mathcal{T}$. 
Interpretation $\mathcal{I}$

Tiling $\tau$

\[
\begin{array}{cccccc}
0 & 2 & 1 & 2 & 2 & 0 \\
1 & 2 & 1 & 0 & 2 & 1 \\
2 & 1 & 1 & 0 & 2 & 2 \\
2 & 2 & 1 & 1 & 0 & 2 \\
0 & 1 & 1 & 0 & 2 & 2 \\
0 & 2 & 2 & 1 & 2 & 2 \\
\end{array}
\]

\[\ldots\]
Conclusion

- Today lecture: tableaux for DLs.
  - Rules for checking concept satisfiability.
  - Rules for checking knowledge base consistency.
  - Termination, soundness, completeness.
  - Undecidability result with role axioms.

- Next week lecture: reasoning about multiagent systems with ATL.
Other topics related to DLs

- More tableaux-style systems and complexity results for $\mathcal{ALC}$ extensions (e.g. for $\mathcal{SROIQ}$, $\mathcal{ALCIQ}$, $\mathcal{ALCOI}$, etc.)

- More fragments with nice computational properties while retaining sufficient expressivity (e.g. $\mathcal{EL}$, $\mathcal{FL}_0$, DL-Lite, etc.)

- Playing with ontologies, ontology editors, etc....

- Query answering with respect to ontologies for large data sets.