Logical Aspects of Artificial Intelligence
Tableaux for DLs & Undecidability

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November 22nd, 2023
Plan of the lecture

- Tableaux calculus for checking $ALC$ concept satisfiability.
- Tableaux calculus for checking $ALC$ knowledge base consistency.
- Undecidability result with role axioms.
- Exercises session.
Recapitulation of the Previous Lecture(s)
**$\mathcal{ALC}$ in a nutshell**

\[ C ::= \top \mid \bot \mid A \mid \neg C \mid C \cap C \mid C \cup C \mid \exists r.C \mid \forall r.C \]

- Interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$.

- TBox $\mathcal{T} = \{ C \sqsubseteq D, \ldots \}$.

- ABox $\mathcal{A} = \{ a : C, (b, b') : r, \ldots \}$.

- Knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. (a.k.a. ontology)

- Decision problems include concept satisfiability, knowledge base consistency, and other problems for classification.
$$
\top^\mathcal{I} \qquad \text{def} \qquad \Delta^\mathcal{I}
$$

$$
\bot^\mathcal{I} \qquad \text{def} \qquad \emptyset
$$

$$
(-C)^\mathcal{I} \qquad \text{def} \qquad \Delta^\mathcal{I} \setminus C^\mathcal{I}
$$

$$
(C_1 \sqcup C_2)^\mathcal{I} \qquad \text{def} \qquad C_1^\mathcal{I} \cup C_2^\mathcal{I}
$$

$$
(C_1 \sqcap C_2)^\mathcal{I} \qquad \text{def} \qquad C_1^\mathcal{I} \cap C_2^\mathcal{I}
$$

$$
(\exists r.C)^\mathcal{I} \qquad \text{def} \qquad \{ a \in \Delta^\mathcal{I} \mid r^\mathcal{I}(a) \cap C^\mathcal{I} \neq \emptyset \}
$$

$$
(\forall r.C)^\mathcal{I} \qquad \text{def} \qquad \{ a \in \Delta^\mathcal{I} \mid r^\mathcal{I}(a) \subseteq C^\mathcal{I} \} 
$$
A few properties about $\mathcal{ALC}$

- Concept satisfiability problem is $\text{PSPACE}$-complete.

- Knowledge base consistency problem is $\text{EXPTIME}$-complete.

- $\mathcal{ALC}$ has many well-known fragments and extensions, some of them to deal with
  - inverse roles,
  - number restrictions,
  - properties on the role interpretations,
  - inclusions between the composition of roles,
  - etc..

- Reduction of decision problems for DLs to first-order logic.
  
  (to modal logics too, but not presented herein)

- Filtration construction leading to an $\text{NEXPTIME}$ upper bound for the $\mathcal{ALC}$ knowledge base consistency problem.
Expansion rules for $\mathcal{ALC}$ ABox consistency

\(\sqcap\)-rule: If $a : C \sqcap D \in \mathcal{A}$ and $\{a : C, a : D\} \not\subseteq \mathcal{A}$ then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{a : C, a : D\}$$

\(\sqcup\)-rule: If $a : C \sqcup D \in \mathcal{A}$ and $\{a : C, a : D\} \cap \mathcal{A} = \emptyset$ then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{a : E\} \quad \text{for some} \ E \in \{C, D\}$$

\(\exists\)-rule: If $a : \exists r. C \in \mathcal{A}$ and there is no $b$ such that $\{(a, b) : r, b : C\} \subseteq \mathcal{A}$ then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{(a, c) : r, c : C\} \quad \text{where} \ c \ \text{is fresh}$$

\(\forall\)-rule: If $\{(a, b) : r, a : \forall r. C\} \subseteq \mathcal{A}$ and $b : C \not\in \mathcal{A}$, then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{b : C\}$$
Today’s objectives

- Termination, soundness, completeness, blocking technique.

- Equivalences between
  - \((\mathcal{T}, \mathcal{A})\) is consistent (for ALC)
  - \(\mathcal{A} \xrightarrow{*} \mathcal{A}'\) for some complete and clash-free ABox \(\mathcal{A}'\) 
    \((\rightarrow\) depends on \(\mathcal{T}\))
  - \(\mathcal{A} \xrightarrow{*} \mathcal{A}'\) for some complete and clash-free ABox \(\mathcal{A}'\) 
    derivable in at most \(f(\text{size}(\mathcal{T}, \mathcal{A}))\) steps.
Example

\[ A = \{(a, b) : s, (a, c) : r\}\cup\{a : A_1 \land \exists s. A_5, a : \forall s. (\neg A_5 \cup \neg A_2), b : A_2, c : A_3 \land \exists s. A_4\}\]

\[ A \overset{*}{\rightarrow} A \cup \{a : A_1, a : \exists s. A_5, a_{\text{new}} : A_5, (a, a_{\text{new}}) : s, b : \neg A_5 \cup \neg A_2, a_{\text{new}} : (\neg A_5 \cup \neg A_2), b : \neg A_5, a_{\text{new}} : \neg A_2, c : A_3, c : \exists s. A_4, c_{\text{new}} : A_4, (c, c_{\text{new}}) : s\}\]

(is it complete?)
Terminology: root vs. tree individuals

- **Tree individuals** are generated by application of the $\exists$-rule.
- If $(a, b) : r$ is added by application of the $\exists$-rule, $b$ is an $r$-successor of $a$.
- Root individuals have no predecessors or ancestors.
Why “Tableaux”?

\[(a, b): s\]
\[(a, c): r\]
\[a: A_3 \cap \exists s. A_5\]
\[a: \forall s. (\neg A_5 \lor A_2)\]
\[b: A_2\]
\[c: A_3 \cap \exists s. A_4\]
\[a: \forall s\]
\[a: \exists s. A_5\]
\[a_{\text{new}}: A_5\]
\[(a, a_{\text{new}}): s\]
\[b: \forall A_5 \lor \forall A_2\]
\[a_{\text{new}}: \forall A_5 \lor \forall A_2\]
\[b: \forall A_2\]
Termination

- **The **$\exists$-weight** of $C$ is the number of its subconcepts of the form $\exists r.D$.

  \[
  w_{\exists}(C) \overset{\text{def}}{=} \text{card}(\{\exists r.D \mid \exists r.D \in \text{sub}(C)\})
  \]

- The definition assumes that $C$ is in NNF.

- **$\forall \exists$-depth** of $C$, written $d_{\forall \exists}(C)$, is the maximal number of imbrications of $\exists r.$ and $\forall s.$ in $C$.

  (a.k.a. quantifier depth, modal depth)

- $d_{\forall \exists}(\exists r. T \sqcup \forall r. \exists s. A) = 2$

- $d_{\forall \exists}(A) = \max\{d_{\forall \exists}(C) \mid a : C \in A\}$. 
Decorating individual names with depth information

- Let $\mathcal{A}$ be an ABox with $W = w_\exists(\mathcal{A})$, $D = d_\forall(\mathcal{A})$ and $N$ is the number of distinct individual names in $\mathcal{A}$.

- Let $\mathcal{A}^0$ be the variant of $\mathcal{A}$ where $a : C$ is replaced by $a^0 : C$.

\(\sqcap\)-rule: If $a^i : C \cap D \in \mathcal{A}$ and $\{a^i : C, a^i : D\} \not\subseteq \mathcal{A}$ then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{a^i : C, a^i : D\}$$

\(\sqcup\)-rule: If $a^i : C \sqcup D \in \mathcal{A}$ and $\{a^i : C, a^i : D\} \cap \mathcal{A} = \emptyset$ then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{a^i : E\} \quad \text{for some } E \in \{C, D\}$$

\(\exists\)-rule: If $a^i : \exists r. C \in \mathcal{A}$ and there is no $b^j$ such that

$$\{(a^i, b^j) : r, b^j : C\} \subseteq \mathcal{A}$$

then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{(a^i, c^{i+1}) : r, c^{i+1} : C\} \quad \text{where } c \text{ is fresh}$$

\(\forall\)-rule: If $\{(a^i, b^j) : r, a^i : \forall r. C\} \subseteq \mathcal{A}$ and $b^j : C \not\in \mathcal{A}$, then

$$\mathcal{A} \rightarrow \mathcal{A} \cup \{b^j : C\}$$
Quantities about $A^0 \rightarrow A'$

- If $a^i : C \in A'$, then $i + d_{\forall \exists}(C) \leq D$.
  Trees from individual names labelled by zero have depth at most $D$.

- $a^i : C \in A'$ implies
  \[
  \text{card}(\{(a^i, b^i) \mid (a^i, b^i) : r \in A'\}) \leq N + W
  \]
  (necessarily either $i = j = 0$ or $j = i + 1$)

The maximum branching degree of nodes in the trees is at most $N + W$. (rough overapproximation)

- $a^i : C \in A'$ implies $C \in \text{sub}(A)$.

- Length of the derivation $A^0 \rightarrow A'$ can be bounded, e.g. at most
  \[
  N \times (N + W)^{D+1} \times ((N + W) \times \text{card(\text{sub}(A)))}
  \]
  (why?)
About $N \times (N + W)^{D+1} \times ((N + W) \times \text{card}(\text{sub}({A})))$

- As there are $N$ root individuals, the rules build $N$ finite trees of maximal depth $D$ and maximal width $(N + W)$.

- A complete tree of depth $D$ and width $(N + W)$ has a number of nodes

$$1 + (N + W) + \cdots (N + W)^D = \frac{(N + W)^{D+1} - 1}{(N + W) - 1} \leq (N + W)^{D+1}$$

- By the subconcept property and since a concept assertion $a : C$ can lead to at most $(N + W)$ derivation steps (actually one for all rules except for the $\forall$-rule), the number of derivations is bounded by

$$N \times (N + W)^{D+1} \times ((N + W) \times \text{card}(\text{sub}({A})))$$
Main algorithm

We shall show that $\mathcal{A}$ is consistent iff $\mathcal{A} \overset{*}{\rightarrow} \mathcal{A'}$ for some complete and clash-free ABox $\mathcal{A'}$.

Existence of $\mathcal{A'}$ amounts to explore a finite tree of bounded depth and bounded degree.
The auxiliary function \( \text{exp} \)

- **Expansion function** \( \text{exp}(\mathcal{A}, R, X) \) taking as arguments
  - an ABox \( \mathcal{A} \),
  - an expansion rule \( R \),
  - a subset \( X \) of \( \mathcal{A} \) (with one or two elements) allowing the application of \( R \)

- … and returning the set of ABoxes obtained from \( \mathcal{A} \) by applying the rule \( R \) with main assertions in \( X \).

\[
\text{exp}\left(\{a : E, a : C \sqcup D\}, \sqcup\text{-rule}, \{a : C \sqcup D\}\right)
\]

is equal to

\[
\{\{a : E, a : C \sqcup D, a : C\}, \{a : E, a : C \sqcup D, a : D\}\}
\]
Algorithm for depth-first search

1: procedure EXPAND(\(\mathcal{A}\))
2: \hspace{1em} if \(\mathcal{A}\) has a clash then return \(\emptyset\)
3: \hspace{1em} end if
4: \hspace{1em} if \(\mathcal{A}\) is clash-free and complete then return \(\mathcal{A}\)
5: \hspace{1em} end if
6: \hspace{1em} for applicable \(R, X\) on \(\mathcal{A}\) and \(\mathcal{A}' \in \exp(\mathcal{A}, R, X)\) do
7: \hspace{2em} if \(\text{EXPAND}(\mathcal{A}') \neq \emptyset\) then return \(\text{EXPAND}(\mathcal{A}')\)
8: \hspace{2em} end if
9: \hspace{1em} end for
10: \hspace{1em} return \(\emptyset\)
11: end procedure
Soundness

Let $\mathcal{A}$ be a finite ABox with at least one concept assertion, complete, clash-free and all the concepts in NNF. Then, $\mathcal{A}$ is consistent.

For each individual name $a$ occurring in $\mathcal{A}$, we write $\text{con}_{\mathcal{A}}(a)$ to denote the set $\{ C \mid a : C \in \mathcal{A} \}$.

Let us define $\mathcal{I} \overset{\text{def}}{=} (\Delta^\mathcal{I},^\mathcal{I})$ as follows.

- $\Delta^\mathcal{I} \overset{\text{def}}{=} \{ a \mid a : C \in \mathcal{A} \}$.
- $a^\mathcal{I} \overset{\text{def}}{=} a$ for all individual names $a$ in $\mathcal{A}$.
- $A^\mathcal{I} \overset{\text{def}}{=} \{ a \mid A \in \text{con}_{\mathcal{A}}(a) \}$ for all concept names $A \in \text{sub}(\mathcal{A})$.
- $r^\mathcal{I} \overset{\text{def}}{=} \{ (a, b) \mid (a, b) : r \in \mathcal{A} \}$.

Let us show that for all $a : C \in \mathcal{A}$, we have $a^\mathcal{I} \in C^\mathcal{I}$. 
Proof by structural induction

▶ The base case with concept assertions $a : A$ is immediate by definition of $A^\mathcal{I}$.

▶ The base case with concept assertions $a : \neg A$ is immediate by definition of $A^\mathcal{I}$ as $A$ is clash-free.

▶ Case $a : C \sqcup D$ in the induction step.
  – As $A$ is complete, $a : C \in A$ or $a : D \in A$.
  – W.l.o.g., suppose $a : C \in A$. By (IH), $a^\mathcal{I} \in C^\mathcal{I}$.
  – By definition of $\cdot^\mathcal{I}$, we conclude $a^\mathcal{I} \in (C \sqcup D)^\mathcal{I}$.

▶ Case $a : \exists r.C$ in the induction step.
  – As $A$ is complete, $\{(a, b) : r, b : C\} \subseteq A$ for some $b$.
  – By definition of $r^\mathcal{I}$, $(a, b) \in r^\mathcal{I}$.
  – By (IH), $b^\mathcal{I} \in C^\mathcal{I}$.
  – By definition of $\cdot^\mathcal{I}$, we conclude $a^\mathcal{I} \in (\exists r.C)^\mathcal{I}$.
Concluding the soundness

- The cases in the induction step for $\cap$-concept assertions and $\forall$-concept assertions are similar.

- If $\text{expand}(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is consistent.

- Indeed, $\text{expand}(\mathcal{A}) \neq \emptyset$ if there is some $\mathcal{A}'$ with $\mathcal{A} \subseteq \mathcal{A}'$ such that $\mathcal{A}'$ is complete and clash-free.

- Consistency of $\mathcal{A}'$ leads to the consistency of $\mathcal{A}$. 
Moving towards completeness

- If $\mathcal{A}$ is consistent, then $\mathcal{A} \rightarrow \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

- Let $\mathcal{I} \overset{\text{def}}{=} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be such that $\mathcal{I} \models \mathcal{A}$.

- If $\mathcal{A}$ is complete, we are done.

- Otherwise, if $\mathcal{A}$ is not complete, we show that there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow \mathcal{A}'$ and $\mathcal{A}'$ is consistent.

- As the length of a derivation from $\mathcal{A}$ is bounded by an exponential in the size of $\mathcal{A}$, there is $\mathcal{A}'$ such that $\mathcal{A} \rightarrow^* \mathcal{A}'$ and $\mathcal{A}'$ is complete, clash-free (and consistent).
Single steps in the completeness proof

- It remains to prove that non-completeness implies the existence of one expansion preserving consistency.

- Guidance from the interpretations to choose disjuncts and tree individuals.

- If the $\sqcup$-rule is applicable on $a : C \sqcup D$, then there is $E \in \{C, D\}$ such that $\mathcal{I} \models \mathcal{A} \cup \{a : E\}$.

- $\mathcal{A} \rightarrow \mathcal{A} \cup \{a : E\}$ and $\mathcal{I} \models \mathcal{A} \cup \{a : E\}$. 
Single steps in the completeness proof (II)

- If the $\exists$-rule is applicable on $a : \exists r. C$, then we use the fact that $a^\mathcal{I} \in (\exists r. C)^\mathcal{I}$.

- There is $a \in \Delta^\mathcal{I}$ such that $a \in C^\mathcal{I}$ and $(a^\mathcal{I}, a) \in r^\mathcal{I}$.

- Let $\mathcal{I}'$ be equal to $\mathcal{I}$ except that $\mathcal{I}'(c) = a$ for some fresh $c$.

- Then, $\mathcal{A} \longrightarrow \mathcal{A} \cup \{ c : C, (a, c) : r \}$ and

$$\mathcal{I}' \models \mathcal{A} \cup \{ c : C, (a, c) : r \}$$

(freshness is required here)
Decision procedure of ABox consistency

- $\mathcal{A}$ is consistent iff $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ for some complete and clash-free ABox $\mathcal{A}'$.

- Derivations $\mathcal{A} \xrightarrow{*} \mathcal{A}'$ have length bounded by an exponential in $\text{size}(\mathcal{A})$.

- Existence of $\mathcal{A}'$ amounts to explore a tree of bounded depth and bounded degree.
Adding a TBox – First properties

▶ \(\mathcal{I} \models C \subseteq D \iff \mathcal{I} \models \top \subseteq \neg C \cup D\).

▶ \(\mathcal{I} \models C \equiv D \iff \mathcal{I} \models \top \subseteq (\neg C \cup D) \cap (\neg D \cup C)\).

▶ In the sequel, GCIs are of the form \(\top \subseteq E\) with \(E\) in NNF.

\(\subseteq\)-rule: If \(a\) in \(\mathcal{A}\), \(\top \subseteq D \in \mathcal{T}\) and \(a : D \not\in \mathcal{A}\), then

\[
\mathcal{A} \rightarrow \mathcal{A} \cup \{a : D\}
\]

▶ The termination argument for ABox consistency does not work anymore.

(W'hy?)
Termination with the blocking technique

Given $\mathcal{A} \xrightarrow{*} \mathcal{A}'$, $a$ is an ancestor of $b$ in $\mathcal{A}'$ iff

$$\{(a_1, a_2) : r_1, \ldots, (a_k, a_{k+1}) : r_k\} \subseteq \mathcal{A}'$$

with $a_1 = a$, $a_{k+1} = b$ and $b$ is a tree individual.

⚠️ The notion of ancestor assumes that one can distinguish the root individuals (individual names from $\mathcal{A}$) from the tree individuals (those introduced by applying the $\exists$-rule).

➤ An individual name $b$ in $\mathcal{A}'$ is **blocked by** $a$ if
  - $a$ is an ancestor of $b$,
  - $\text{con}_{\mathcal{A}'}(b) \subseteq \text{con}_{\mathcal{A}'}(a)$.

➤ An individual name $b$ is **blocked in** $\mathcal{A}'$ iff it is blocked by some individual name or, one or more of its ancestors is blocked in $\mathcal{A}'$. 
b blocked by a
Expansion rules with blocking

⊓-rule: If $a : C \cap D \in A$, $a$ is not blocked and $\{a : C, a : D\} \not\subseteq A$ then $A \rightarrow A \cup \{a : C, a : D\}$.

⊔-rule: If $a : C \cup D \in A$, $a$ is not blocked and $\{a : C, a : D\} \cap A = \emptyset$ then $A \rightarrow A \cup \{a : E\}$ for some $E \in \{C, D\}$.

∃-rule: If $a : \exists r.C \in A$, $a$ is not blocked and there is no $b$ such that $\{(a, b) : r, b : C\} \subseteq A$ then

$$A \rightarrow A \cup \{(a, c) : r, c : C\} \quad \text{where } c \text{ is fresh}$$

∀-rule: If $\{(a, b) : r, a : \forall r.C\} \subseteq A$, $a$ is not blocked and $b : C \not\in A$, then $A \rightarrow A \cup \{b : C\}$.

⊑-rule: If $a$ in $A$, $\top \subseteq D \in T$, $a$ is not blocked and $a : D \not\in A$, then $A \rightarrow A \cup \{a : D\}$. 
Termination

- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCI's of the form $\top \subseteq D$.

- $\mathbf{N}$: number of root individuals in $\mathcal{A}$, $M = \text{card}({\text{sub}(\mathcal{K})})$, $W = w_\exists(\mathcal{K})$.

- $\mathcal{A} \rightarrow^* \mathcal{A}'$ and $a : C \in \mathcal{A}'$ imply
  \[
  \text{card}(\{(a, b) \mid (a, b) : r \in \mathcal{A}'\}) \leq \mathbf{N} + \mathbf{W}
  \]

- $\mathcal{A} \rightarrow^* \mathcal{A}'$ and $a : C \in \mathcal{A}'$ imply $C \in \text{sub}(\mathcal{K})$.
  ("subconcept property")

- $\{ (a_1, a_2) : r_1, \ldots, (a_k, a_{k+1}) : r_k \} \subseteq \mathcal{A}'$ and $a_2$ is a tree individual imply $k \leq 2^M$.

- The length of the derivation $\mathcal{A} \rightarrow^* \mathcal{A}'$ is at most
  \[
  \mathbf{N} \times (\mathbf{N} + \mathbf{W})^{(2^M+1)} \times ((\mathbf{N} + \mathbf{W}) \times M + \text{card}(\mathcal{T}))
  \]
Soundness (or why blocking is safe)

- $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with concepts in NNF, and GCIs of the form $\top \subseteq D$.

- $\mathcal{A} \rightarrow^* \mathcal{A}'$ with $\mathcal{A}'$ complete and clash-free.

- We construct $\mathcal{A}''$ as the ABox made of the following assertions

$$\{a : C \mid a : C \in \mathcal{A}', \ a \text{ is not blocked}\} \cup$$

$$\{(a, b) : r \mid (a, b) : r \in \mathcal{A}', \ b \text{ is not blocked}\} \cup$$

$$\{(a, b') : r \mid (a, b) : r \in \mathcal{A}', \ a \text{ is not blocked and } b \text{ is blocked by } b'\}$$
Construction of $A''$
Properties of $\mathcal{A}''$

- $\mathcal{A} \subseteq \mathcal{A}''$ as root individuals cannot be blocked and $\mathcal{A} \subseteq \mathcal{A}'$.

- None of the individual names occurring in $\mathcal{A}''$ is blocked.

- For all $a$ in $\mathcal{A}''$, we have $\con_{\mathcal{A}''}(a) = \con_{\mathcal{A}'}(a)$.

  (left as an exercise)

- $\mathcal{A}''$ is complete and clash-free.
Proof: $\mathcal{A}''$ is complete

\[(\star) \con_{\mathcal{A}''}(a) = \con_{\mathcal{A}'}(a) \text{ for all } a \in \mathcal{A}''\]

\[\textbf{Suppose } a : C \cap D \in \mathcal{A}''.\]
By $(\star)$, $a : C \cap D \in \mathcal{A}'$.
As $\mathcal{A}'$ is complete, $\{a : C, a : D\} \subseteq \mathcal{A}'$.
By $(\star)$, $\{a : C, a : D\} \subseteq \mathcal{A}''$.

\[\textbf{Suppose that } a : C \in \mathcal{A}'' \text{ and } T \subseteq D \in T.\]
By $(\star)$, $a : C \in \mathcal{A}'$.
As $\mathcal{A}'$ is complete, $a : D \in \mathcal{A}'$.
By $(\star)$, $a : D \in \mathcal{A}''$.\]
Case with the $\exists$-rule

- Suppose that $a : \exists r. C \in A''$.
  By ($\star$), $a : \exists r. C \in A'$ and $a$ not blocked.
  By completeness of $A'$, there is $b$ such that
  $\{(a, b) : r, b : C\} \subseteq A'$.

- If $b$ is not blocked, then $\{(a, b) : r, b : C\} \subseteq A''$.

- As $a$ is not blocked, if $b$ is blocked, then $b$ is blocked by $b'$ in $A'$ and $b'$ is not blocked.

- By definition of $A''$, $(a, b') : r \in A''$.

- $\text{con}_{A'}(b) \subseteq \text{con}_{A'}(b')$ (blocking). By ($\star$),
  \[ C \in \text{con}_{A'}(b) \subseteq \text{con}_{A'}(b') = \text{con}_{A''}(b') \]
  So, $b' : C \in A''$.

- Case with the $\forall$-rule left as an exercise.
More about the soundness proof

- $\mathcal{A} \overset{*}{\rightarrow} \mathcal{A}'$ with $\mathcal{A}'$ complete and clash-free and $\mathcal{A}''$ computed as above.

- Let us define $\mathcal{I} \overset{\text{def}}{=} (\Delta \mathcal{I}, \cdot \mathcal{I})$ from $\mathcal{A}''$ as follows.
  - $\Delta \mathcal{I} \overset{\text{def}}{=} \{ a \mid a : C \in \mathcal{A}'' \}$.
  - $a \mathcal{I} \overset{\text{def}}{=} a$ for all individual names $a$ in $\mathcal{A}''$.
  - $A \mathcal{I} \overset{\text{def}}{=} \{ a \mid A \in \text{con}_{\mathcal{A}''}(a) \}$ for all concept names $A \in \text{sub}(\mathcal{A}'')$.
  - $r \mathcal{I} \overset{\text{def}}{=} \{ (a, b) \mid (a, b) : r \in \mathcal{A}'' \}$.

(Previous construction with $\mathcal{A}''$ instead)

- One can show that for all $a : C \in \mathcal{A}''$, we have $a \mathcal{I} \in C \mathcal{I}$.

(Left as an exercise.)
The final step about soundness

- It remains to check that $\mathcal{I} \models (\mathcal{T}, \mathcal{A})$.

- One can show that for all $a : C \in \mathcal{A}''$, we have $a^\mathcal{I} \in C^\mathcal{I}$.

- Consequently, $\mathcal{I} \models \mathcal{A}$ as $\mathcal{A} \subseteq \mathcal{A}''$.

- Moreover, $\mathcal{I} \models \top \subseteq C$ for all $\top \subseteq C \in \mathcal{T}$.

- $a \in \Delta^\mathcal{I}$
  $\rightarrow a : C \in \mathcal{A}''$ ($\mathcal{A}''$ is complete)
  $\rightarrow a \in C^\mathcal{I}$ (see above)
Completeness (bis)

- If \( \mathcal{K} = (\mathcal{T}, \mathcal{A}) \) is consistent, then \( \mathcal{A} \xrightarrow[*]{} \mathcal{A}' \) for some complete and clash-free ABox \( \mathcal{A}' \).

- Let \( \mathcal{I} \overset{\text{def}}{=} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}) \) be such that \( \mathcal{I} \models (\mathcal{T}, \mathcal{A}) \).

- If \( \mathcal{A} \) is complete, we are done.

- Otherwise (\( \mathcal{A} \) is not complete), we show there is \( \mathcal{A}' \) such that \( \mathcal{A} \xrightarrow{} \mathcal{A}' \) and \( \mathcal{A}' \) is consistent.

- As the length of a derivation from \( \mathcal{A} \) is bounded, there is \( \mathcal{A}' \) such that \( \mathcal{A} \xrightarrow[*]{} \mathcal{A}' \) and \( \mathcal{A}' \) is complete, clash-free (and consistent).

- One can prove that non-completeness implies the existence of one expansion preserving consistency.
Complexity issues

- **$\mathcal{ALC}$** concept satisfiability in **PSPACE**, knowledge base consistency in **EXPTIME**.

- The algorithm for ABox consistency runs in exponential space:
  - Because of the nondeterministic $\sqcap$-rule, exponentially many ABoxes may be generated.
  - Complete ABoxes may be exponentially large.

- **PSPACE** bound for ABox consistency can be regained by exploring the tree-like interpretations in a depth-first manner having only one path at a time.
Recapitulation: Tableaux for $\mathcal{ALC}$ knowledge base consistency

- Tableaux-based algorithm to decide $\mathcal{ALC}$ knowledge base consistency.

- All other standard decision problems can be handled too.

- Termination is guaranteed thanks to the blocking technique.

- In the worst-case, exponential space is used but optimisations exist to meet the optimal upper bound $\text{EXPTIME}$.

- Tableaux can be extended to richer variants of $\mathcal{ALC}$ (with inverses, nominals, number restrictions, etc.)
Undecidability with Role Inclusion Axioms
Many developments to extend $\textit{ALC}$ while preserving the decidability status / complexity of the main decision problems.

Many developments to study fragments of $\textit{ALC}$ (or variants) to identify tractable fragments.

It is also important to identify undecidable extensions.
Tiling system

- **Tiling system**: \((T, H, V, t_0)\) where
  - \(T\) is a finite set of **tile types** and \(t_0 \in T\),
  - \(H, V \subseteq T \times T\) are two relations referred to as the **horizontal**, resp. **vertical matching relation**.

- **A set of tile types (a.k.a. tiles)**

  \[
  t_1 = \begin{bmatrix}
  2 & 0 \\
  1 & 2
  \end{bmatrix}
  \quad t_2 = \begin{bmatrix}
  1 & 2 \\
  2 & 1
  \end{bmatrix}
  \quad t_3 = \begin{bmatrix}
  0 & 2 \\
  1 & 0
  \end{bmatrix}
  \quad t_4 = \begin{bmatrix}
  2 & 1 \\
  0 & 2
  \end{bmatrix}
  \]

- **...with its matching relations**
  - \(H = \{(t_1, t_3), (t_1, t_4), (t_2, t_1), (t_3, t_2), (t_4, t_1)\}\)
  - \(V = \{(t_1, t_2), (t_1, t_4), (t_2, t_3), (t_4, t_1), (t_4, t_2)\}\)
A tiling for the \(([0, 3] \times [0, 2])\)-arena

\[
\text{tiling } \tau : [0, 3] \times [0, 2] \rightarrow T
\]
The \((\infty \times \infty)\)-tiling problem.

**Input:** A tiling system \((T, H, V, t_0)\).

**Question:** Is there a tiling \(\tau : \mathbb{N} \times \mathbb{N} \rightarrow T\) such that for all \(i, j \in \mathbb{N}\),

- (hori) if \(\tau(i, j) = t\) and \(\tau(i + 1, j) = t'\), then \((t, t') \in H\),
- (verti) if \(\tau(i, j) = t\) and \(\tau(i, j + 1) = t'\), then \((t, t') \in V\)

The \((\infty \times \infty)\)-tiling problem is undecidable.
Complexity about $ALC$ problems

- Concept satisfiability problem is PSPACE-complete.

- PSPACE-hardness by reduction from $(n \times n)$-tiling game problem.

- EXPTIME-complete knowledge base consistency problem. EXPTIME-hardness from $(n \times \infty)$-tiling game problem.
A standard undecidability result

- $\mathcal{ALC} +$ role axioms $r \circ s \sqsubseteq q$ and $q \sqsubseteq r \circ s$ has undecidable knowledge base consistency problem.

  (actually CBox consistency is undecidable)

- Reduction from $(\infty \times \infty)$-tiling problem.

- $\mathcal{ALC} +$ local role value maps $r \circ s \sqsubseteq q$ and $q \sqsubseteq r \circ s$ has undecidable concept satisfiability problem.

  (not presented herein)
An undecidable extension of \( \mathcal{ALC} \)

- Let us consider the extension of \( \mathcal{ALC} \) in which we allow role axioms of the form

\[
  r \circ s \sqsubseteq q \quad q \sqsubseteq r \circ s,
\]

\[
  \mathcal{I} \models r \circ s \sqsubseteq q \iff r^\mathcal{I} \circ s^\mathcal{I} \subseteq q^\mathcal{I} \quad \mathcal{I} \models q \sqsubseteq r \circ s \iff q^\mathcal{I} \subseteq r^\mathcal{I} \circ s^\mathcal{I}
\]

- Role axioms \( r \circ s \equiv s \circ r \) can be encoded by introducing a fresh role name \( q \):

\[
  \{ r \circ s \sqsubseteq q, q \sqsubseteq r \circ s, s \circ r \sqsubseteq q, q \sqsubseteq s \circ r \}
\]

(correctness left as an exercise)

- Reduction from the \((\infty \times \infty)\)-tiling problem to knowledge base consistency for such an \( \mathcal{ALC} \) extension.
The reduction

- Given a tiling system $\mathcal{T} = (T, H, V, t_0)$, we introduce two role names $r_x$ and $r_y$.

- We build a TBox $\mathcal{T}^T$ such that $\mathcal{T}$ is a positive instance of the $(\infty \times \infty)$-tiling problem iff $\mathcal{T}^T$ is consistent.

- Every individual has a horizontal and a vertical successor:

  $\top \sqsubseteq \exists r_x.T \cap \exists r_y.T$

- Every individual belongs to a unique tile type.

  $\top \sqsubseteq \bigcup_{t \in T} (t \cap \bigcap_{t' \neq t} \neg t')$

- Tile types of adjacent individuals satisfy the matching relations:

  $\top \sqsubseteq \bigcup_{(t, t') \in H} (t \cap \forall r_x.t') \cap \bigcup_{(t, t') \in V} (t \cap \forall r_y.t')$
The properties

- The set of $r_x r_y$-successors is equal to the set of $r_y r_x$-successors.

\[ r_x \circ r_y \equiv r_y \circ r_x \]

- $\mathcal{T}_T$ is made of the above GCIs and role axioms.

- $\mathcal{T}_T$ is consistent iff $\top$ is a positive instance.

- TBox consistency problem for $\mathcal{ALC}$ augmented with role axioms of the form $r \circ s \sqsubseteq q$ and $q \sqsubseteq r \circ s$ is undecidable.
Correctness proof (or how to extract a grid)

Let $\mathcal{I}$ be an interpretation satisfying the TBox $\mathcal{T}_T$.

We define a map $f : \mathbb{N} \times \mathbb{N} \rightarrow \Delta^\mathcal{I}$ such that for all $i, j$

- $(f(i, j), f(i + 1, j)) \in r^\mathcal{I}_x$
- $(f(i, j), f(i, j + 1)) \in r^\mathcal{I}_y$

Then, we define $\tau : \mathbb{N} \times \mathbb{N} \rightarrow T$ from $f$ as follows:

$$\tau(i, j) \overset{\text{def}}{=} \text{unique } t \text{ such that } f(i, j) \in t^\mathcal{I}$$

Unicity of $t$ guaranteed by $\mathcal{I} \models T \subseteq \bigcup_{t \in T} (t \cap \prod_{t' \neq t} \neg t')$.

Afterwards, easy to check $\tau$ is a tiling as $\mathcal{I} \models \mathcal{T}_T$. 
How to define $f$ while maintaining properties?

- $f(0, 0)$ is chosen arbitrarily in $\Delta^\mathcal{I}$ (non-empty).

- As $\mathcal{I} \models \top \subseteq \exists r_x. \top$, when $f(i, i)$ is already defined, pick $\alpha \in \Delta^\mathcal{I}$ such that
  - $(f(i, i), \alpha) \in r^\mathcal{I}_x$,
  - $f(i + 1, i) \overset{\text{def}}{=} \alpha$

- As $\mathcal{I} \models \top \subseteq \exists r_y. \top$, when $f(i + 1, i)$ is already defined, pick $\beta \in \Delta^\mathcal{I}$ such that
  - $(f(i + 1, i), \beta) \in r^\mathcal{I}_y$,
  - $f(i + 1, i + 1) \overset{\text{def}}{=} \beta$
More cases for defining $f$

- As $\mathcal{I} \models r_x \circ r_y \equiv r_y \circ r_x$, when

$$f(i, j), f(i + 1, j), f(i + 1, j + 1)$$

are defined and $f(i, j + 1)$ undefined, pick $a \in \Delta^\mathcal{I}$ such that

- $(f(i, j), a) \in r^\mathcal{I}_y$
- $(a, f(i + 1, j + 1)) \in r^\mathcal{I}_x$
- $f(i, j + 1) \overset{\text{def}}{=} a$

- When $f(i, j), f(i, j + 1), f(i + 1, j + 1)$ are defined and $f(i + 1, j)$ undefined, pick $a \in \Delta^\mathcal{I}$ such that

$$f(i, j + 1) \overset{r_x}{\rightarrow} f(i + 1, j + 1)$$

- $(f(i, j), a) \in r^\mathcal{I}_x,$
- $(a, f(i + 1, j + 1)) \in r^\mathcal{I}_y,$
- $f(i + 1, j) \overset{\text{def}}{=} a$

- With these four cases, how to build $f$ on $\mathbb{N} \times \mathbb{N}$?
Construction of the map $f$: a bit of organisation
The other direction (easy)

Let $\mathcal{T} = (T, H, V, t_0)$ be a tiling system and $\tau : \mathbb{N} \times \mathbb{N} \rightarrow T$ be a tiling.

Interpretation $\mathcal{I} \stackrel{\text{def}}{=} (\Delta \mathcal{I}, \cdot \mathcal{I})$:

$\Delta \mathcal{I} \stackrel{\text{def}}{=} \mathbb{N} \times \mathbb{N}$

$r_x \mathcal{I} \stackrel{\text{def}}{=} \{((i, j), (i + 1, j)) \mid i, j \in \mathbb{N}\}$

$r_y \mathcal{I} \stackrel{\text{def}}{=} \{((i, j), (i, j + 1)) \mid i, j \in \mathbb{N}\}$

$t \mathcal{I} \stackrel{\text{def}}{=} \{(n, m) \mid \tau(n, m) = t\}$ for every $t \in T$

It is easy to check $\mathcal{I}$ satisfies all the GCIs and the role axioms from $\mathcal{T}_\mathcal{T}$. 
Tiling $\tau$

```
0 2 1 2
1 2 1 0
2 2 0 1
1 2 0 2
2 0 1 2
2 2 0 1
0 1 2 2
2 2 0 1
2 2 0 1
```

Interpretation $\mathcal{I}$

```
\begin{align*}
k_3 (2,0) & \xrightarrow{r_y} k_4 \xrightarrow{r_x} (4,1) \xrightarrow{r_x} (2,1) \xrightarrow{r_x} (3,1) \\
k_2 (4,0) & \xrightarrow{r_y} (4,1) \xrightarrow{r_x} (2,0) \xrightarrow{r_x} (3,1) \xrightarrow{r_x} \cdots
\end{align*}
```
Conclusion

▶ Today lecture: tableaux for DLs.

- Rules for checking concept satisfiability.
- Rules for checking knowledge base consistency.
- Termination, soundness, completeness.
- Undecidability result with role axioms.

▶ Next week lecture: reasoning about multiagent systems with ATL.
Other topics related to DLs

- More tableau-style systems and complexity results for ALC extensions (e.g. for SROIQ, ALCIQ, ALCOI, etc.)

- Fragments with nice computational properties while retaining sufficient expressivity (e.g. $\mathcal{EL}$, $\mathcal{FL}_0$, DL-Lite, etc.)

- Playing with ontologies, ontology editors, etc..

- Query answering with respect to ontologies for large data sets.