2

REVERSAL-BOUNDED COUNTER MACHINES

(Lectures given on 16/10/15 and on 06/11/15 by S. Demri)

2.1 The Effects of Bounding the Number of Reversals

Most reachability problems for Presburger counter machines are undecidable since counter machines are Turing-complete devices. A way to overcome this negative result is to restrict the class of runs for such machines so that decidability can be regained. An obvious way to restrict the runs is to require that the length of the runs is bounded by a bound b. Other types of bound exist and in the class of *reversal-bounded counter machines* introduced in this chapter, the runs are restricted differently so that the number of *reversals* in a run is bounded by a bound r. A reversal for a counter occurs in a run when there is an alternation from nonincreasing mode to nondecreasing mode and vice-versa. A counter machine is reversal-bounded whenever there is a bound r such that every run witnesses a number of reversals bounded by r. Reversal-bounded counter machines have been first studied in (Ibarra, 1978) and several extensions have been considered in the literature, see e.g. (Finkel and Sangnier, 2008).

For instance, in the sequence below, there are three reversals identified by an upper line:

$$0011223334444\overline{3}33222\overline{3}33444455555\overline{4}.$$

Similarly, the sequence 0011122222333333334444 has no reversal. Figure 2.1 presents schematically the behavior of a counter with 5 reversals. A counter machine is *reversal-bounded* whenever there is $r \geq 0$ such that for all the runs from a given

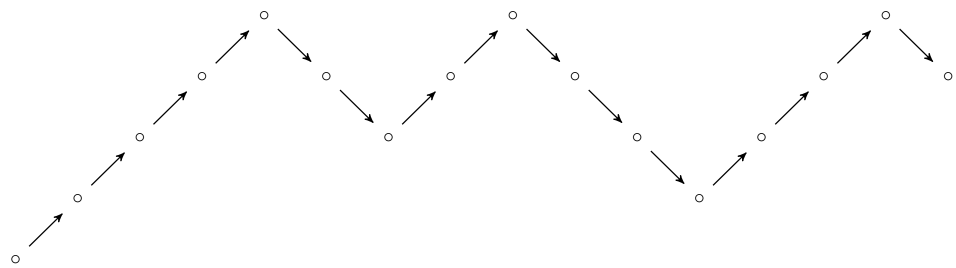

Figure 2.1: Five reversals in a row

initial configuration, every counter makes no more than r reversals. A formal definition will follow, but before going any further, it is worth pointing out a few peculiarities of this subclass. Firstly, reversal-boundedness is defined for initialized counter machines (a counter machine augmented with an initial configuration) and the bound r usually depends on the initial configuration. Secondly, this class is not defined from the full class of counter machines by imposing syntactic restrictions but rather semantically (see Section 2.2.1). For example, in flat counter machines, the syntactic restriction consists in requiring that the control graph of the counter machine is flat, i.e., every control state belongs to at most one simple cycle. Similarly, VASS are counter machines for which operations are restricted to translations.

A major property of such systems is that reachability sets are computable Presburger sets (Theorem 2.10). We present a proof of this result in the chapter that is different from the proof in (Ibarra, 1978) and that relies on developments from (Gurari and Ibarra, 1981). Apart from presenting this essential property, the chapter investigates decidability/complexity issues summarized as follows.

1. The reversal-boundedness detection problem is undecidable for counter machines and is ExpSpace-complete for vector addition systems with states.

2. The reachability problem for counter machines with a given bounded number of reversals is NExpTime-complete. Note that decidability was already a consequence of the fact that reachability sets of reversal-bounded counter machines can be effectively represented by Presburger formulae.

3. The control state repeated reachability problem with bounded number of reversals and the $\exists$-Presburger infinitely often problem are shown decidable (and NExpTime-complete), see e.g. (Dang et al., 2001).

4. All above-mentioned decidability problems are obtained with counter machines in which a counter value can only be compared to a constant and guards are closed under Boolean connectives. We also explain why the reachability problem with bounded number of reversals becomes undecid-

able if equalities and inequalities between counters are allowed in guards (see Exercise 2.9 and Theorem 2.16).

5. We present an alternative notion of reversal-boundedness introduced in (Finkel and Sangnier, 2008), called *weak reversal-boundedness*, that captures the one from (Ibarra, 1978) and we show that all properties about semilinearity still hold.

6. The universality problem for reversal-bounded one-counter machines equipped with an alphabet is also shown undecidable, see Exercise 2.12.

So, in this chapter, we consider runs in which the number of reversals is bounded and we show that the sets of reachable configurations with such restrictions are computable Presburger sets. Moreover, we are able to characterise the complexity of the reachability problem when the number of reversals is bounded. Several extensions are introduced in the chapter; equalities and inequalities in guards lead to undecidability whereas weak reversal-boundedness preserves all the nice properties about standard reversal-boundedness from (Ibarra, 1978). Last but not least, infinite repetition of a semilinear property can be verified when reversal-bounded runs are considered.

2.2 What is Reversal-Boundedness?

2.2.1 Counter machines in this chapter

In the present chapter, we consider counter machines $\mathcal{M} = \langle Q, T, C \rangle$ such that

- Q is a finite set of control states,
- C is a finite set of counters $\{\mathtt{x}_1, \ldots, \mathtt{x}_d\}$ for some $d \geq 1$,
- T is a finite set of transitions from $Q \times \Sigma \times Q$ where the operations in Σ are defined as follows. Each operation in Σ is defined as a pair $\langle g, \mathbf{a} \rangle$ where $\mathbf{a} \in \mathbb{Z}^d$ is an update (as for VASS) and g is a guard built over the following grammar:
$$g ::= \top \mid \bot \mid \mathtt{x} \sim k \mid g \wedge g \mid g \vee g \mid \neg g$$
where $\mathtt{x} \in C$, $\sim \in \{\leq, \geq, =\}$ and $k \in \mathbb{N}$.

Note that we have omitted the (infinite) set of operations Σ in the structure defining counter machines since it is implicit and does not lead to any confusion. Below, we write $\mathcal{C}$ to denote this class of counter machines. Observe that $\mathcal{C}$ contains Minsky machines and VASS, to cite a few classes. The developments in the present chapter about reversal-boundedness are performed for this class of counter machines, unless otherwise stated. In some places (for instance in Section 2.3.4 and in Exercise 2.11), we deal with an extended class of counter machines.

2.2.2 Formal definition of reversal-boundedness

Let $\mathcal{M} = \langle Q, T, C \rangle$ be a counter machine in $\mathcal{C}$. From a run

$$\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle, \ldots$$

of $\mathcal{M}$, in order to describe the behavior of counters varying along ρ, we define a sequence of *mode vectors* $\mathfrak{md}_0, \mathfrak{md}_1, \ldots$ (of the same length as ρ) such that each $\mathfrak{md}_i$ belongs to $\{\mathrm{INC}, \mathrm{DEC}\}^d$. Intuitively, each value in a mode vector records whether a counter is currently in an increasing phase or in an decreasing phase. We are now ready to define the sequence $\mathfrak{md}_0, \mathfrak{md}_1, \ldots$ associated with ρ.

- By convention, $\mathfrak{md}_0$ is the unique vector in $\{\mathrm{INC}\}^d$.
- For all $j \geq 0$ and for all $i \in [1, d]$, we have
  1. $\mathfrak{md}_{j+1}(i) \stackrel{\text{def}}{=} \mathfrak{md}_j(i)$ when $\mathbf{x}_j(i) = \mathbf{x}_{j+1}(i)$,
  2. $\mathfrak{md}_{j+1}(i) \stackrel{\text{def}}{=} \mathrm{INC}$ when $\mathbf{x}_{j+1}(i) - \mathbf{x}_j(i) > 0$,
  3. $\mathfrak{md}_{j+1}(i) \stackrel{\text{def}}{=} \mathrm{DEC}$ when $\mathbf{x}_{j+1}(i) - \mathbf{x}_j(i) < 0$.

Now, let $Rev_i \stackrel{\text{def}}{=} \{j \in [0, |\rho| - 1] : \mathfrak{md}_j(i) \neq \mathfrak{md}_{j+1}(i)\}$.

Definition 2.1. Let $\mathcal{M} = \langle Q, T, C \rangle$ be a counter machine with $d \geq 1$ counters, $i \in [1, d]$ and $r \in \mathbb{N}$. We say that the run ρ in $\mathcal{M}$ is *r-reversal-bounded* with respect to i $\stackrel{\text{def}}{\Leftrightarrow}$ $\mathrm{card}(Rev_i) \leq r$. The run ρ is *r-reversal-bounded* $\stackrel{\text{def}}{\Leftrightarrow}$ for every $i \in [1, d]$, we have $\mathrm{card}(Rev_i) \leq r$. An initialized counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is *r-reversal-bounded* $\stackrel{\text{def}}{\Leftrightarrow}$ every run from $\langle q, \mathbf{x} \rangle$ is r-reversal-bounded. An initialized counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is *reversal-bounded* $\stackrel{\text{def}}{\Leftrightarrow}$ there is some $r \geq 0$ such that every run from $\langle q, \mathbf{x} \rangle$ is r-reversal-bounded.

Figure 2.2 contains a counter machine $\mathcal{M}$ such that any initialized counter machine of the form $\langle \mathcal{M}, \langle q_1, \mathbf{x} \rangle \rangle$ with $\mathbf{x} \in \mathbb{N}^2$ is reversal-bounded.

Reversal-boundedness for counter machines is very appealing because reachability sets are semilinear. Indeed, given an initialized counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ that is r-reversal-bounded for some $r \geq 0$, for each state q', $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$ is a computable Presburger set, see Theorem 2.10. This means that one can compute effectively a Presburger formula that characterizes precisely the reachable configurations whose state is q'.

A counter machine $\mathcal{M}$ is *uniformly reversal-bounded* iff there is $r \geq 0$ such that for every initial configuration, the initialized counter machine is r-reversal-bounded. The question of checking whether a counter machine $\mathcal{M}$ is uniformly reversal-bounded can be reduced to reversal-boundedness. The proof for this property is left as an exercise (see Exercise 2.1). One can check that the counter machine in Figure 2.2 is not uniformly reversal-bounded. Indeed, any configuration that reaches the control state q_{11} leads to non-reversal-boundedness because

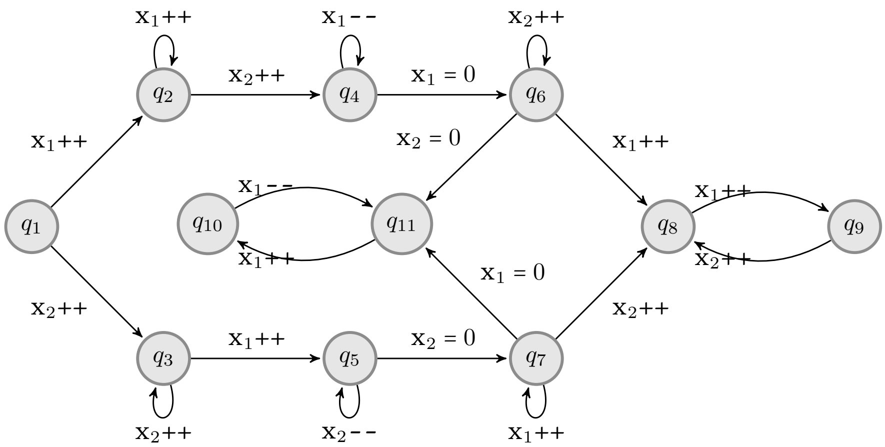

Figure 2.2: A counter machine that bounds the numbers of reversals

of the cycle between q_{10} and q_{11} that increments and decrements the first counter. Similarly, any configuration with control state q_6 and such that the second counter is equal to zero leads to non-reversal-boundedness.

In the sequel, when we consider a uniformly reversal-bounded counter machine or a reversal-bounded initialized counter machine, it comes with an a priori given maximal number of reversals $r \geq 0$. Remember that in full generality, we show that the reversal-boundedness detection problem is undecidable (see Theorem 2.17). Nevertheless, the situation is not that bad, since the problem restricted to VASS is decidable and can be solved in ExpSpace, see details in Section 2.4. Hence, for VASS, in case of reversal-boundedness, the value r can be effectively computed.

Alternatively, given an arbitrary counter machine and a bound $r \geq 0$, it is possible to build a new counter machine such that each counter has at most r reversals on each run, possibly at the cost of increasing exponentially the cardinal of the set of control states (see the proof of Theorem 2.12). It is sufficient to take the product between $\mathcal{M}$ and a finite-state automaton with number of control states in $\mathcal{O}(r^d)$ where d is the number of counters.

2.3 Reachability Sets are Computable Presburger Sets

In this section, we show that reachability sets in reversal-bounded (initialized) counter machines are computable Presburger sets. Moreover, when uniform reversal-boundedness is satisfied, one can show that the reachability relation is also computable and semilinear. Effectiveness refers here to the possibility to construct Presburger formulae defining exactly those sets or binary relations. The proof below is conceptually simple and it does not involve other intermediate results, apart from the fact that it uses a proof technique already used in Section 1.6.2.

2.3.1 Preliminaries

Let $\mathcal{M} = \langle Q, T, C \rangle$ be a counter machine with d counters. Without any loss of generality, we can assume that the guards in $\mathcal{M}$ are negation-free. Indeed, $\neg(\mathtt{x} \geq k)$ is equivalent to $\mathtt{x} \leq k-1$ when $k > 0$ (otherwise if $k = 0$, then it is equivalent to $\perp$). Similarly, $\neg(\mathtt{x} \leq k)$ is equivalent to $\mathtt{x} \geq k+1$. We write AG to denote the set of atomic guards of the form $\mathtt{x} \sim k$ occurring in $\mathcal{M}$ and $\mathcal{K} = \{k_1, \ldots, k_K\}$ to denote the set of distinct constants k occurring in atomic guards of the form $\mathtt{x} \sim k$ in AG, augmented by the value zero. We pose that $K = \mathrm{card}(\mathcal{K})$. Below, we assume that $0 = k_1 < k_2 < \cdots < k_K$ and we write $\mathcal{I}$ to denote the set of intervals below:

$$\mathcal{I} \stackrel{\text{def}}{=} \{[k_1,k_1],[k_1+1,k_2-1],[k_2,k_2],[k_2+1,k_3-1],[k_3,k_3],\ldots,$$
$$[k_K,k_K],[k_K+1,+\infty)\} \setminus \{\emptyset\}.$$

In the definition of the set $\mathcal{I}$, we possibly remove $\emptyset$ since $[k_j+1, k_{j+1}-1]$ is empty when $k_{j+1} = k_j + 1$. So, $\mathcal{I}$ contains at most $2K$ intervals and at least $K+1$ intervals. Furthermore, we consider a natural linear ordering $\leq$ on the intervals in $\mathcal{I}$ so that

$$[k_1,k_1] \leq [k_1+1,k_2-1] \leq [k_2,k_2] \leq [k_2+1,k_3-1] \leq [k_2,k_2] \leq \ldots \leq$$
$$\leq [k_K,k_K] \leq [k_K+1,+\infty)\}.$$

The above relationships should be understood to hold only if the intervals are non-empty. An *interval map* $\mathfrak{im}$ is a map of the form $\mathfrak{im} : C \to \mathcal{I}$ understood as a symbolic way to represent counter values. This abstracts a map $C \to \mathbb{N}$ by only taking into account to which elements of $\mathcal{I}$ each counter value belongs to.

Given a guard $g \in AG$ and an interval map $\mathfrak{im}$, we write $\mathfrak{im} \vdash g$ with the following inductive definition:

1. $\mathfrak{im} \vdash g_1 \vee g_2 \stackrel{\text{def}}{\Leftrightarrow} \mathfrak{im} \vdash g_1$ or $\mathfrak{im} \vdash g_2$.
2. $\mathfrak{im} \vdash g_1 \wedge g_2 \stackrel{\text{def}}{\Leftrightarrow} \mathfrak{im} \vdash g_1$ and $\mathfrak{im} \vdash g_2$.
3. $\mathfrak{im} \vdash \mathtt{x} = k \stackrel{\text{def}}{\Leftrightarrow} \mathfrak{im}(\mathtt{x}) = [k,k]$.
4. $\mathfrak{im} \vdash \mathtt{x} \geq k \stackrel{\text{def}}{\Leftrightarrow} \mathfrak{im}(\mathtt{x}) \subseteq [k,+\infty)$.
5. $\mathfrak{im} \vdash \mathtt{x} \leq k \stackrel{\text{def}}{\Leftrightarrow} \mathfrak{im}(\mathtt{x}) \subseteq [0,k]$.

Note that there is no clause for negated guards because without any loss of generality, we have seen that negation can be discarded.

The relation $\vdash$ is simply a symbolic satisfaction relation between interval maps and guards. Since interval maps and guards are built over the same set of constants, completeness is obtained as stated in the property below:

($\mathcal{P}_1$) $\mathfrak{im} \vdash g$ can be checked in polynomial time in the sum of the respective sizes of $\mathfrak{im}$ and g (for some reasonably succinct encoding in which natural numbers can be encoded in binary).

($\mathcal{P}_2$) $\mathfrak{im} \vdash g$ iff for all $\mathfrak{f} : C \to \mathbb{N}$ and for all $\mathrm{x} \in C$, we have $\mathfrak{f}(\mathrm{x}) \in \mathfrak{im}(\mathrm{x})$ implies $\mathfrak{f} \models g$ (in Presburger arithmetic).

The proofs for the properties ($\mathcal{P}_1$) and ($\mathcal{P}_2$) are left as an exercise.

A *guarded mode* $\mathfrak{gmd}$ is a pair $\langle \mathfrak{im}, \mathfrak{md} \rangle$ where $\mathfrak{im}$ is an interval map and $\mathfrak{md} \in \{\mathrm{INC}, \mathrm{DEC}\}^d$. A transition $t = q \xrightarrow{\langle g, \mathbf{a} \rangle} q'$ is *compatible* with the guarded mode $\mathfrak{gmd} = \langle \mathfrak{im}, \mathfrak{md} \rangle \stackrel{\text{def}}{\Leftrightarrow}$

1. $\mathfrak{im} \vdash g$,
2. for every $i \in [1, d]$,
   - $\mathfrak{md}(i) = \mathrm{INC}$ implies $\mathbf{a}(i) \geq 0$,
   - $\mathfrak{md}(i) = \mathrm{DEC}$ implies $\mathbf{a}(i) \leq 0$.

Let us generalize some definitions and concepts from Section 1.6.2. A *path* π is a finite sequence of transitions from T of the form

$$q_1 \xrightarrow{\langle g_1, \mathbf{a}_1 \rangle} q_1', \ldots, q_n \xrightarrow{\langle g_n, \mathbf{a}_n \rangle} q_n'$$

so that for every $i \in [1, n]$, we have $q_i' = q_{i+1}$. Let $\pi = t_1 \cdots t_n$ be a path such that each transition t_j has the update $\mathbf{a}_j \in \mathbb{Z}^d$. The *effect* of π is the update $\mathfrak{ef}(\pi) \stackrel{\text{def}}{=} \sum_j \mathbf{a}_j \in \mathbb{Z}^d$.

A *simple loop* sl is a non-empty path that starts and ends by the same state and these are the only states that are repeated in sl. We say that sl *loops* on its first state (equal to its last state). The number of simple loops is therefore bounded by $\mathrm{card}(T)^{\mathrm{card}(Q)}$. We assume an arbitrary total linear ordering $\prec$ on simple loops.

We write $\mathfrak{sc}(\mathcal{M})$ to denote the maximal absolute value among the updates $\mathbf{a}$ in $\mathcal{M}$. The value $\mathfrak{sc}(\mathcal{M})$ is called the *scale* of $\mathcal{M}$. Assuming that the size of $\mathcal{M}$ is N, we have $\mathfrak{sc}(\mathcal{M}) \leq 2^N$ (all the integers in $\mathcal{M}$ are encoded with a binary representation).

Lemma 2.2. *The effect* $\mathfrak{ef}(sl)$ *of a simple loop* sl *is in*

$$[-\mathrm{card}(Q)\mathfrak{sc}(\mathcal{M}), \mathrm{card}(Q)\mathfrak{sc}(\mathcal{M})]^d$$

This means also that the number of effects from simple loops in $\mathcal{M}$ is bounded by $(1 + 2 \times \mathrm{card}(Q)\mathfrak{sc}(\mathcal{M}))^d$.

An *extended path* $\mathbf{P}$ is an expression of the form below:

$$\pi_0 \; S_1 \; \pi_1 \; \cdots \; S_\alpha \; \pi_\alpha$$

where the S_i's are (finite) and non-empty sets of simple loops, the π_i's are non-empty paths and

1. if S occurs just before a path π, then all the simple loops in S loop on the first state of π,
2. similarly, if S occurs just after a path π, then all the simple loops in S loop on the last state of π.

An extended path generalizes the notion of path in which simple loops in the sets S_i's can be visited a non-zero number of times but respecting the arbitrary linear ordering on simple loops. A guarded mode induces a restriction of a counter machine by considering only a subset of transitions from $\mathcal{M}$, namely those that are compatible with the guarded mode. We say that a path [resp. loop, simple loop, extended path] is *compatible* with the guarded mode $\mathfrak{gmd} = \langle \mathfrak{im}, \mathfrak{md} \rangle \overset{\text{def}}{\Leftrightarrow}$ all its transitions are compatible with $\mathfrak{gmd}$.

Given an extended path $\mathbf{P}$, we introduce a few notions.

- The *skeleton* of $\mathbf{P}$ is the path $\pi_0 \cdots \pi_\alpha$,
- Given a set of simple loops $S = \{sl_1, \ldots, sl_m\}$ with $sl_1 \prec \cdots \prec sl_m$, we write $e(S)$ to denote the regular expression

$$(sl_1)^+ \cdots (sl_m)^+$$

  So, each simple loop is taken at least once. Indeed, we want to make explicit that each simple loop sl is used in S, otherwise it is always possible to exclude sl from S, leading to another legitimate extended path. We write $e(\mathbf{P})$ to denote the regular expression defined as follows:

$$\pi_0 \cdot e(S_1) \cdots e(S_\alpha) \cdot \pi_\alpha$$

We write $\mathrm{L}(e)$ to denote the language generated by the regular expression e. One can observe that $\mathrm{L}(e(\mathbf{P}))$ is a bounded and regular language for any extended path $\mathbf{P}$. For the sake of simplicity, we write $\mathrm{L}(\mathbf{P})$ instead of $\mathrm{L}(e(\mathbf{P}))$. A finite run $\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \cdots \xrightarrow{t_l} \langle q_l, \mathbf{x}_l \rangle$ *respects* the extended path $\mathbf{P} \overset{\text{def}}{\Leftrightarrow} \pi = t_1 \cdots t_l \in \mathrm{L}(\mathbf{P})$.

2.3.2 Runs in normal form

Given a non-empty r-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \cdots \xrightarrow{t_l} \langle q_l, \mathbf{x}_l \rangle$, we aim at showing that the path as a finite word belongs to a bounded regular language. To do so, we divide the run ρ into several subruns such that the number of reversals on each subrun is zero (i.e., a reversal can only occur when passing from one subrun to a next one) and moreover, all the counter values of each subrun satisfy exactly the same atomic guards. That is why we have introduced the notion of guarded mode since it contains an interval map and a mode for each counter in C.

A *global reversal phase* is a finite sequence of transitions such that each transition in it is compatible with some guarded mode $\langle \mathfrak{im}, \mathfrak{md} \rangle$, for some mode $\mathfrak{md} \in$

$\{\text{INC}, \text{DEC}\}^d$. So, in a run respecting a global reversal phase, the number of reversals is zero for all the counters.

Lemma 2.3. *Any r-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0 \rangle \cdots \langle q_l, \mathbf{x}_l \rangle$ can be divided as a sequence of subruns $\rho = \rho_1 \cdot \rho_2 \cdots \rho_L$ such that each ρ_i respects a global reversal phase and $L \leq (d \times r) + 1$.*

The proof is by an easy verification. A *local guard phase* is a finite sequence of transitions such that each transition in it is compatible with some guarded mode $\langle \mathfrak{im}, \mathfrak{md} \rangle$. Hence, in a run respecting a local guard phase, not only the number of reversals is zero for all the counters but also the counter values satisfy the same atomic guards.

Lemma 2.4. *Any r-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0 \rangle \cdots \langle q_l, \mathbf{x}_l \rangle$ can be divided as a sequence of subruns $\rho = \rho_1 \cdot \rho_2 \cdots \rho_{L'}$ such that each ρ_i respects a local guard phase and $L' \leq ((d \times r) + 1) \times 2Kd$.*

Proof. By Lemma 2.3, we have seen that ρ can be divided in at most $(d \times r) + 1$ subruns respecting a global reversal phase.

It remains to show that each such subrun can be divided in at most $2Kd$ subruns respecting a local guard phase. Actually, this is due to the following property.

Let $\mathbf{a} \in \mathbb{Z}^d$. We define the binary relation $\preceq_{\mathbf{a}}$ on the set of interval maps so that $\mathfrak{im} \preceq_{\mathbf{a}} \mathfrak{im}' \overset{\text{def}}{\Leftrightarrow}$ for every $i \in [1, d]$, we have

- $\mathfrak{im}(\mathrm{x}_i) \leq \mathfrak{im}'(\mathrm{x}_i)$ if $\mathbf{a}(i) \geq 0$,
- $\mathfrak{im}'(\mathrm{x}_i) \leq \mathfrak{im}(\mathrm{x}_i)$ if $\mathbf{a}(i) \leq 0$,
- $\mathfrak{im}'(\mathrm{x}_i) = \mathfrak{im}(\mathrm{x}_i)$ if $\mathbf{a}(i) = 0$.

We write $\mathfrak{im} \prec_{\mathbf{a}} \mathfrak{im}'$ when $\mathfrak{im} \preceq_{\mathbf{a}} \mathfrak{im}'$ and $\mathfrak{im} \neq \mathfrak{im}'$. Property ($\mathcal{P}_3$) below states that sequences of strictly increasing interval maps have polynomially bounded length, even though the number of interval maps is in $\mathcal{O}(K^d)$.

($\mathcal{P}_3$) Let $\mathbf{a} \in \mathbb{Z}^d$ and $\mathfrak{im}_1 \prec_{\mathbf{a}} \mathfrak{im}_2 \prec_{\mathbf{a}} \cdots \prec_{\mathbf{a}} \mathfrak{im}_\beta$. Then, $\beta \leq 2Kd$.

The proof of ($\mathcal{P}_3$) is left as an exercise. Indeed, in a subrun respecting a global reversal phase, each counter is compared against at most K constants and all the counters have a monotonous behaviour (in increasing mode or in decreasing). Hence, each counter during the global reversal phase can visit at most $2K$ distinct intervals in $\mathcal{I}$, whence the bound $2Kd$ for the maximal number of local guard phases. □

Below, a *sequence* of extended paths is understood as being of the form $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$ with the proviso that each $\mathbf{P}_i$ is an extended path compatible with some guarded mode and the expression $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$ can be also viewed as an extended path by itself (possibly by concatenating paths), i.e. it is compatible with the control graph of $\mathcal{M}$.

Lemma 2.5. *Any r-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0\rangle \cdots \langle q_l, \mathbf{x}_l\rangle$ respects a sequence of extended paths $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$ with $L' \leq ((d \times r) + 1) \times 2Kd$.*

A *small* $\pi_0\ S_1\ \pi_1\ \cdots\ \pi_{\alpha-1}\ S_\alpha\ \pi_\alpha$, with $\alpha \geq 1$, is an extended path such that

1. π_0 and π_α have at most $2 \times \mathrm{card}(Q)$ transitions,
2. $\pi_1, \ldots, \pi_{\alpha-1}$ have at most $\mathrm{card}(Q)$ transitions,
3. for each state $q \in Q$, there is at most one set S containing simple loops on q.

So, the length of the skeleton is bounded by $\mathrm{card}(Q)(3 + \mathrm{card}(Q))$. Note that the set of small extended paths is finite, even though its cardinal can be exponential in the size of $\mathcal{M}$. We also consider *degenerated* small extended paths made of paths of length at most $3 \times \mathrm{card}(Q)$. Usually, this case is omitted in the proofs since it can be easily obtained from the non-degenerated case (i.e. when $\alpha \geq 1$).

Proposition 2.6. *Let $\rho = \langle q_0, \mathbf{x}_0\rangle \cdots \langle q_l, \mathbf{x}_l\rangle$ be a run respecting an extended path $\mathbf{P}$ compatible with some guarded mode $\mathfrak{gmd}$. Then, there is small extended path $\mathbf{P}'$ compatible with $\mathfrak{gmd}$ and a run $\rho' = \langle q_0, \mathbf{x}_0\rangle \cdots \langle q_l, \mathbf{x}_l\rangle$ (possibly different from ρ) such that ρ' respects $\mathbf{P}'$.*

Proof. Let $\rho = \langle q_0, \mathbf{x}_0\rangle \xrightarrow{t_1} \cdots \xrightarrow{t_l} \langle q_l, \mathbf{x}_l\rangle$ be a run of $\mathcal{M}$ respecting an extended path $\mathbf{P}$ compatible with some guarded mode $\mathfrak{gmd}$. So, $\pi = t_1 \cdots t_l \in \mathrm{L}(\mathbf{P})$. We shall build a small extended path $\mathbf{P}'$ such that $\mathbf{P}'$ is compatible with $\mathfrak{gmd}$ and there is a run ρ' respecting $\mathbf{P}'$ that starts and ends by the same configurations as ρ.

To do so, we define a sequence of extended paths $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_\beta$ such that

- all the $\mathbf{P}_i$'s are extended paths compatible with $\mathfrak{gmd}$ and there is a run ρ_i respecting $\mathbf{P}_i$ that starts and ends by the same configurations,
- $\mathbf{P}_0$ is equal to $t_1 \cdots t_l$ viewed as an extended path,
- $\mathbf{P}_\beta$ is a small extended path,
- $\mathbf{P}_{i+1}$ is obtained from $\mathbf{P}_i$ by removing a simple loop on q and possibly adding it to a set of simple loops S already in $\mathbf{P}_i$ or by creating one if none exists.

So, at the end of this process, $\mathbf{P}_\beta$ is a small extended path and there is a run ρ_β respecting $\mathbf{P}_\beta$ that starts by $\langle q_0, \mathbf{x}_0\rangle$ and ends by $\langle q_l, \mathbf{x}_l\rangle$.

It remains to explain how to build $\mathbf{P}_{i+1}$ from $\mathbf{P}_i$. We assume that $\mathbf{P}_i$ has the form below

$$\pi_0\ S_1\ \pi_1\ \cdots\ S_\alpha\ \pi_\alpha$$

where

(a) $\alpha \leq \mathrm{card}(Q)$,

(b) each path in $\pi_0, \ldots, \pi_{\alpha-1}$ has length less than $\mathrm{card}(Q)$,

(c) each state has at most one S_i containing simple loops on it.

Obviously, $\mathbf{P}_0$ verifies these conditions since it is degenerated. $\mathbf{P}_{i+1}$ will satisfy the same condition except that we require that the length of the final path of $\mathbf{P}_{i+1}$ strictly decreases. Now, let us define $\mathbf{P}_{i+1}$ from $\mathbf{P}_i$.

Case 1: $\mathbf{P}_i$ is a small extended path. We are done and $\mathbf{P}_i$ is the final extended path of the sequence.

Case 2: $\pi_\alpha = \pi \cdot sl \cdot \pi'$ where sl is a simple loop on q, $\pi\pi' \neq \varepsilon$ and S_γ already contains simple loops on q ($\gamma \leq \alpha$). Then, $\mathbf{P}_{i+1}$ is equal to the extended path below:

$$\pi_0 \ \cdots \ S_{\gamma-1} \ \pi_{\gamma-1} \ (S_\gamma \cup \{sl\}) \ \cdots \ \pi_{\alpha-1} \ S_\alpha \ (\pi\pi')$$

Case 3: $\pi_\alpha = \pi \cdot sl \cdot \pi'$ where sl is a simple loop on q and the first one occurring in $\pi \cdot sl$, $\pi\pi' \neq \varepsilon$, and no S_γ already contains simple loops on q. Then, $\mathbf{P}_{i+1}$ is equal to the extended path below:

$$\pi_0 \ \cdots \ S_\alpha \ \pi \ \{sl\} \ \pi'$$

In that case, we create a new set of simple loop(s).

It remains to show that there is a run ρ_{i+1} respecting $\mathbf{P}_{i+1}$ that starts by $\langle q_0, \mathbf{x}_0 \rangle$ and ends by $\langle q_l, \mathbf{x}_l \rangle$. Satisfaction of the conditions (a)–(c) are by an easy verification. In order to show the former property, we need to use the fact that all the transitions in $\mathbf{P}_{i+1}$ are compatible with $\mathfrak{gmd}$ (by construction), the counter values have a monotonous behaviour (increase or decrease) and the atomic guards are convex.

We deal with the Case 2 below, Case 3 admits a similar analysis and it is left as an exercise. Let ρ_i be a run respecting $\mathbf{P}_i$, starting by the configuration $\langle q_0, \mathbf{x}_0 \rangle$ and ending by the configuration $\langle q_l, \mathbf{x}_l \rangle$. The extended path $\mathbf{P}_i$ is of the form

$$\pi_0 \ S_1 \ \pi_1 \ \cdots \ S_\alpha \ (\pi \cdot sl \cdot \pi')$$

and the extended path $\mathbf{P}_{i+1}$ is of the form

$$\pi_0 \ \cdots \ S_{\gamma-1} \ \pi_{\gamma-1} \ (S_\gamma \cup \{sl\}) \ \cdots \ \pi_{\alpha-1} \ S_\alpha \ (\pi\pi')$$

Suppose that $S_\gamma = S^1_\gamma \uplus S^2_\gamma$, and for all $sl' \in S^1_\gamma$ [resp. $sl' \in S^2_\gamma$], we have $sl' \prec sl$ [resp. $sl \prec sl'$]. Since $\mathbf{P}_i$ is compatible with the guarded mode $\mathfrak{gmd} = \langle \mathfrak{im}, \mathfrak{md} \rangle$, for every $j \in [1, d]$, we have:

- $\mathfrak{md}(j) = \mathrm{INC}$ implies that for all counter values $\mathbf{x} \in \mathbb{N}^d$ occurring in the run ρ_i, we get that $\mathbf{x}_0(j) \leq \mathbf{x}(j) \leq \mathbf{x}_l(j)$,

- $\mathfrak{md}(j) = \text{DEC}$ implies that for all counter values $\mathbf{x} \in \mathbb{N}^d$ occurring in the run ρ_i, we get that $\mathbf{x}_l(j) \leq \mathbf{x}(j) \leq \mathbf{x}_0(j)$.

Moreover, assuming that $\mathbf{y} \in \mathbb{N}^d$ is the penultimate vector of counter values in ρ, we know for all counter values $\mathbf{x} \in \mathbb{N}^d$ occurring in the run ρ until that occurrence of $\mathbf{y}$, for every atomic guard $\mathtt{x}_j \sim k$ in AG, we have $\mathfrak{im} \vdash \mathtt{x}_j \sim k$ iff $\mathbf{x}(j) \sim k$ iff $\mathbf{x}_0(j) \sim k$ iff $\mathbf{y}(j) \sim k$ (partly thanks to Property ($\mathcal{P}_2$)).

In order to build a run ρ_{i+1} similar to ρ_i that respects $\mathbf{P}_{i+1}$, that starts by $\langle q_0, \mathbf{x}_0 \rangle$ and that ends by $\langle q_l, \mathbf{x}_l \rangle$, we need to decompose the run ρ_i in the following way in order to explain how to build ρ_{i+1}.

The run ρ_i can be divided as follows. Each subrun $\rho_i^\star$ respects a factor of $\mathbf{P}_i$ (we are a bit liberal here with the notion of respect):

$$\rho = \overbrace{\rho_1^\star}^{\pi_0 \cdots S_{\gamma-1}\ \pi_{\gamma-1}\ S_\gamma^1} \cdot \overbrace{\rho_2^\star}^{S_\gamma^2\ \pi_\gamma \cdots \pi_{\alpha-1}\ S_\alpha\ \pi} \cdot \overbrace{\rho_3^\star}^{sl} \cdot \overbrace{\rho_4^\star}^{\pi'}$$

For each subrun $\rho_j^\star$, we write $\langle q_0^j, \mathbf{x}_0^j \rangle$ [resp. $\langle q_f^j, \mathbf{x}_f^j \rangle$] to denote its first [resp. last] configuration. For example, by definition we have $\mathbf{x}_f^1 = \mathbf{x}_0^2$. In order to build ρ_i', we introduce two sequences of configurations $\rho_3^{\star\star}$ and $\rho_2^{+\mathfrak{ef}(sl)}$ that will happen to be runs. $\rho_3^{\star\star}$ is the sequence of configurations obtained from the initial configuration $\langle q_0^2, \mathbf{x}_0^2 \rangle$ by firing the transitions of the simple loop sl. Similarly, $\rho_2^{+\mathfrak{ef}(sl)}$ is the sequence of configurations obtained from the last configuration of $\rho_3^{\star\star}$ by firing the sequence of transitions used for $\rho_2^\star$. Observe that $\rho_2^\star$ and $\rho_2^{+\mathfrak{ef}(sl)}$ have the same length and for any configuration $\langle q, \mathbf{x} \rangle$ in $\rho_2^\star$, say at position h, the configuration at position h in $\rho_2^{+\mathfrak{ef}(sl)}$ is exactly $\langle q, \mathbf{x} + \mathfrak{ef}(sl) \rangle$.

Let us consider the sequences of configurations ρ_{i+1} as defined below:

$$\rho_{i+1} = \overbrace{\rho_1^\star}^{\pi_0 \cdots S_{\gamma-1}\ \pi_{\gamma-1}\ S_\gamma^1} \cdot \overbrace{\rho_3^{\star\star}}^{sl} \cdot \overbrace{\rho_2^{+\mathfrak{ef}(sl)}}^{S_\gamma^2\ \pi_\gamma \cdots \pi_{\alpha-1}\ S_\alpha\ \pi} \cdot \overbrace{\rho_4^\star}^{\pi'}$$

Note that the sequence of configurations respects the updates on the transitions. In order to check that ρ_{i+1} is a run, it remains to show that transitions in $\rho_3^{\star\star}$ and in $\rho_2^{+\mathfrak{ef}(sl)}$ can be fired by respecting the guards. Suppose that $\mathfrak{md}(j) = \text{INC}$ for some $j \in [1, d]$ (the case $\mathfrak{md}(j) = \text{DEC}$ admits a similar development). Every vector of counter values $\mathbf{y}$ from a configuration in $\rho_3^{\star\star}$ satisfies the following inequalities:

$$\mathbf{x}_0(j) = \mathbf{x}_0^1(j) \leq \mathbf{x}_f^1(j) = \mathbf{x}_0^2(j) \leq \mathbf{y}(j) \leq \mathbf{x}_0^4(j) \leq \mathbf{x}_f^4(j) = \mathbf{x}_l(j)$$

By convexity of the atomic guards $\mathtt{x}_j \sim k$ in AG, $\mathbf{y}(j) \sim k$ iff $\mathbf{y}'(j) \sim k$ where $\mathbf{y}'$ is the corresponding vector of counter values in the run $\rho_3^\star$ (at the same position). So, $\rho_3^{\star\star}$ is indeed a run of $\mathcal{M}$ respecting sl. Similary, one can show that $\rho_2^{+\mathfrak{ef}(sl)}$ is a run respecting $S_\gamma^2 \pi_\gamma \cdots \pi_{\alpha-1}\ S_\alpha\ \pi$, which concludes the proof. □

A *small* sequence of extended paths is a sequence of extended paths $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$ such that $L' \leq ((d \times r) + 1) \times 2Kd$ and not only each extended path of the sequence compatible with a unique guarded mode but also the extended path is itself small.

Theorem 2.7. *For any r-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0 \rangle \cdots \langle q_l, \mathbf{x}_l \rangle$, there is an r-reversal-bounded run ρ' between the same configurations that respects a small sequence of extended paths.*

Let $\mathbf{P}$ be a small extended path compatible with the guarded mode $\mathfrak{gmd} = \langle \mathfrak{im}, \mathfrak{md} \rangle$, say $\mathbf{P}$ is of the form below:

$$\pi_0 \; \{sl_1^1, \ldots, sl_1^{n_1}\} \; \pi_1 \; \cdots \; \{sl_\alpha^1, \ldots, sl_\alpha^{n_\alpha}\} \; \pi_\alpha$$

where q_0 is the first control state in π_0 and q_f is the last control state in π_α.

Lemma 2.8. *There is a Presburger formula $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_d, \mathsf{y}_1, \ldots, \mathsf{y}_d)$ of exponential size in $|\mathcal{M}|$ such that $[\![\varphi]\!] = \{\langle \mathbf{x}_0, \mathbf{y} \rangle :$* there is a run $\langle q_0, \mathbf{x}_0 \rangle \xrightarrow{*} \langle q_f, \mathbf{y} \rangle$ respecting $\mathbf{P}\}$.

Proof. Let $\pi_\alpha = \pi'_\alpha \cdot t$, so that t is the last transition of π_α. The formula φ states the following properties:

1. the initial counter values belong to the right intervals induced by $\mathfrak{im}$,
2. the counter values for the penultimate configuration $\langle q'_f, \mathbf{y}' \rangle$ belong to the right intervals induced by $\mathfrak{im}$,
3. the values for $\bar{\mathsf{y}}$ are obtained from $\bar{\mathsf{x}}$ by considering the effects of the paths π_i plus a finite amount of times the effects of each simple loop occurring in $\mathbf{P}$.

Note that (1) and (2) are sufficient to guarantee that every other configuration $\mathbf{x}$ in the run $\langle q_0, \mathbf{x}_0 \rangle \xrightarrow{*} \langle q_f, \mathbf{y} \rangle$, possibly except $\mathbf{y}$, belong to the right intervals induced by $\mathfrak{im}$. Indeed, if $\mathfrak{md}(i) =$ INC, then $\mathbf{x}_0(i) \leq \mathbf{x}(i) \leq \mathbf{y}'(i)$ and by convexity of the guards in AG, we get that $\mathbf{x}(i)$ satisfies the same atomic guards. A similar analysis can be made when $\mathfrak{md}(i) =$ DEC.

So, the formula φ is of the form below:

$$\exists \, \mathsf{z}_1^1, \ldots, \mathsf{z}_1^{n_1}, \ldots, \mathsf{z}_\alpha^1, \ldots, \mathsf{z}_\alpha^{n_\alpha}$$

$$(\mathsf{z}_1^1 \geq 1) \wedge \cdots \wedge (\mathsf{z}_1^{n_1} \geq 1) \wedge \cdots \wedge (\mathsf{z}_\alpha^1 \geq 1) \wedge \cdots \wedge (\mathsf{z}_\alpha^{n_\alpha} \geq 1) \wedge$$

$$(\bar{\mathsf{y}} = \bar{\mathsf{x}} + \mathfrak{ef}(\pi_0) + \cdots + \mathfrak{ef}(\pi_\alpha) + \sum_{i,j} \mathsf{z}_i^j \, \mathfrak{ef}(sl_i^j)) \wedge$$

$$(\bigwedge_{\mathfrak{im} \vdash \mathsf{x}_c \sim k} \mathsf{x}_c \sim k) \wedge (\bigwedge_{\text{not } \mathfrak{im} \vdash \mathsf{x}_c \sim k} \neg(\mathsf{x}_c \sim k)) \wedge$$

$$(\bigwedge_{j\in[1,d]} (\mathsf{x}_j \in \mathfrak{im}(\mathtt{x}_j) \wedge (\mathsf{y}_j \in \mathfrak{im}(\mathtt{x}_j)))$$

$$(\bigwedge_{\mathfrak{im}\vdash \mathtt{x}_c \sim k} (\mathsf{x}_c + \mathfrak{ef}(\pi_0)(c) + \cdots + \mathfrak{ef}(\pi_{\alpha-1})(c) + \mathfrak{ef}(\pi'_\alpha)(c) + \sum_{i,j} \mathsf{z}_i^j \mathfrak{ef}(sl_i^j)(c)) \sim k) \wedge$$

$$(\bigwedge_{\text{not } \mathfrak{im}\vdash \mathtt{x}_c \sim k} \neg(\mathsf{x}_c + \mathfrak{ef}(\pi_0)(c) + \cdots + \mathfrak{ef}(\pi_{\alpha-1})(c) + \mathfrak{ef}(\pi'_\alpha)(c) + \sum_{i,j} \mathsf{z}_i^j \mathfrak{ef}(sl_i^j)(c) \sim k))$$

It remains to define what we mean by '$\mathsf{z}_j \in \mathfrak{im}(\mathtt{x}_j)$': $\mathsf{z}_j \in [l, l']$ stands for $l \leq \mathsf{z}_j \wedge \mathsf{z}_j \leq l'$ whereas $\mathsf{z}_j \in [k_K + 1, +\infty)$ stands for $k_K + 1 \leq \mathsf{z}_j$. □

The formula φ in Lemma 2.8 has size polynomial in $|\mathcal{M}|$ and in the size of $\mathbf{P}$. The size of $\mathbf{P}$ is itself exponential in $|\mathcal{M}|$. This is the best we can hope for since the number of simple loops can be obviously exponential in the size of the control graph of $\mathcal{M}$.

Lemma 2.9. *Let $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$ be a sequence of small extended paths. There is a Presburger formula $\varphi(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ such that*

$$[\![\varphi]\!] = \{\langle \mathbf{x}, \mathbf{y}\rangle : \text{ there is a run } \langle q_0, \mathbf{x}\rangle \xrightarrow{*} \langle q_f, \mathbf{y}\rangle \text{ respecting } \mathbf{P}_1 \cdots \mathbf{P}_{L'}\}$$

Proof. The proof is by an easy verification by using the formulae from Lemma 2.8 (L' times) and to take advantage of existential first-order quantifications for $(L' - 1)$ intermediary configurations. Indeed, for each small extended path $\mathbf{P}_i$, let $\varphi_i(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ be the formula constructed in the proof of Lemma 2.8. The formula φ is then defined as follows:

$$\exists\ \bar{\mathsf{z}_0}, \ldots, \bar{\mathsf{z}_{L'}}\ (\bar{\mathsf{x}} = \bar{\mathsf{z}_0}) \wedge (\bar{\mathsf{y}} = \bar{\mathsf{z}_{L'}}) \wedge$$

$$\varphi_1(\bar{\mathsf{z}_0}, \bar{\mathsf{z}_1}) \wedge \varphi_2(\bar{\mathsf{z}_1}, \bar{\mathsf{z}_2}) \wedge \cdots \varphi_{L'-1}(\bar{\mathsf{z}_{L'-2}}, \bar{\mathsf{z}_{L'-1}}) \wedge \varphi_{L'}(\bar{\mathsf{z}_{L'-1}}, \bar{\mathsf{z}_{L'}}). \quad \square$$

The formula φ in Lemma 2.9 is of exponential size in $log(r) + |\mathcal{M}|$. Since the number of small sequences of extended paths is finite and actually double exponential in $log(r) + |\mathcal{M}|$, we get the following theorem.

Theorem 2.10. *Let $\langle \mathcal{M}, \langle q, \mathbf{x}\rangle\rangle$ be an initialized counter machine that is r-reversal-bounded for some $r \geq 0$. For each state $q' \in Q$, the set $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x}\rangle \xrightarrow{*} \langle q', \mathbf{y}\rangle\}$ is a computable Presburger set.*

Theorem 2.10 is clearly a consequence of Theorem 2.7 and of its corollaries.

Proof. Let us consider the formula $\varphi(\bar{\mathsf{y}})$ below:

$$\exists\ \bar{\mathsf{x}}\ (\bigwedge_{i\in[1,d]} \mathbf{x}(i) = \mathsf{x}_i) \wedge \bigvee_{\text{small seq. } \sigma = \mathbf{P}_1 \cdots \mathbf{P}_{L'}} \varphi_\sigma(\bar{\mathsf{x}}, \bar{\mathsf{y}})$$

where $\varphi_\sigma(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ is the Presburger formula for the small sequence of extended paths $\sigma = \mathbf{P}_1 \cdots \mathbf{P}_{L'}$ obtained from Lemma 2.9. Moreover, in the disjunction, we assume that $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$ starts by the state q and ends by the state q'. Moreover, if $q' = q$, we add the disjunct $(\bigwedge_{i \in [1,d]} \mathbf{x}(i) = \mathsf{y}_i)$.

Finally, note that the generalized disjunction is finite since the number of small sequences of extended paths is finite and bounded by $2^{p(log(r),|\mathcal{M}|)}$ for some polynomial $p(\cdot,\cdot)$. Indeed, a small sequence has length at most $((d \times r) + 1) \times 2Kd$ and the number of small extended paths is exponential in the size of $\mathcal{M}$. □

Theorem 2.11. *Let $\mathcal{M}$ be a counter machine that is uniformly r-reversal-bounded for some $r \geq 0$. For all states q, q', one can compute a Presburger formula $\varphi(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ such that $[\![\varphi]\!] = \{\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{N}^{2d} : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$.*

Indeed, it is sufficient to consider the formula $\bigvee_{\text{small seq. } \sigma = \mathbf{P}_1 \cdots \mathbf{P}_{L'}} \varphi_\sigma(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ from the proof of Theorem 2.10.

2.3.3 Reachability problem with bounded number of reversals

Let us consider the following problem.
Reachability problem with bounded number of reversals:

Input: a counter machine $\mathcal{M}$, a bound $r \in \mathbb{N}$, an initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ and a final configuration $\langle q_f, \mathbf{x}_f \rangle$,

Question: Is there a finite run of $\mathcal{M}$ with initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ and final configuration $\langle q_f, \mathbf{x}_f \rangle$ such that each counter has at most r reversals?

Observe that when $\langle \mathcal{M}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ is r'-reversal-bounded for some $r' \leq r$, we get an instance of the reachability problem with initial configuration $\langle q_0, \mathbf{x_0} \rangle$.

Note that Theorem 2.12 below is also a consequence of Theorem 2.7. Neverthelesss, the alternative proof we propose below is interesting for its own sake: it shows how to transform a counter machine into a r-reversal-bounded counter machine whose runs are exactly the r-reversal-bounded runs of the original counter machine.

Theorem 2.12. *The reachability problem with bounded number of reversals is decidable.*

Proof. Here is the decidability proof that uses Theorem 2.10. Let $\mathcal{M} = \langle Q, T, C \rangle$, $r \in \mathbb{N}$, $\langle q_0, \mathbf{x}_0 \rangle$ and $\langle q_f, \mathbf{x}_f \rangle$ be an instance of the reachability problem with bounded number of reversals. First, we build a counter machine $\mathcal{M}' = \langle Q', T', C \rangle$ with $Q' = Q \times \{\text{DEC}, \text{INC}\}^d \times [0, r]^d$.

By construction of $\mathcal{M}'$, we guarantee that $\langle \mathcal{M}', \langle \langle q_0, \text{INC}, \mathbf{0} \rangle, \mathbf{x}_0 \rangle \rangle$ is r-reversal-bounded. Indeed, for each counter, we count the number of reversals and by construction of $\mathcal{M}'$ we enforce that it is bounded by r on each run. The set of transitions T' is defined as follows: $\langle q, \mathfrak{md}, \sharp\mathfrak{alt} \rangle \xrightarrow{\langle g, \mathbf{a} \rangle} \langle q', \mathfrak{md}', \sharp\mathfrak{alt}' \rangle \in T' \overset{\text{def}}{\Leftrightarrow}$

$q \xrightarrow{\langle g,\mathbf{a}\rangle} q' \in T$ and for every $i \in [1,d]$, the relation described by the following table is verified. The values of two first columns induce values for the two last columns (when it is possible, see e.g. the condition $\sharp\mathfrak{alt}(i) < r$).

$\mathbf{a}$	$\mathfrak{md}(i)$	$\mathfrak{md}'(i)$	$\sharp\mathfrak{alt}'(i)$
$\mathbf{a}(i) < 0$	DEC	DEC	$\sharp\mathfrak{alt}(i)$
$\mathbf{a}(i) < 0$	INC	DEC	$\sharp\mathfrak{alt}(i) + 1$ and $\sharp\mathfrak{alt}(i) < r$
$\mathbf{a}(i) > 0$	INC	INC	$\sharp\mathfrak{alt}(i)$
$\mathbf{a}(i) > 0$	DEC	INC	$\sharp\mathfrak{alt}(i) + 1$ and $\sharp\mathfrak{alt}(i) < r$
$\mathbf{a}(i) = 0$	DEC	DEC	$\sharp\mathfrak{alt}(i)$
$\mathbf{a}(i) = 0$	INC	INC	$\sharp\mathfrak{alt}(i)$

By construction, $\mathcal{M}'$ is uniformly r-reversal-bounded and the properties below are equivalent:

1. there is a run of $\mathcal{M}$ with initial configuration $\langle q_0, \mathbf{x}_0\rangle$ and final configuration $\langle q_f, \mathbf{x}_f\rangle$ such that each counter has at most r reversals,
2. $\langle\langle q_f, \mathfrak{md}, \sharp\mathfrak{alt}\rangle, \mathbf{x}_f\rangle$ is reachable from $\langle\langle q_0, \mathrm{INC}, \mathbf{0}\rangle, \mathbf{x_0}\rangle$ in $\mathcal{M}'$ for some $\mathfrak{md}$, $\sharp\mathfrak{alt}$.

The number of distinct pairs $\langle\mathfrak{md}, \sharp\mathfrak{alt}\rangle$ is bounded by $2^d \times (r+1)^d$ and therefore (1.) is equivalent to the existence of $\langle\mathfrak{md}, \sharp\mathfrak{alt}\rangle$ among a finite set such that

3. $\langle\langle q_f, \mathfrak{md}, \sharp\mathfrak{alt}\rangle, \mathbf{x}_f\rangle$ is reachable from $\langle\langle q_0, \mathrm{INC}, \mathbf{0}\rangle, \mathbf{x_0}\rangle$ in $\mathcal{M}'$.

By Theorem 2.10, the set

$$X_{\langle\mathfrak{md},\sharp\mathfrak{alt}\rangle} = \{\mathbf{x}' \in \mathbb{N}^d : \langle\langle q_0, \mathrm{INC}, \mathbf{0}\rangle, \mathbf{x}_0\rangle \xrightarrow{*} \langle\langle q_f, \mathfrak{md}, \sharp\mathfrak{alt}\rangle, \mathbf{x}'\rangle\}$$

is a computable Presburger set. This means that one can construct a Presburger formula $\varphi_{\langle\mathfrak{md},\sharp\mathfrak{alt}\rangle}$ such that $[\![\varphi_{\langle\mathfrak{md},\sharp\mathfrak{alt}\rangle}]\!] = X_{\langle\mathfrak{md},\sharp\mathfrak{alt}\rangle}$ and checking whether $\mathbf{x} \in X_{\langle\mathfrak{md},\sharp\mathfrak{alt}\rangle}$ amounts to verify the satisfiability of the formula

$$(\bigwedge_{i=1}^{d} \mathsf{x}_i = \mathbf{x}(i)) \wedge \varphi_{\langle\mathfrak{md},\sharp\mathfrak{alt}\rangle}.$$

Since the satisfiability problem for Presburger arithmetic is decidable (see Chapter 1), we get an algorithm to solve the reachability problem with a bounded number of reversals. Indeed, it amounts to checking satisfiability of some Presburger formula made a disjunction with at most $2^d(r+1)^d$ disjuncts. □

Theorem 2.12 is interesting but does not help much to understand the computational complexity of the reachability problem with bounded number of reversals. However, the complexity can be nailed down thanks to the following developments. First, let us make use of Lemma 2.9.

Lemma 2.13. *If there is a run from $\langle q_0, \mathbf{x}_0 \rangle$ to $\langle q_f, \mathbf{x}_f \rangle$ such that each counter has at most r reversals, then there is an r-reversal-bounded run between these configurations respecting a small sequence of extended paths such that each simple loop is visited at most a doubly-exponential number of times in $log(r) + |\mathbf{x}_0| + |\mathbf{x}_f| + |\mathcal{M}|$.*

We recall that the size of $\mathbf{x} \in \mathbb{N}^d$ is defined so that $|\mathbf{x}| \in \mathcal{O}(d \times log(m))$ where m is the maximal value among the components of $\mathbf{x}$. Similarly, the size of a counter machine $\mathcal{M}$ uses a reasonably succinct encoding with integers encoded in binary.

Proof. Let ρ be an r-reversal-bounded run from $\langle q_0, \mathbf{x}_0 \rangle$ to $\langle q_f, \mathbf{x}_f \rangle$. By Theorem 2.7, there is an r-reversal-bounded run ρ' between the same configurations that respects a small sequence of extended paths $\mathbf{P}_1 \cdots \mathbf{P}_{L'}$.

Let $\varphi(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ be the Presburger formula for that sequence. The formula $\varphi(\bar{\mathsf{x}}, \bar{\mathsf{y}})$ is equivalent to an existential formula (in prenex normal form, only the existential quantifier occurs) and it is of size exponential in $log(r) + |\mathcal{M}|$. Note that most of the existentially quantified variables are related to the number of times simple loops are visited. So, the formula

$$(\bigwedge_{j \in [1,d]} (\mathsf{x}_j = \mathbf{x}_0(j) \wedge \mathsf{y}_j = \mathbf{x}_f(j)) \wedge \varphi(\bar{\mathsf{x}}, \bar{\mathsf{y}})$$

is satisfiable, which is equivalent to the satisfiability of a quantifier-free formula φ' by removing the quantifications.

By Theorem 1.9, that formula φ' is satisfiable with values at most exponential in the size of φ'. Consequently, each simple loop is visited at most a double exponential amount of times. □

Since in a small sequence of extended paths, there are at most $((d \times r) + 1) \times 2Kd$ extended paths, and each extended path has at most $\mathrm{card}(T)^{\mathrm{card}(Q)}$ simple loops and at most $\mathrm{card}(Q)(3 + \mathrm{card}(Q))$ transitions, that do not occur in simple loops, if there is an r-reversal-bounded run from $\langle q_0, \mathbf{x}_0 \rangle$ to $\langle q_f, \mathbf{x}_f \rangle$, then there is such a run of length at most double exponential in $log(r) + |\mathcal{M}| + |\mathbf{x}_0| + |\mathbf{x}_f|$. This means that a nondeterministic exponential space algorithm can guess such a run and therefore the reachability problem with bounded number of reversals is in ExpSpace by Savitch's Theorem. This can be improved: the runs are much more structured which will allow us to show NExpTime-completeness.

Theorem 2.14. *The reachability problem with bounded number of reversals is in NExpTime assuming that all natural numbers are encoded in binary.*

Proof. Let $\mathcal{M}$, r, $\langle q_0, \mathbf{x}_0 \rangle$ and $\langle q_f, \mathbf{x}_f \rangle$ be an instance of size N for the reachability problem with bounded number of reversals. We have $N \in \mathcal{O}(|\mathcal{M}| + log(r) + |\mathbf{x}_0| + |\mathbf{x}_f|)$. We have seen that there is an r-reversal-bounded run from $\langle q_0, \mathbf{x}_0 \rangle$ to $\langle q_f, \mathbf{x}_f \rangle$ iff there is an r-reversal-bounded run ρ between these configurations that respects a small sequence of extended paths, that is of length at most double exponential in N and each simple loop on that sequence is taken at most a double exponential number of times.

Such a sequence has at most $((d \times r) + 1) \times 2Kd$ small extended paths. Each extended path is compatible with a guarded mode and it has at most $\mathrm{card}(T)^{\mathrm{card}(Q)}$ simple loops and at most $1 + \mathrm{card}(Q)$ paths of length at most $3 \times \mathrm{card}(Q)$. Note that these are rough bounds that also take into account the degenerated small extended paths.

The NExpTime algorithm below guesses on-the-fly the small sequence of extended paths and computes the effect of taking a path of length at most $3 \times \mathrm{card}(Q)$ or a simple loop compatible with a guarded mode a double exponential number of times. Computing the effect of taking such an amount of times a simple loop can be computed in exponential-time because the natural numbers are encoded in binary. We do not compute the full run but only the intermediate configurations after firing a path or a simple loop.

The number of paths of length at most $3 \times \mathrm{card}(Q)$ or the number of simple loops visited along the small sequence of extended paths is bounded by:

$$G = ((d \times r) + 1) \times 2Kd \times (\mathrm{card}(T)^{\mathrm{card}(Q)} + \mathrm{card}(Q) + 1)$$

Here is the algorithm:

1. $\langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle := \langle q_0, \mathbf{x}_0 \rangle$; Guess $\alpha \leq G$; $\beta := 1$;
2. While $\beta \leq \alpha$ do
   (a) Guess either a path π of length at most $3 \times \mathrm{card}(Q)$ or, a simple loop sl and a guarded mode $\mathfrak{gmd} = \langle \mathfrak{im}, \mathfrak{md} \rangle$ and γ of double exponential value in N such that sl is compatible with $\mathfrak{gmd}$;
   (b) If a simple loop is guessed in (a), then check that $\mathbf{x}_{\mathrm{cur}}$ and $\mathbf{x}_{\mathrm{cur}} + (\gamma - 1)\mathfrak{ef}(sl) + sl^{\setminus \mathrm{last}}$ are in the right intervals: for every $i \in [1, d]$, $\mathbf{x}_{\mathrm{cur}}(i)$ and $(\mathbf{x}_{\mathrm{cur}} + (\gamma - 1)\mathfrak{ef}(sl) + \mathfrak{ef}(sl^{\setminus \mathrm{last}}))(i)$ belong to $\mathfrak{im}(\mathtt{x}_i)$ where $sl^{\setminus \mathrm{last}}$ equals sl minus its last transition.
   (c) If a path π is guessed in (a), then check that the sequence of transitions in π can be fired from $\langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle$ and set $\langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle := \langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle + \mathfrak{ef}(\pi)$. The effect of a path is the sum of the updates of its transitions;
   (d) $\beta := \beta + 1$;
3. Return $(\langle q_{\mathrm{cur}}, \mathbf{x}_{\mathrm{cur}} \rangle = \langle q_f, \mathbf{x}_f \rangle)$.

Checking that the algorithm runs in nondeterministic exponential-time is then by an easy verification. What is missing above, is a means to check that the number of reversals is indeed bounded by r and this can be done similarly to what is presented in the proof of Theorem 2.12. So strictly speaking, the above algorithm should be completed and additional variables should be introduced to count the number of reversals per counter. □

Lemma 2.15. *The reachability problem with bounded number of reversals is NExpTime-complete.*

Proof. It remains to establish the NExpTime lower bound. NExpTime-hardness is shown by simulating a nondeterministic Turing machine running in exponential time. This is actually easy by using the simulation of Turing machines with Minsky machines equipped with three counters. Each step in the Turing machine is simulated by an exponential amount of steps in the counter machines. □

2.3.4 A simple undecidable extension

In this section, we consider the class of counter machines $\mathcal{C}^+$ that extends the class $\mathcal{C}$ so that the atomic guards of the form $\mathtt{x} \sim k$ ($\mathtt{x} \in C$ and $k \in \mathbb{N}$) are extended to atomic guards ot the form $\sum_i a_i \mathtt{x}_i \sim k$ where the a_i's and k are in $\mathbb{Z}$. For example, in counter machines in $\mathcal{C}^+$, equalities $\mathtt{x}_i = \mathtt{x}_{i'}$ and inequalities $\mathtt{x}_i \neq \mathtt{x}_{i'}$ can occur as guards.

Theorem 2.16. *The reachability problem with bounded number of reversals for the class $\mathcal{C}^+$ is undecidable.*

Proof. (sketch) Undecidability can be shown even if r is restricted to zero (no reversal) and the only guards in transitions are equalities or inequalities.

To prove this result, we present a reduction from the halting problem for Minsky machines. Indeed, assuming that guards of the form $\mathtt{x}_i = \mathtt{x}_{i'}$ and $\mathtt{x}_i \neq \mathtt{x}_{i'}$ are allowed, each counter $\mathtt{x}_i$ from the Minsky machine provides two increasing counters $\mathtt{x}_i^{inc}$ and $\mathtt{x}_i^{dec}$, that counts the number of increments on $\mathtt{x}_i$ and the number of decrements, respectively. Zero-test for counter $\mathtt{x}_i$ is then simulated by a test $\mathtt{x}_i^{inc} = \mathtt{x}_i^{dec}$. Similarly, before incrementing $\mathtt{x}_i^{dec}$ (simulating a decrement in M), we test whether $\mathtt{x}_i^{inc} \neq \mathtt{x}_i^{dec}$. See Exercise 2.9. □

2.4 The Reversal-Boundedness Detection Problem

Since reversal-boundedness is not defined from counter machines by a syntactic criterion, the following problem makes sense and indeed it happens to be undecidable.

Reversal-boundedness detection problem

Input: Initialized counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ of dimension d and $i \in [1, d]$.

Question: Is $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ reversal-bounded with respect to the counter $\mathtt{x}_i$?

Theorem 2.17. *Reversal-boundedness detection problem is undecidable.*

Proof. Let $\mathcal{M}$ be a Minsky machine $\mathcal{M} = \langle Q, T, C \rangle$ with counters $C = \{\mathtt{x}_1, \mathtt{x}_2\}$, initial state $q_0 \in Q$ and halting state $q_H \in Q$ such that $q_0 \neq q_H$. No transition starts at q_H since it is the halting state. Moreover, the transitions in T have one of the forms below:

- $q_1 \xrightarrow{\mathtt{x}_i\texttt{++}} q_2$ with $i \in [1,2]$,
- $q_1 \xrightarrow{\mathtt{x}_i=0} q_2$ and $q_1 \xrightarrow{\mathtt{x}_i\texttt{--}} q_3$ with $i \in [1,2]$, that is, either counter $\mathtt{x}_i$ is decremented or a zero-test is performed on it.

Note that the Minsky machine $\mathcal{M}$ is deterministic and therefore either the Minsky machine has a unique infinite run (and never visits the halting state) or it has a unique finite run (and halts at q_H).

Let us define $\mathcal{M}' = \langle Q, T', C \cup \{\mathtt{x}_3\} \rangle$ and $\langle q_0', \mathbf{x}_0' \rangle$:

- T' contains all the transitions of T, but with no update on the new counter $\mathtt{x}_3$.
- T' contains two additionnal transitions that break reversal-boundedness of the counter $\mathtt{x}_3$, namely $q_H \xrightarrow{\mathtt{x}_3\texttt{++}} q_H$ and $q_H \xrightarrow{\mathtt{x}_3\texttt{--}} q_H$. The control state q_H is not anymore halting.
- $q_0' \stackrel{\text{def}}{=} q_0$ and $\mathbf{x}_0'$ is equal to $\mathbf{x}_0$ on the 2 first counters and $\mathbf{x}_0'(3) = 0$.

The only reason for $\langle \mathcal{M}', \langle q_0', \mathbf{x}_0' \rangle \rangle$ not being reversal-bounded with respect to the counter $\mathtt{x}_3$ is to reach the control state q_H. It is easy to show that the Minsky machine $\mathcal{M}$ halts iff $\mathcal{M}'$ is reversal-bounded with respect to the counter $\mathtt{x}_3$ from the initial configuration $\langle q_0, \mathbf{0} \rangle$. ☐

By contrast, the reversal-boundedness detection problem for VASS is decidable.

Theorem 2.18. *The reversal-boundedness detection problem restricted to vector addition systems with states is* ExpSpace*-complete.*

Proof. ExpSpace-hardness is obtained by reducing the control state reachability problem for VASS. Let $\mathcal{M} = \langle Q, T, C \rangle$ be a VASS with d counters, $\langle q_0, \mathbf{x}_0 \rangle$ be a configuration and q_f be a control state. We build a VASS $\mathcal{M}' = \langle Q', T', C \cup \{\mathtt{x}_{d+1}\} \rangle$ with one more counter $\mathtt{x}_{d+1}$ and a configuration $\langle q_0', \mathbf{x}_0' \rangle$ such that $\langle q_0, \mathbf{x}_0 \rangle \xrightarrow{*} \langle q_f, \mathbf{x}_f \rangle$ for some $\mathbf{x}_f \in \mathbb{N}^d$ iff $\langle \mathcal{M}', \langle q_0', \mathbf{x}_0' \rangle \rangle$ is not reversal-bounded with respect to the new counter $\mathtt{x}_{d+1}$. Since the control state reachability problem for VASS is ExpSpace-hard and coExpSpace= ExpSpace, we get that the reversal-boundedness detection problem restricted to VASS is ExpSpace-hard. Let us define $\mathcal{M}' = \langle Q, T', C \cup \{\mathtt{x}_{d+1}\} \rangle$ and $\langle q_0', \mathbf{x}_0' \rangle$:

- T' contains all the transitions of T, but with no update on the new counter $\mathtt{x}_{d+1}$.

- T' contains two additionnal transitions that break reversal-boundedness of the counter $\mathtt{x}_{d+1}$, namely $q_f \xrightarrow{\mathtt{x}_{d+1}\mathtt{++}} q_f$ and $q_f \xrightarrow{\mathtt{x}_{d+1}\mathtt{--}} q_f$.
- $q_0' \stackrel{\text{def}}{=} q_0$ and $\mathbf{x}_0'$ is equal to $\mathbf{x}_0$ on the d first counters and $\mathbf{x}_0'(d+1) = 0$.

The only reason for $\langle \mathcal{M}', \langle q_0', \mathbf{x}_0' \rangle \rangle$ not being reversal-bounded with respect to the counter $\mathtt{x}_{d+1}$ is to reach the control state q_f.

In order to show that the reversal-boundedness detection problem is in ExpSpace, it is sufficient to provide a logarithmic-space reduction to the place-boundedness problem for VASS that is known to be in ExpSpace. Indeed, given a VASS $\mathcal{M} = \langle Q, T, C \rangle$ and an initial configuration $\langle q_0, \mathbf{x}_0 \rangle$, one can build a VASS $\mathcal{M}' = \langle Q', T', C \cup \{\mathtt{x}_{d+1}\} \rangle$ with $Q' = Q \times \{\text{DEC}, \text{INC}\}$ such that $\langle \mathcal{M}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ is reversal-bounded with respect to the counter $\mathtt{x}_i$ iff $\langle \mathcal{M}', \langle q_0', \mathbf{x}_0' \rangle \rangle$ is bounded with respect to the counter $\mathtt{x}_{d+1}$ where $q_0' = \langle q_0, \text{INC} \rangle$, $\mathbf{x}_0'$ restricted to the d first counters is $\mathbf{x_0}$ and $\mathbf{x}_0'(d+1) = 0$. In $\mathcal{M}'$, the number of reversals for the counter $\mathtt{x}_i$ is recorded in the value of the counter $\mathtt{x}_{d+1}$. □

2.5 Decidable Repeated Reachability Problems

In this section, we show how to reduce the control state repeated reachability problem to the reachability problem when reversal-bounded counter machines are involved. Let us consider the following problem.

Control state repeated reachability problem with bounded number of reversals:

Input: a counter machine $\mathcal{M}$, a bound $r \in \mathbb{N}$, an initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ and an accepting control state q_f,

Question: Is there an infinite run of $\mathcal{M}$ with initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ such that each counter has at most r reversals and q_f is repeated infinitely often?

Before solving the above problem, let us introduce a simple variant for which we already have decidability.

Control state reachability problem with bounded number of reversals:

Input: a counter machine $\mathcal{M}$, a bound $r \in \mathbb{N}$, an initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ and an accepting control state q_f,

Question: Is there a finite run of $\mathcal{M}$ with initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ such that each counter has at most r reversals and the final control state is q_f?

As a consequence of Theorem 2.10, we get the following result.

Lemma 2.19. *Control state reachability problem with bounded number of reversals is decidable.*

Indeed, reachability sets are computable Presburger sets. Let us take advantage of this to establish Lemma 2.20.

Lemma 2.20. *Control state repeated reachability problem with bounded number of reversals is decidable.*

Proof. Let $\langle \mathcal{M}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ be an initialized counter machine, $r \geq 0$, with $\mathcal{M} = \langle Q, T, C \rangle$ and $q_f \in Q$.

We propose an algorithm to answer the following question: is there an infinite r-reversal-bounded run starting at $\langle q_0, \mathbf{x}_0 \rangle$ such that the control state q_f is repeated infinitely often? We reduce it to an instance of the control state reachability problem with bounded number of reversals, which is decidable by Lemma 2.19. Let $k_{max} \in \mathbb{N}$ denote the maximal constant k occurring in an atomic guard of the form $\mathrm{x} \sim k$ in $\mathcal{M}$.

Let ($\star$) be the desired property:

($\star$) There is an r-reversal-bounded infinite run from $\langle q_0, \mathbf{x}_0 \rangle$ such that q_f is repeated infinitely often.

Let ($\star\star$) be the property below:

($\star\star$) There exist an r-reversal-bounded finite run $\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle \cdots \xrightarrow{t_l} \langle q_l, \mathbf{x}_l \rangle$, $l' \in [0, l-1]$ and $C_= \subseteq C$ such that

(a) $q_l = q_{l'} = q_f$,

(b) for all $\mathrm{x}_i \in C_=$ and $j \in [l'+1, l]$, $\mathbf{x}_{j-1}(i) = \mathbf{x}_j(i)$,

(c) for all $\mathrm{x}_i \in (C \setminus C_=)$ and $j \in [l'+1, l]$, $\mathbf{x}_{j-1}(i) \leq \mathbf{x}_j(i)$,

(d) for all $\mathrm{x}_i \in (C \setminus C_=)$, we have $k_{max} < \mathbf{x}_{l'}(i)$,

(e) for all $\mathrm{x}_i \in C_=$, have $\mathbf{x}_{l'}(i) \leq k_{max}$.

Below, we show that ($\star$) and ($\star\star$) are equivalent, which allows us to reduce control state repeated reachability to control state reachability. Indeed, checking ($\star\star$) amounts to introduce $\mathrm{card}(\mathcal{P}([1, d]))$ copies of $\mathcal{M}$ (one for each subset of C).

First, let us show that ($\star$) and ($\star\star$) are equivalent. Suppose ($\star$). There exists an infinite r-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle \xrightarrow{t_2} \langle q_2, \mathbf{x}_2 \rangle \cdots$ such that q_f is repeated infinitely often. Let $C^\rho_=$ be the subset of C that contains exactly the counters whose values are less or equal to k_{max}, apart from a finite prefix. Since ρ is r-reversal-bounded, there exists $I \geq 0$ such that for some $n \geq I$, no counters in $C \setminus C^\rho_=$ is decremented and their values are strictly greater than k_{max} and all the counters in $C^\rho_=$ have a constant value less or equal to k_{max}. Since q_f is repeated infinitely often, there are $I \leq l' < l$ such that $q_l = q_{l'} = q_f$ and (b)-(e) hold.

Now suppose that there exist an r-reversal-bounded finite run

$$\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle \cdots \xrightarrow{t_l} \langle q_l, \mathbf{x}_l \rangle,$$

$l' \in [0, l-1]$ and $C_= \subseteq C$ witnessing the satisfaction of ($\star\star$). It is then easy to show that the ω-sequence of transitions $t_1 \cdots t_{l'}(t_{l'+1} \cdots t_l)^\omega$ allows us to define an infinite r-reversal-bounded run ρ' that extends ρ. It is clear that in ρ' the control state q_f is repeated infinitely often. Guards on transitions are satisfied by the counter values because of conditions (c),(d) and (e) and values for counters in $(C \setminus C_=)$ are non-negative thanks to (c) and (d).

We construct a reversal-bounded counter machine $\mathcal{M}' = \langle Q', T', C \rangle$ such that ($\star\star$) iff there is a finite r-reversal-bounded run from $\langle q_0, \mathbf{x}_0 \rangle$ that reaches the control state q_{new}. It remains to define the counter machine $\mathcal{M}'$. It is made of the original version of $\mathcal{M}$ (called below the *original copy*) augmented with 2^d copies of $\mathcal{M}$; each copy corresponds to a possible set $C_= \subseteq C$ in ($\star\star$). By the $C_=$*-copy*, we mean the restriction of $\mathcal{M}$ such that:

- no transition in the $C_=$-copy modifies a counter from $C_=$,
- no transition in the $C_=$-copy decrements a counter in $(C \setminus C_=)$.

For each $C_= \subseteq C$, by definition, the control states of the $C_=$-copy are pairs in $Q \times \{C_=\}$. The second component simply indicates to which copy belongs the control state.

In order to simulate the subrun $\langle q_{l'}, \mathbf{x}_{l'} \rangle \cdots \langle q_l, \mathbf{x}_l \rangle$ for the satisfaction of ($\star\star$) in $\mathcal{M}$, we nondeterministically move from the original copy to some $C_=$-copy in $\mathcal{M}'$ (and therefore we choose which counters remain constant below k_{max} forever). To do so, for every set $C_= \subseteq C$, we consider in $\mathcal{M}'$ a transition from q_f to $\langle q_f, C_= \rangle$ whose task is to check that

1. all counters in $C_=$ have values less or equal to k_{max},
2. all counters in $(C \setminus C_=)$ have values strictly greater than k_{max} (and the transition has no effect). Of course, the guard of such a transition is the following one:

$$(\bigwedge_{\mathtt{x} \in (C \setminus C_=)} \mathtt{x} \geq (k_{max} + 1)) \wedge (\bigwedge_{\mathtt{x} \in C_=} \mathtt{x} \leq k_{max}).$$

   Such a transition has also no effect on the number of reversals.

As soon as in the $C_=$-copy, we reach again a control state whose first component is q_f, we may jump to the final control state q_{new}. Note that in $\mathcal{M}'$, it is sufficient to look for a r-reversal-bounded run. □

Theorem 2.21. *The control state repeated reachability problem with bounded number of reversals is NExpTime-complete.*

Proof. NExpTime-hardness follows from the NExpTime-hardness of the reachability problem with bounded number of reversals (Theorem 2.15). In order to obtain the NExpTime upper bound, it is sufficient to consider the proof of Lemma 2.20.

From $\mathcal{M}$, $\langle q_0, \mathbf{x_0} \rangle$, q_f and $r \geq 0$ (instance of size N), we construct a counter machine $\mathcal{M}' = \langle Q', T', C \rangle$ such that the control state q_{new} can be reached from $\langle q_0, \mathbf{x}_0 \rangle$ for some r-reversal-bounded run iff there is an infinite r-reversal-bounded run from $\langle q_0, \mathbf{x}_0 \rangle$ such that q_f is repeated infinitely often. Since $\mathcal{M}'$ essentially restricts the behaviours of $\mathcal{M}$ (by guessing at some stage a set of counters $C_= \subseteq [1, d]$), q_{new} can be reached from $\langle q_0, \mathbf{x}_0 \rangle$ with an r-reversal-bounded run sharing the structural properties of small runs from the proof of Theorem 2.15, whence the NExpTime upper bound. □

Lemma 2.20 can be extended so that, instead of repeating infinitely often control states, properties on counters definable in Presburger arithmetic are repeated infinitely often. Let us introduce the following problem.
∃-Presburger infinitely often problem

Input: Initialized counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ with d counters, $r \geq 0$ and a Presburger formula on counters $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_d)$.

Question: Is there an infinite r-reversal-bounded run from $\langle q, \mathbf{x} \rangle$ such that infinitely often $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_d)$ holds?

The complement of the above problem is defined as follows. The ∀-Presburger-almost-always problem is defined analogously:

Input: Initialized counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ with d counters, $r \geq 0$ and a Presburger formula on counters $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_d)$.

Question: Is it the case that every infinite r-reversal-bounded run from $\langle q, \mathbf{x} \rangle$ satisfies that after some position, all the future positions satisfy $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_d)$?

Theorem 2.22. *The ∃-Presburger infinitely often problem and the ∀-Presburger-almost-always problem are decidable.*

The proof is indeed a generalization of the proof of Lemma 2.20. Exercise 2.6 is dedicated to the proof of Theorem 2.22.

2.6 Weak Reversal-Boundedness

An interesting refinement of reversal-boundedness consists in counting the number of reversals when they occur for a counter value above a given bound B (see Figure 2.3). For instance, finiteness of the reachability set implies reversal-boundedness in that sense, which we shall call *weak reversal-boundedness*. Let $\mathcal{M} = \langle Q, T, C \rangle$ be a counter machine and a bound $B \in \mathbb{N}$. As in Section 2.2.2, from a run $\rho = \langle q_0, \mathbf{x}_0 \rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1 \rangle, \ldots$, we define a sequence of mode vectors $\mathfrak{md}_0, \mathfrak{md}_1, \ldots$ such that each $\mathfrak{md}_i$ belongs to $\{\text{INC}, \text{DEC}\}^d$. Now, let $Rev_i^B \stackrel{\text{def}}{=} \{j \in [0, |\rho| - 1] : \mathfrak{md}_j(i) \neq \mathfrak{md}_{j+1}(i), \{\mathbf{x}_j(i), \mathbf{x}_{j+1}(i)\} \not\subseteq [0, B]\}$.

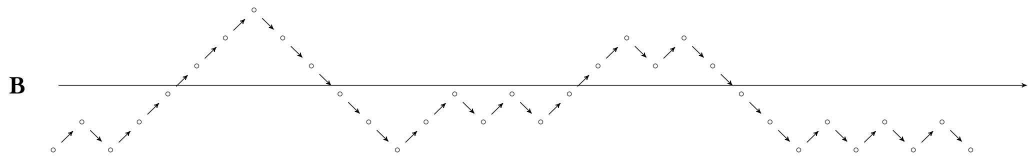

Figure 2.3: A 4-reversal-B-bounded run

Given $B \geq 0$ and $r \geq 0$, the initialized counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is r-*reversal-B-bounded* $\overset{\text{def}}{\Leftrightarrow}$ for every finite run ρ starting at $\langle q, \mathbf{x} \rangle$, $\text{card}(Rev_i^B) \leq r$ for every $i \in [1, d]$. Initialized counter machine $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is *weakly reversal-bounded* $\overset{\text{def}}{\Leftrightarrow}$ there are $r, B \geq 0$ such that $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is r-reversal-B-bounded. Observe that whenever r-reversal-boundedness coincides with r-reversal-0-boundedness. Figure 2.3 illustrates weak reversal-boundedness.

Weak reversal-boundedness for counter machines is very appealing because reachability sets are still semilinear as stated below.

Theorem 2.23. *Let $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ be an initialized counter machine that is weakly r-reversal-B-bounded for some given $r, B \geq 0$. For each control state q', the set $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$ is a computable Presburger set.*

Proof. We show that reachability sets for weak reversal-bounded counter machines can be expressed as reachability sets of reversal-bounded counter machines. Let $\mathcal{M} = \langle Q, T, C \rangle$ be a weak reversal-bounded counter machines with bounds $r, B \geq 0$. Without any loss of generality, we can assume that B is greater or equal to any constant k occurring in an atomic guard $\mathtt{x} \sim k$ and to any absolute value of any $\mathbf{a}(i)$ with $i \in [1, d]$ and $\mathbf{a}$ is an update in $\mathcal{M}$. Indeed, r-reversal-B-boundedness implies r-reversal-B'-boundedness for any $B' > B$.

Let us define the counter machine $\mathcal{M}' = \langle Q', T', C \rangle$ such that $Q' = Q \times [0, B]^d$. In the counter machine $\mathcal{M}'$, we encode in the control states the fact that a counter value is below B. For instance, in a configuration $\langle q, \mathbf{v}, \mathbf{x} \rangle$ with $\mathbf{v}(i) = \alpha < B$, we have $\mathbf{x}(i) = 0$ and the intended counter value from $\mathcal{M}$ is precisely α. In that way, updating the counters below B does not create any reversal since the counter value in $\mathcal{M}'$ remains equal to zero. By contrast, in a configuration $\langle q, \mathbf{v}, \mathbf{x} \rangle$ with $\mathbf{v}(i) = B$, $\mathbf{x}(i)$ can take any value and the intended counter value from $\mathcal{M}$ is precisely $B + \mathbf{x}(i)$. It remains to implement that principle for transitions in T', at the level of updates of course but also at the level of guards.

Given a guard g from $\mathcal{M}$ and $\mathbf{v} \in [0, B]^d$, we write $[g]_{\mathbf{v}}$ to denote the corresponding guard in $\mathcal{M}'$ inductively defined as follows:

- $[\mathtt{x}_i \sim k]_{\mathbf{v}} \overset{\text{def}}{=} \mathbf{v}(i) \sim k$,
- $[\cdot]_{\mathbf{v}}$ is homomorphic for Boolean connectives.

It is easy to check that $[g]_{\mathbf{v}}$ is equivalent either to $\top$ or to $\bot$. Moreover, it is simple to determine whether $[g]_{\mathbf{v}}$ is equivalent to $\top$.

Let us define a map $\mathfrak{f} : (Q \times \mathbb{N}^d) \to ((Q \times [0, B]^d) \times \mathbb{N}^d)$ (between configurations of the two machines $\mathcal{M}$ and $\mathcal{M}'$) such that $\mathfrak{f}(\langle q, \mathbf{x} \rangle) \stackrel{\text{def}}{=} \langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle$ with

1. $q = q'$,
2. for every $i \in [1, d]$, if $\mathbf{x}(i) < B$ then $\mathbf{v}(i) = \mathbf{x}(i)$ and $\mathbf{x}'(i) = 0$, otherwise $\mathbf{x}'(i) = \mathbf{x}(i) - B$ and $\mathbf{v}(i) = B$.

The partial map $\mathfrak{f}^{-1}$ is defined in such a way that $\mathfrak{f}^{-1}(\langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle)$ is defined when there is $\langle q, \mathbf{x} \rangle \in Q \times \mathbb{N}^d$ such that $\mathfrak{f}(\langle q, \mathbf{x} \rangle) = \langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle$. In other words, $\mathfrak{f}^{-1}$ is defined whenever ($\mathbf{x}'(i) > 0$ implies $\mathbf{v}(i) = B$).

It is worth noting that for all $\langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle$ such that $\mathfrak{f}(\langle q, \mathbf{x} \rangle) = \langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle$ and for all guards g in $\mathcal{M}$, we have $\mathbf{x} \models g$ iff $[g]_{\mathbf{v}}$ is equivalent to $\top$. This is the place where we use the assumption that B is greater or equal to any constant k occurring in an atomic guard $\mathtt{x} \sim k$.

For each transition $q \xrightarrow{\langle g, \mathbf{a} \rangle} q'$ in T, we consider all the transitions

$$\langle q, \mathbf{v} \rangle \xrightarrow{\langle g', \mathbf{a}' \rangle} \langle q', \mathbf{v}' \rangle$$

with $g' \stackrel{\text{def}}{=} [g]_{\mathbf{v}}$ (possibly augmented by atomic guards, see below) and verifying the following conditions for all $i \in [1, d]$:

- $\mathbf{v}(i) < B$ and $\mathbf{v}(i) + \mathbf{a}(i) < B$ implies $\mathbf{v}'(i) = \mathbf{v}(i) + \mathbf{a}(i)$ and $\mathbf{a}'(i) = 0$. This is the case when a counter is incremented/decremented in $\mathcal{M}$ but below the bound B. The value for counter $\mathtt{x}_i$ remains equal to zero in $\mathcal{M}'$.
- $\mathbf{v}(i) < B$ and $\mathbf{v}(i) + \mathbf{a}(i) \geq B$ implies $\mathbf{v}'(i) = B$ and $\mathbf{a}'(i) = \mathbf{v}(i) + \mathbf{a}(i) - B$. This is the case when a counter is incremented in $\mathcal{M}$ from a value below the bound B to a value above the bound B. The value for counter $\mathtt{x}_i$ is incremented but not as much as $\mathbf{a}(i)$ because there is an implicit shift of B when $\mathbf{v}'(i) = B$.
- $\mathbf{v}(i) = B$ and $\mathbf{a}(i) \geq 0$ implies $\mathbf{v}'(i) = B$ and $\mathbf{a}'(i) = \mathbf{a}(i)$. This is the case when a counter is incremented in $\mathcal{M}$ from a value above the bound B. The value for counter $\mathtt{x}_i$ is incremented by $\mathbf{a}(i)$ too in $\mathcal{M}'$.
- $\mathbf{v}(i) = B$ and $\mathbf{a}(i) < 0$ and the value for counter $\mathtt{x}_i$ (say $\alpha \in [0, -\mathbf{a}(i) - 1]$) is strictly less than $\mathbf{a}(i)$ imply $\mathbf{v}'(i) = B + \alpha + \mathbf{a}(i)$ and $\mathbf{a}'(i) = -\alpha$. We add $\mathtt{x}_i = \alpha$ to the guard g'. This is the case when a counter is decremented in $\mathcal{M}$ from a value above the bound B to a value below the bound B. This is the place where we use the assumption $-\mathbf{a}(i) \leq B$.
- $\mathbf{v}(i) = B$ and $\mathbf{a}(i) < 0$ and the value for counter $\mathtt{x}_i$ is not strictly less than $-\mathbf{a}(i)$ imply $\mathbf{v}'(i) = B$ and $\mathbf{a}'(i) = \mathbf{a}(i)$. This is the case when a counter is decremented in $\mathcal{M}$ from a value above the bound B to a value above the bound B. Note that there is no need to add $\mathtt{x}_i \geq -\mathbf{a}(i)$ to the guard g'.

First, observe that for every $r' \geq 0$, we have $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is weakly reversal-bounded with respect to r' and B iff $\langle \mathcal{M}', \mathfrak{f}(\langle q, \mathbf{x} \rangle) \rangle$ is r'-reversal-bounded. So, $\langle \mathcal{M}', \mathfrak{f}(\langle q, \mathbf{x} \rangle) \rangle$ is r-reversal-bounded Moreover, for every $q' \in Q$, we have that $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$ is equal to the finite union below

$$\bigcup_{\mathbf{v} \in [0,B]^d} \{\pi_2(\mathfrak{f}^{-1}(\langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle)) : \mathfrak{f}(\langle q, \mathbf{x} \rangle) \xrightarrow{*} \langle \langle q', \mathbf{v} \rangle, \mathbf{x}' \rangle\}$$

where π_2 is the projection on the second component. The proof is by an easy induction on the run lengths. Since $\langle \mathcal{M}', \mathfrak{f}(\langle q, \mathbf{x} \rangle) \rangle$ is r-reversal-bounded, each set occurring in the union is computable and semilinear, and Presburger arithmetic has finite disjunction, whence $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$ is a computable Presburger set (see Theorem 2.10). Indeed, for every $\mathbf{v}$, there is a formula $\varphi_{\mathbf{v}}(\mathsf{y}_1, \ldots, \mathsf{y}_d)$ such that

$$[\![\varphi_{\mathbf{v}}]\!] = \{\mathbf{y} \in \mathbb{N}^d : \mathfrak{f}(\langle q, \mathbf{x} \rangle) \xrightarrow{*} \langle \langle q', \mathbf{v} \rangle, \mathbf{y} \rangle\}.$$

Consequently, the set $\{\mathbf{y} \in \mathbb{N}^d : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{y} \rangle\}$ is characterised by the formula $\varphi(\mathsf{z}_1, \ldots, \mathsf{z}_d)$ defined below:

$$\bigvee_{\mathbf{v}} \exists\, \mathsf{y}_1, \ldots, \mathsf{y}_d \; (\varphi_{\mathbf{v}}(\mathsf{y}_1, \ldots, \mathsf{y}_d) \wedge$$

$$\bigwedge_{i \in [1,d]} ((\mathbf{v}(i) = B \Rightarrow \mathsf{z}_i = \mathsf{y}_i + B) \wedge (\mathbf{v}(i) < B \Rightarrow \mathsf{z}_i = \mathbf{v}(i)))).$$

□

Theorem 2.18 can be adapted to weak reversal-boundedness; reference to the proof can be found in the Bibliographic Notes.

Theorem 2.24. *The weak reversal-boundedness detection problem restricted to vector addition systems with states is* ExpSpace*-complete.*

Exercises

Exercise 2.1. Show that the question of checking whether a counter machine $\mathcal{M}$ is uniformly reversal-bounded can be reduced to reversal-boundedness of an initialized counter machine.

Exercise 2.2. A set $X \subseteq \mathbb{N}^d$ is *downward closed* $\overset{\text{def}}{\Leftrightarrow}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, ($\mathbf{x} \in X$ and $\mathbf{y} \preceq \mathbf{x}$ imply $\mathbf{y} \in X$).

1. Given a VASS $\mathcal{M} = \langle Q, T, C \rangle$ and $q \in Q$, show that the set $\{\mathbf{x} \in \mathbb{N}^d : \langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is reversal-bounded$\}$ is downward closed.

2. Show that the set $\{\mathbf{x} \in \mathbb{N}^d : \langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is reversal-bounded$\}$ is semilinear.

Exercise 2.3. Let us consider the reversal-bounded counter machine in Figure 2.2.

1. Is $\langle \mathcal{M}, \langle q_1, \mathbf{0} \rangle \rangle$ reversal-bounded?
2. For which q, every $\langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is reversal-bounded?
3. Let $\mathbf{x} \in \mathbb{N}^2$ and φ be the Presburger formula

$$\varphi = (\mathsf{x}_1 \geq 2 \wedge \mathsf{x}_2 \geq 1 + \mathbf{x}(2) \wedge (\mathsf{x}_2 - \mathbf{x}(2)) + 1 \geq \mathsf{x}_1) \vee$$

$$(\mathsf{x}_2 \geq 2 \wedge \mathsf{x}_1 \geq 1 + \mathbf{x}(1) \wedge (\mathsf{x}_1 - \mathbf{x}(1)) + 1 \geq \mathsf{x}_2)$$

   Show that $[\![\varphi]\!]$ is equal to $\{\mathbf{y} \in \mathbb{N}^2 : \langle q_1, \mathbf{x} \rangle \xrightarrow{*} \langle q_9, \mathbf{y} \rangle\}$.

4. Find a Presburger formula φ' such that $[\![\varphi']\!] = \{\mathbf{y} \in \mathbb{N}^2 : \langle q_1, \mathbf{0} \rangle \xrightarrow{*} \langle q_6, \mathbf{y} \rangle\}$.
5. Show that for every q, $\{\mathbf{x} \in \mathbb{N}^2 : \langle \mathcal{M}, \langle q, \mathbf{x} \rangle \rangle$ is reversal-bounded$\}$ is semilinear.

Exercise 2.4.

1. Show the properties $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ in Section 2.3.1.
2. Show the property $(\mathcal{P}_3)$ in the proof of Lemma 2.4.

Exercise 2.5. Show that the problem below can be solved in NExpTime:

Input: a counter machine $\mathcal{M}$, a bound $r \in \mathbb{N}$, an initial configuration $\langle q_0, \mathbf{x}_0 \rangle$, $q_f \in Q$ and a linear set $X \subseteq \mathbb{N}^d$ defined by a basis $\mathbf{b}$ and the periods $\mathbf{p}_1, \ldots, \mathbf{p}_N$ (possibly none). All integers are encoded in binary.

Question: Is there a finite run of $\mathcal{M}$ with initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ and final configuration $\langle q_f, \mathbf{x}_f \rangle$ such that each counter has at most r reversals and $\mathbf{x}_f \in X$?

Exercise 2.6. Let $\mathcal{M} = \langle Q, T, C \rangle$ be a counter machine and $\langle q_0, \mathbf{x}_0 \rangle$ be an initial configuration. The goal of this exercise is to show that deciding whether there is an r-reversal-bounded infinite run from $\langle q_0, \mathbf{x}_0 \rangle$ such that counter values belong to a given semilinear set infinitely often can be decided in NExpTime.

1. Let $X \subseteq \mathbb{N}^d$ be a linear set characterised by the basis $\mathbf{b}$ and the periods $\mathbf{p}_1, \ldots, \mathbf{p}_N$ (possibly there are no periods). Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be an infinite sequence of elements in X. Show that there are $l' < l$ and $\mathbf{a}, \mathbf{c} \in \mathbb{N}^N$ such that

   (I) $\mathbf{x}_{l'} \preceq \mathbf{x_l}$,

   (II) $\mathbf{x}_{l'} = \mathbf{b} + \sum_{k \in [1,N]} \mathbf{a}(k)\mathbf{p}_k$ and $\mathbf{x}_l = \mathbf{b} + \sum_{k \in [1,N]} \mathbf{c}(k)\mathbf{p}_k$ and $\mathbf{a} \preceq \mathbf{c}$.

2. Design a uniformly 0-reversal-bounded counter machine with d counters such that for some state $q_0, q_f \in Q$, for all $\mathbf{x} \in \mathbb{N}^d$, we have $\mathbf{x} \in X$ iff there is a run from $\langle q_0, \mathbf{x} \rangle$ to $\langle q_f, \mathbf{0} \rangle$.

3. Design a uniformly 1-reversal-bounded counter machine with $2d$ counters such that for some state $q_0, q_f \in Q$, for all $\mathbf{x} \in \mathbb{N}^{2d}$ such that the restriction to $\mathbf{x}$ to the d last counters equal to $\mathbf{0}$, we have, the restriction of $\mathbf{x}$ to the d first counters belongs to X iff there is a run from $\langle q_0, \mathbf{x}\rangle$ to $\langle q_f, \mathbf{x}\rangle$.

4. Design a uniformly 1-reversal-bounded counter machine with $4d$ counters such that for some state $q_0, q_f \in Q$, for all $\mathbf{x} \in \mathbb{N}^{4d}$ such that the restriction to $\mathbf{x}$ to the $2d$ last counters equal to $\mathbf{0}$, we have, (there are $\lambda_1, \ldots, \lambda_N \in \mathbb{N}$ such that for all $i \in [1, d]$, $\mathbf{x}(d+i) - \mathbf{x}(i) = \lambda_1 \mathbf{p}_1(i) + \cdots \lambda_N \mathbf{p}_N(i)$) iff there is a run from $\langle q_0, \mathbf{x}\rangle$ to $\langle q_f, \mathbf{x}\rangle$.

5. Let $r \geq 0$. Show that the conditions below are equivalent:

   ($\star$) there is an infinite r-reversal-bounded run from $\langle q_0, \mathbf{x}_0\rangle$ such that counter values belong to X infinitely often

   ($\star\star$) There exist a finite r-reversal-bounded run $\rho = \langle q_0, \mathbf{x}_0\rangle \xrightarrow{t_1} \langle q_1, \mathbf{x}_1\rangle \cdots \xrightarrow{t_l} \langle q_l, \mathbf{x}_l\rangle$, $l' \in [0, l-1]$ and $C_= \subseteq C$ such that

   (a) $q_l = q_{l'}$,
   (b) $\mathbf{x_{l'}}, \mathbf{x_l} \in X$,
   (c) (I) and (II) above,
   (d) for all $\mathtt{x}_i \in C_=$ and $j \in [l'+1, l]$, $\mathbf{x}_j(i) - \mathbf{x}_{j-1}(i) = 0$,
   (e) for all $\mathtt{x}_i \in (C \setminus C_=)$ and $j \in [l'+1, l]$, $\mathbf{x}_{j-1}(i) \leq \mathbf{x}_j(i)$,
   (f) for all $\mathtt{x}_i \in (C \setminus C_=)$, we have $k_{max} < \mathbf{x}_{l'}(i)$.
   (g) for all $\mathtt{x}_i \in C_=$, have $\mathbf{x}_{l'}(i) \leq k_{max}$.

   k_{max} denotes the maximal constant k occurring in an atomic guard of the form $\mathtt{x} \sim k$ in $\mathcal{M}$.

6. Design a reduction from ($\star\star$) to an instance of the reachability problem with bounded number of reversals.

7. Show that checking whether an initialized counter machine has an infinite r-reversal-bounded run visiting infinitely often a semilinear set (encoded with bases and periods in binary representation) can be decided in NExpTime.

8. Conclude that the $\exists$-Presburger infinitely often problem is decidable (see Theorem 2.22).

Exercise 2.7. We have seen in Section 1.6 that for any finite-state automaton $\mathcal{A}$ over the alphabet Σ, equipped with a linear ordering of the letters, say with k letters, one can compute a formula $\varphi_{\mathcal{A}}(\mathtt{x}_1, \ldots, \mathtt{x}_k)$ in FO($\mathbb{N}$) such that $\Pi(\mathrm{L}(\mathcal{A})) = [\![\varphi_{\mathcal{A}}]\!]$. Moreover, it is possible to build a quantifier-free formula $\varphi_{\mathcal{A}}$ in polynomial time in the size of $\mathcal{A}$ (see e.g. (Seidl et al., 2004)). By using this standard property and developments from the current chapter, prove that the reachability problem with bounded number of reversals and r encoded in unary is in NP.

Exercise 2.8. Show that the weak reversal-boundedness detection problem restricted to VASS is ExpSpace-hard.

Exercise 2.9. We consider an extended class of counter machines in which equalities and inequalities between counters are allowed (diagonal constraints). Guards on transitions are therefore on the form below:

$$g ::= \mathrm{x} \sim k \mid g \wedge g \mid \neg g \mid \mathrm{x} = \mathrm{x}' \mid \mathrm{x} \neq \mathrm{x}'$$

Show that the reachability problem with bounded number of reversals for this class of counter machines is undecidable even if the number of reversals is zero (Theorem 2.16).

Exercise 2.10. Complete the proof of Proposition 2.6 (Case 3).

Exercise 2.11. A *parameterized counter machine* $\mathcal{M}$ is defined as a counter machine from $\mathcal{C}$ except that atomic guards are either $\top$, $\perp$, $\mathrm{x} \sim k$ (as for counter machines in $\mathcal{C}$) or $\mathrm{x} \sim \mathrm{z}$ where z is a *parameter* from a set PAR. Strictly speaking a parameterized counter machine is not a counter machine unless the parameters take a concrete value. A *concretization map* C for $\mathcal{M}$ is a map C : PAR $\rightarrow \mathbb{N}$. Given a parameterized counter machine $\mathcal{M}$ and a concretization map C, we can easily define a counter machine in $\mathcal{C}$ by interpreting the parameters via the map C. The reachability problem with bounded number of reversals for the class of parameterized counter machines is defined as follows:

Input: a parameterized counter machine $\mathcal{M}$, a bound $r \in \mathbb{N}$, an initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ and a final configuration $\langle q_f, \mathbf{x}_f \rangle$,

Question: Are there a concretization C : PAR $\rightarrow \mathbb{N}$ and a finite run of $\mathcal{M}$ with initial configuration $\langle q_0, \mathbf{x}_0 \rangle$ and final configuration $\langle q_f, \mathbf{x}_f \rangle$ such that each counter has at most r reversals and the parameters in $\mathcal{M}$ are interpreted by C?

1. By adapting the developments from Section 2.3, show that the reachability problem with bounded number of reversals for the class of parameterized counter machines is decidable.
2. What is the complexity of the problem ?

Exercise 2.12. A counter machine $\mathcal{M}$ in the class $\mathcal{C}$ can be viewed as a language acceptor by labelling each transition by a letter a from a finite alphabet Σ, and by defining a set of initial states and a set of final states. An initial configuration contains an initial state and all its counters equal to zero. A final configuration contains a final state. The language of finite words in Σ^* accepted by such augmented counter machines $\mathcal{M}$ is written $\mathrm{L}(\mathcal{M})$. Given $r \geq 0$, we write $\mathrm{L}^r(\mathcal{M})$ to denote the subset of $\mathrm{L}(\mathcal{M})$ such that the accepted words in $\mathrm{L}^r(\mathcal{M})$ are obtained from r-reversal-bounded runs. The *universality problem* consists in checking whether the language $\mathrm{L}^r(\mathcal{M})$ defined by a counter machine is Σ^*. It is known that the problem is PSpace-complete for finite-state automata (i.e. for counter machines without counters).

Let $\mathcal{M}$ be a Minsky machine $\mathcal{M} = \langle Q, T, C \rangle$ with $C = \{\mathrm{x}_1, \mathrm{x}_2\}$, $q_0 \neq q_H \in Q$ and no transition starts at q_H, i.e. q_H is the halting state and q_0 is the initial state. Moreover, the transitions in T have one of the forms below:

- $q_1 \xrightarrow{\mathrm{x}_i\texttt{++}} q_2$ with $i \in [1, 2]$,
- $q_1 \xrightarrow{\mathrm{x}_i = 0} q_2$ and $q_1 \xrightarrow{\mathrm{x}_i\texttt{--}} q_3$ with $i \in [1, 2]$, that is, either counter x_i is decremented or a zero-test is performed on it.

Note that the Minsky machine $\mathcal{M}$ is deterministic. Finite runs of $\mathcal{M}$ are encoded by finite words over the alphabet $\Sigma = Q \uplus \{a, \sharp\}$ so that a counter value n is encoded by the word a^n of length n. More precisely, let ρ be the run below:

$$\langle q_0, n_0, m_0\rangle, \ldots, \langle q_\gamma, n_\gamma, m_\gamma\rangle \in (Q \times \mathbb{N}^2)$$

The word u_ρ in Σ^* is the word below:

$$q_0 \cdot a^{n_0} \cdot \sharp \cdot a^{m_0} \cdot \cdots \cdot q_\gamma \cdot a^{n_\gamma} \cdot \sharp \cdot a^{m_\gamma}$$

Below, let us define necessary conditions on words $u \in \Sigma^*$ for being the encoding of an halting run.

(REG) The word u belongs to the language of the regular expression below:

$$q_0 \cdot \sharp \cdot (Q \cdot a^* \cdot \sharp \cdot a^*)^+ \cdot q_H \cdot a^* \cdot \sharp \cdot a^*$$

(INC$_1$) For every transition of the form $q_1 \xrightarrow{\mathtt{x}_1\texttt{++}} q_2$, if u contains a factor of the form

$$q_1 \cdot a^i \cdot \sharp \cdot a^j \cdot q' \cdot a^{i'} \cdot \sharp \cdot a^{j'}$$

then $q' = q_2$, $i' = i + 1$ and $j' = j$.

(INC$_2$) For every transition of the form $q_1 \xrightarrow{\mathtt{x}_2\texttt{++}} q_2$, if u contains a factor of the form

$$q_1 \cdot a^i \cdot \sharp \cdot a^j \cdot q' \cdot a^{i'} \cdot \sharp \cdot a^{j'}$$

then $q' = q_2$, $i' = i$ and $j' = j + 1$.

(DEC$_1$) For all transitions of the form $q_1 \xrightarrow{\mathtt{x}_1\texttt{=0}} q_2$ and $q_1 \xrightarrow{\mathtt{x}_1\texttt{--}} q_3$ (zero-test on counter $\mathtt{x}_1$), if u contains a factor of the form

$$q_1 \cdot \sharp \cdot a^j \cdot q' \cdot a^{i'} \cdot \sharp \cdot a^{j'}$$

then $q' = q_2$, $i' = 0$ and $j' = j$.

(DEC$_1'$) For all transitions of the form $q_1 \xrightarrow{\mathtt{x}_1\texttt{=0}} q_2$ and $q_1 \xrightarrow{\mathtt{x}_1\texttt{--}} q_3$ (zero-test on counter $\mathtt{x}_1$), if u contains a factor of the form

$$q_1 \cdot a \cdot a^i \cdot \sharp \cdot a^j \cdot q' \cdot a^{i'} \cdot \sharp \cdot a^{j'}$$

then $q' = q_3$, $i' = i$ and $j' = j$.

1. Define conditions (DEC$_2$) and (DEC$_2'$) analogously to the conditions (DEC$_1$), (DEC$_1'$) so that for all words $u \in \Sigma^*$, we have u satisfies the conditions (REG), (INC$_1$), (INC$_2$), (DEC$_1$), (DEC$_1'$), (DEC$_2$), (DEC$_2'$) iff u is the encoding of an halting run for $\mathcal{M}$.
2. For each condition (C) among (REG), (INC$_1$), (INC$_2$), (DEC$_1$), (DEC$_1'$), (DEC$_2$), (DEC$_2'$), build a one-counter machine $\mathcal{M}_{\mathrm{C}}$ with alphabet that is uniformly 1-reversal-bounded and, distinguished states q_0^{C} and q_f^{C} such that for all words $u \in \Sigma^*$, we have u does not satisfy the condition (C) iff there is a run from $\langle q_0^{\mathrm{C}}, 0\rangle$ to some final configuration $\langle q_f^{\mathrm{C}}, n\rangle$ that accepts u.
3. Build a uniformly 1-reversal-bounded one-counter machine $\mathcal{M}'$ with alphabet Σ such that $\mathrm{L}(\mathcal{M}') = \Sigma^*$ iff the Minsky machine $\mathcal{M}$ has no halting run.
4. Conclude that the universality problem with $r = 1$ and restricted to one-counter machines (with alphabet) is undecidable.

Bibliographic Notes

Reversal-bounded counter machines. The class of reversal-bounded counter machines has been introduced and studied in (Ibarra, 1978), partly inspired by similar restrictions on multistack machines (Baker and Book, 1974). Theorem 2.10 that states that every reachability set of an initialized reversal-bounded counter machine is a Presburger set, is shown in (Ibarra, 1978). Reversal-bounded multipushdown machines that extend reversal-bounded counter machines with stacks instead of counters have been also studied in (Baker and Book, 1974; Ibarra, 1978). For instance, developments about multihead pushdown machines recognizing bounded languages (Ibarra, 1974) can lead to semilinearity of reachability sets for initialized reversal-bounded counter machines. Other classes of counter machines with reachability sets that are computable Presburger sets can be found in (Finkel and Sutre, 2000; Leroux and Sutre, 2005; Bozga et al., 2009).

The proof of Theorem 2.10 presented in the chapter relies on developments from (Gurari and Ibarra, 1981) and it uses a proof *à la Rackoff* (Rackoff, 1978) (see Theorem 2.15). Even though the main intentions in (Gurari and Ibarra, 1981) are related to optimal complexity upper bounds, semilinearity of reachability sets can be derived too. It is worth noting that the class of counter machines considered in the chapter is a bit larger than the class considered in (Ibarra, 1978). Indeed, we allow comparisons between a counter value and any constant as well as any Boolean combination and the updates are those from VASS instead of being restricted to updates in $\{-1, 0, +1\}$.

The notion of reversal-boundedness from (Ibarra, 1978) has been also relaxed, for example by allowing a free counter (Howell and Rosier, 1987) (i.e., one counter has no constraints on the number of reversals) or by counting the reversals only above a given bound (Finkel and Sangnier, 2008; Sangnier, 2008). In both cases, semilinearity of the reachability sets is still preserved (see e.g., Section 2.6 about weak reversal-boundedness). The case with one free counter is not treated in the chapter, see e.g. (Ibarra, 1978; Howell and Rosier, 1987) for formal developments. Note that results in (Gurari and Ibarra, 1981) are extended to the case with a single free counter in (Howell and Rosier, 1987). Reversal-bounded counter machines have been also studied as computational devices, see e.g. (Chan, 1981). Moreover, decidable reachability problems for parameterized reversal-bounded initialized counter machines can be found in (Ibarra et al., 2002, Section 4) (see also Exercise 2.11).

Ibarra's proof in a nutshell. The first part of the proof in (Ibarra, 1978) amounts to showing that we can restrict ourselves to 1-reversal-bounded counter machines at the cost of introducing additional counters; this restriction is indeed based on (Baker and Book, 1974) for reversal-bounded multistack machines. Then, the second part shows that reachability sets for 1-reversal-bounded counter machines are computable Presburger sets, essentially based on Parikh's Theorem (Parikh, 1966) restricted to regular languages. Section 1.6 provides the main ingredient of the proof establishing that the commutative image of any regular language is computable and semilinear. Semilinearity is obtained by expressing Birkhoff's equations about control states augmented by connectivity constraints. This analysis allows us to conclude that when a counter machine is uniformly reversal-bounded, then the reachability relation is computable and semilinear.

The problem of deciding whether a counter machine is reversal-bounded has been shown undecidable in (Ibarra, 1978) (see Theorem 2.17).

Complexity of the reachability problem with bounded number of reversals. Even though the reachability problem with bounded number of reversals is not explicitly considered in (Gurari and Ibarra, 1981; Howell and Rosier, 1987), the proof for the lower bound in Theorem 2.15 is due to (Howell and Rosier, 1987) whereas the proof for the upper bound in Theorem 2.15 is due to (Gurari and Ibarra, 1981) (and it uses small solutions of inequation systems as in (Rackoff, 1978)). The lower bound is obtained by encoding computations of a Turing machine running in non-deterministic exponential time by a counter machine, by using ideas similar to (Minsky, 1967). The reachability problem with bounded number of reversals is in NP when the number of reversals r is encoded in unary (Gurari and Ibarra, 1981).

Weak reversal-boundedness. The proof in (Ibarra, 1978) extends to weak reversal-boundedness and Theorem 2.23 is shown in (Finkel and Sangnier, 2008; Sangnier, 2008); whenever a counter value is below B, this information is encoded in the control state which provides a reduction to (standard) reversal-boundedness. Moreover, a breakthrough has been done in (Finkel and Sangnier, 2008) by establishing that checking whether a vector addition systems with states is weakly reversal-bounded is decidable. The decidability proof in (Finkel and Sangnier, 2008) provides a decision procedure that requires nonprimitive recursive time in the worst-case since Karp and Miller tree needs to be built (Karp and Miller, 1969; Valk and Vidal-Naquet, 1981). A complexity analysis can be found in (Demri, 2013).

Reversal-boundedness detection problem. Even though reversal-boundedness detection problem is undecidable in full generality (Ibarra, 1978) (see Theorem 2.17), it is shown in (Finkel and Sangnier, 2008) that the problem is decidable for counter automata without zero-tests, and more generally for vector addition systems with states (by adapting in the obvious way the concept of reversal-boundedness to VASS). More recently, it has been shown that the reversal-boundedness detection problem restricted to VASS is ExpSpace-complete (Demri, 2013).

Beyond reachability. Reversal-bounded counter machines admit decidable problems that go beyong simple reachability. For instance, control state repeated reachability problem with bounded number of reversals and $\exists$-Presburger infinitely often problem have been shown decidable (and NExpTime-complete). Indeed, In (Dang et al., 2001), it is shown that $\exists$-Presburger-infinitely often problem for reversal-bounded counter machines (with guards made of Boolean combinations of the form $\mathsf{x}_i \sim k$) is decidable. Moreover, $\exists$-Presburger-always problem for reversal-bounded counter automata is undecidable (Dang et al., 2001).

The NExpTime upper bound established for the reachability problem with bounded number of reversals can be extended to richer classes of counter systems and to richer specification languages such as LTL with arithmetical constraints (Bersani and Demri, 2011; Hague and Lin, 2011). Note that decidability with both extensions is possible thanks to the introduction a new concept for reversal-boundedness that makes explicit the role of arithmetical terms (Bersani and Demri, 2011) and it captures previous notions on reversal-boundedness. In (Hague and Lin, 2011), operational models extending pushdown systems with counters and clocks are considered; a version of reversal-bounded LTL model-checking is shown to be co-NExpTime (Hague and Lin, 2011, Theorem 2). A prototypical implementation and experimental results are also presented in (Hague and Lin, 2011).

The proof of (Hague and Lin, 2011, Theorem 1) share common features with our proof of Theorem 2.15, at least in the use of counter modes. In both cases Presburger formulae are built: the proof of Theorem 2.15 is based on a run analysis whereas the proof in (Hague and Lin, 2011, Theorem 1) builds directly the formula.

Finally, in (Kopczynski and To, 2010, Theorem 22), ExpTime upper bound for LTL model-checking over reversal-bounded counter automata is shown but the logical language has no arithmetical constraint and the number of reversals r is encoded in unary (see also (To, 2010)).

Extensions. Theorem 2.16, stating that the reachability problem with bounded number of reversals becomes undecidable as soon as the guards are linear constraints, is established in (Ibarra et al., 2002), which seems to destroy any hope to obtain semilinearity for reversal-bounded counter machines with guards comparing counter values. However, a new notion of reversal-boundedness is also introduced in (Bersani and Demri, 2011) so that reversal-boundedness is related to terms occurring in guards and semilinearity and NExpTime-completeness of the reachability problem with bounded number of reversals can be proved.

Universality problem. The undecidability of the universality problem for 1-reversal-bounded one-counter machines with alphabet is shown in (Ibarra, 1979), see also (Greibach, 1969; Baker and Book, 1974). Exercise 2.12 provides a proof by reduction from the halting problem for Minsky machines (instead of from Turing machines as done in (Ibarra, 1979)).