TD: Logical Aspects of Artificial Intelligence
Tableaux for DLs (28/09/2022)

⋆⋆ Exercises related to previous sessions ⋆⋆

Exercise 1. Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be an interpretation (for $\mathcal{ALC}$), $\mathfrak{a} \in \Delta^\mathcal{I}$, and $(\mathfrak{a}, 1), (\mathfrak{a}, 2) \notin \Delta^\mathcal{I}$ (fresh individuals).

1. Given $X \subseteq \Delta^\mathcal{I}$, we write $[X]$ to denote the set below

$$(X \setminus \{\mathfrak{a}\}) \cup \{(\mathfrak{a}, i) \mid i \in \{1, 2\},\ \mathfrak{a} \in X\}$$

   ($[X]$ is obtained from X by replacing $\mathfrak{a}$ by $(\mathfrak{a}, 1), (\mathfrak{a}, 2)$)

   a. Show that X is non-empty iff $[X]$ is non-empty.
   b. Show that $X \subseteq Y$ implies $[X] \subseteq [Y]$.

2. Let $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ be the interpretation defined from $\mathcal{I}$ as follows:

   - $\Delta^\mathcal{J} \stackrel{\text{def}}{=} [\Delta^\mathcal{I}]$.
   - $A^\mathcal{J} \stackrel{\text{def}}{=} [A^\mathcal{I}]$, for all concept names A.
   - For all $\mathfrak{b} \in \Delta^\mathcal{J} \cap \Delta^\mathcal{I}$, $r^\mathcal{J}(\mathfrak{b}) \stackrel{\text{def}}{=} [r^\mathcal{I}(\mathfrak{b})]$, for all role names r.
   - For all $(\mathfrak{a}, i) \in \Delta^\mathcal{J}$ ($i \in \{1, 2\}$), $r^\mathcal{J}((\mathfrak{a}, i)) \stackrel{\text{def}}{=} [r^\mathcal{I}(\mathfrak{a})]$, for all role names r.
   - $\mathcal{I}$ and $\mathcal{J}$ agree on the interpretation of the individual names when the value is in $\Delta^\mathcal{I} \setminus \{\mathfrak{a}\}$.

   Show that for all concepts C, we have $C^\mathcal{J} = [C^\mathcal{I}]$.

3. Prove that if $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is consistent, then there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$ and for all distinct individual names a and b occurring in $\mathcal{A}$, we have $a^\mathcal{I} \neq b^\mathcal{I}$.

Exercise 2. $\mathcal{EL}$ is a fragment of $\mathcal{ALC}$ in which the $\mathcal{EL}$ concept are defined from $C ::= \top \mid A \mid C \sqcap D \mid \exists r.C$. $\mathcal{EL}$ knowledge bases are defined as for $\mathcal{ALC}$ except that only $\mathcal{EL}$ concepts are allowed. Show that every $\mathcal{EL}$ knowledge base is consistent.

Exercise 3. (from exam 2020/2021) An $\mathcal{ALC}$ TBox $\mathcal{T}$ is said to be **simple** if it contains GCIs of the form

$$A \sqsubseteq B \quad A_1 \sqcap A_2 \sqsubseteq B \quad A \sqsubseteq \exists r.B \quad \exists r.A \sqsubseteq B$$

where A, the A_i's and B are arbitrary concept names or $\top$ and r is an arbitrary role name. In the sequel, by convention, we consider that $\top$ is a special concept name (instead of a truth constant) whose interpretation is always the full interpretation domain $\Delta^{\mathcal{I}}$ (assuming that the interpretation is $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$). Given a simple TBox $\mathcal{T}$, we write $\mathcal{S}(\mathcal{T})$ to denote the set of concept and role names occurring in $\mathcal{T}$ with the addition of $\top$ (when $\top$ is not already in $\mathcal{T}$). Below, let us consider a fixed simple TBox $\mathcal{T}$.

1. We introduce rules to deduce new GCIs from $\mathcal{T}$. In the rules below, A, the A_i's and B are concept names in $\mathcal{S}(\mathcal{T})$, the C_i's use only concept names and role names from $\mathcal{S}(\mathcal{T})$ too. We write $\mathcal{T} \vdash C \sqsubseteq D$ when $C \sqsubseteq D$ can be derived from $\mathcal{T}$ by applying the rules below.

   $$\frac{C \sqsubseteq D \in \mathcal{T}}{\mathcal{T} \vdash C \sqsubseteq D} \in\text{-rule} \qquad \frac{}{\mathcal{T} \vdash A \sqsubseteq A} \text{id-rule} \qquad \frac{}{\mathcal{T} \vdash A \sqsubseteq \top} \top\text{-rule}$$

   $$\frac{\mathcal{T} \vdash A \sqsubseteq A_1,\ \ \mathcal{T} \vdash A \sqsubseteq A_2,\ \ \mathcal{T} \vdash A_1 \sqcap A_2 \sqsubseteq B}{\mathcal{T} \vdash A \sqsubseteq B} \sqcap\text{-rule}$$

   $$\frac{\mathcal{T} \vdash A \sqsubseteq \exists r.A_1,\ \ \mathcal{T} \vdash A_1 \sqsubseteq B}{\mathcal{T} \vdash A \sqsubseteq \exists r.B} \exists\text{-rule}$$

   $$\frac{\mathcal{T} \vdash C_1 \sqsubseteq C_2,\ \ \mathcal{T} \vdash C_2 \sqsubseteq C_3}{\mathcal{T} \vdash C_1 \sqsubseteq C_3} \text{trans-rule}$$

   under the condition: $\{C_1 \sqsubseteq C_3\}$ is simple.

   Show that

   $$\{\exists r.B \sqsubseteq B_1, A_1 \sqsubseteq B_2, B_1 \sqcap B_2 \sqsubseteq A_2, A_1 \sqsubseteq \exists r.A_1, \top \sqsubseteq B\} \ \vdash \ A_1 \sqsubseteq A_2.$$

2. A simple TBox $\mathcal{T}^c$ is **complete** $\overset{\text{def}}{\Leftrightarrow}$ for all $C \sqsubseteq D$, $\mathcal{T}^c \vdash C \sqsubseteq D$ implies $C \sqsubseteq D \in \mathcal{T}^c$. Given a simple TBox $\mathcal{T}$, show that there is a smallest complete and simple $\mathcal{T}^c$ such that $\mathcal{T} \subseteq \mathcal{T}^c$ ('smallest' refers to set-inclusion). Moreover, evaluate the time required to compute $\mathcal{T}^c$ from $\mathcal{T}$.

3. Prove that if $\mathcal{T} \vdash C \sqsubseteq D$, then $\mathcal{T}$ and $\mathcal{T} \cup \{C \sqsubseteq D\}$ are satisfied by exactly the same interpretations. Conclude $C \sqsubseteq D \in \mathcal{T}^c$ (with $\mathcal{T}^c$ from Question 2.) implies $\mathcal{T} \models C \sqsubseteq D$.

4. Let $\mathcal{I}$ be the interpretation defined as follows (depending on $\mathcal{T}$ via $\mathcal{T}^c$).

   - $\Delta^{\mathcal{I}}$ is the set of concept names from $\mathcal{S}(\mathcal{T})$ (including $\top$).
   - $A^{\mathcal{I}} \stackrel{\text{def}}{=} \{B \in \Delta^{\mathcal{I}} \mid B \sqsubseteq A \in \mathcal{T}^c\}$ for all concept names A in $\mathcal{S}(\mathcal{T})$.
   - $r^{\mathcal{I}} \stackrel{\text{def}}{=} \{(A, B) \mid A \sqsubseteq \exists r.B \in \mathcal{T}^c\}$ for all role names r in $\mathcal{S}(\mathcal{T})$.

   Verify that $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$. Show that $\mathcal{I} \models \mathcal{T}^c$.

5. Conclude that for all $A, B \in \mathcal{S}(\mathcal{T})$, we have $A \sqsubseteq B \notin \mathcal{T}^c$ implies $\mathcal{T} \not\models A \sqsubseteq B$.

6. Show that given an arbitrary simple TBox $\mathcal{T}$ and $A, B \in \mathcal{S}(\mathcal{T})$, checking whether $\mathcal{T} \models A \sqsubseteq B$ can be done in polynomial time in the size of $\mathcal{T}$.

$\star\star$ Exercises related to today session$\star\star$

Exercise 4. By using the tableaux calculus for $\mathcal{ALC}$ and its properties, show that the concept below is not satisfiable.

$$(\forall r.((\neg A) \sqcup B)) \sqcap (\forall r.A) \sqcap \exists r.\neg B$$

(use the property that $\mathcal{A}$ is consistent iff there is a complete and clash-free ABox $\mathcal{A}'$ derivable from $\mathcal{A}$ with the tableaux rules)

Exercise 5. Show that

$$\{(a, b) : s, (a, c) : r\} \cup \{a : A_1 \sqcap \exists s.A_5, a : \forall s.\neg A_5 \sqcup \neg A_2, b : A_2, c : A_3 \sqcap \exists s.A_4\}$$

is a consistent ABox.

Exercise 6. Using the tableaux calculus for $\mathcal{ALC}$ with blocking, show that the knowledge base $(\mathcal{T}, \mathcal{A})$ below is consistent.

$$\mathcal{T} = \{\top \sqsubseteq (\neg A) \sqcup \exists r.B\} \quad \mathcal{A} = \{a : A \sqcap B, a : \forall r.\forall r.C\}$$

Exercise 7. Complete the case for the $\forall$-rule in the soundness proof of the tableau proof system for $\mathcal{ALC}$ without TBoxes.

Exercise 8. Complete the case for the $\forall$-rule to show that $\mathcal{A}''$ is complete in the soundness proof for the tableau proof system for $\mathcal{ALC}$ with TBoxes (and with blocking technique).

Exercise 9. Complete in the soundness proof for $\mathcal{ALC}$ with TBoxes the property: for all $a : C \in \mathcal{A}''$, we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

Exercise 10. Let us consider the extension of $\mathcal{ALC}$ TBoxes by allowing role axioms of the form $\texttt{Disj}(r, s)$ such that $\mathcal{I} \models \texttt{Disj}(r, s)$ iff $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$. Extend the tableau-style proof system for $\mathcal{ALC}$ knowledge base consistency and prove termination, soundness and completeness.

Exercise 11. (Tableaux with RIAs) Let us consider the extension of $\mathcal{ALC}$ TBoxes by allowing role inclusion axioms $r \sqsubseteq s$. Extend the tableau-style proof system for $\mathcal{ALC}$ knowledge base consistency and prove termination, soundness and completeness.

Exercise 12. Let us propose an alternative definition for blocking that does not rely on the ancestor-relation but rather on the notion of age. Whenever the $\exists$-rule is applied, the **age** of the fresh individual name c is equal to the number of previous applications of the $\exists$-rule plus one. The age of the root individuals is equal to zero. An individual name b in some ABox $\mathcal{A}'$ derived from $\mathcal{A}$ is **blocked by** a if

- the age of a is strictly less that the age of b,
- $\mathsf{con}_{\mathcal{A}'}(b) \subseteq \mathsf{con}_{\mathcal{A}'}(a)$,
- a is not blocked.

Show that the tableau-style proof system for $\mathcal{ALC}$ knowledge base consistency with this notion of blocking provides also a decision procedure.

Exercise 13. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base with

$$\mathcal{T} = \{A \sqsubseteq \neg\forall r.\neg A\} \quad \mathcal{A} = \{a : (A \sqcap (\exists r.B)) \sqcap (\neg\forall r.B)\}$$

1. Using the tableaux calculus for $\mathcal{ALC}$, show that $\mathcal{K}$ is consistent.
2. Based on the derivation from Question 1., define an interpretation satisfying $\mathcal{K}$.

Exercise 14. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base for $\mathcal{ALC}$ with role axioms, with $\mathcal{T} = \mathcal{T}_{GCI} \cup \{r \circ s \equiv s \circ r\}$. $\mathcal{T}_{GCI}$ is made of GCIs. Show that $\mathcal{K}$ is consistent iff $\mathcal{K}' = (\mathcal{T}_{GCI} \cup \mathcal{T}'_{RA}, \mathcal{A})$ is consistent with $\mathcal{T}'_{RA}$ equal to $\{r \circ s \sqsubseteq q, q \sqsubseteq r \circ s, s \circ r \sqsubseteq q, q \sqsubseteq s \circ r\}$ for some new role name q.