TD: Logical Aspects of Artificial Intelligence
Introduction to DLs & Properties (21/09/2022)

$\star\star$ Exercises related to the previous session $\star\star$

Exercise 1. Let $\mathcal{K} = (\mathcal{T} \cup \{A \sqsubseteq C\}, \mathcal{A})$ be a knowledge base such that A is a concept name and B is a concept name that does not occur in $\mathcal{K}$ (B is "new"). Show that $\mathcal{K}$ is consistent iff $\mathcal{K}' = (\mathcal{T} \cup \{A \equiv B \sqcap C\}, \mathcal{A})$ is consistent.

Exercise 2. Let $\mathcal{T}^\star = \{A_1 \equiv C_1, \ldots, A_m \equiv C_m\}$ be an $\mathcal{ALC}$ TBox satisfying the following properties.

- Every A_i is a concept name, and $A_i \equiv C_i$ is an abbreviation for $A_i \sqsubseteq C_i$ and $C_i \sqsubseteq A_i$.
- For all $i, j \in [1, m]$, if A_j occurs in C_i, then $j > i$.
- If $i \neq j \in [1, m]$, then A_i and A_j are syntactically distinct.

Such a TBox $\mathcal{T}^\star$ is called **acyclic**.

1. Briefly define an acyclic graph from $\mathcal{T}^\star$, which would justify the terminology "$\mathcal{T}^\star$ is acyclic".
2. Given an interpretation $\mathcal{I}$, show that there exists an interpretation $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{T}^\star$, the interpretations of the role names and concept names different from $\{A_1, \ldots, A_m\}$ are identical in $\mathcal{I}$ and $\mathcal{J}$.
3. Design an algorithm that takes as input a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with acyclic $\mathcal{T}$ and returns an ABox $\mathcal{A}'$ such that $\mathcal{K}$ is consistent iff $(\emptyset, \mathcal{A}')$ is consistent, and $\mathcal{A}'$ contains no A_i's. The proof for the soundness of the algorithm is not requested.
4. Explain why your algorithm terminates and analyse its computational complexity.

Exercise 3. (Exponential-size interpretations) Define a family of concepts $(C_n)_{n\geq 1}$ such that each C_n is of polynomial size in n (for a fixed polynomial), C_n is satisfiable, and the interpretations satisfying C_n have at least 2^n individuals in its domains.

Exercise 4. (Infinite models) Let $\mathcal{ALCIN}$ be the extension of $\mathcal{ALC}$ with unqualified number restrictions and inverse roles. Let $C = \neg A \sqcap \exists r.A$ and $\mathcal{T} = \{A \sqsubseteq \exists r.A, \top \sqsubseteq (\leq 1\ r^-)\}$. Show that for all interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $C^{\mathcal{I}} \neq \emptyset$ and $\mathcal{I} \models \mathcal{T}$, $\Delta^{\mathcal{I}}$ is infinite.

$\star\star$ Exercises related to today session$\star\star$

Exercise 5. Let us consider the translation map $\mathfrak{t}$ into first-order logic. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation.

1. Let C be a complex concept in $\mathcal{ALC}$. Show that for all $\mathfrak{a} \in \Delta^{\mathcal{I}}$, we have $\mathfrak{a} \in C^{\mathcal{I}}$ iff $\mathcal{I}, \rho[\mathtt{x} \leftarrow \mathfrak{a}] \models \mathfrak{t}(C, \mathtt{x})$ where ρ is a first-order assignment.

2. Show that $\mathcal{I} \models \mathcal{K}$ iff $\mathcal{I} \models \mathfrak{t}(\mathcal{K})$.

Exercise 6. (Model-checking in PTIME) Let $\mathcal{I}$ be an interpretation with finite domain and C be an $\mathcal{ALC}$ concept. Recapitulate the main arguments to show that the algorithm seen in the lecture to compute $C^{\mathcal{I}}$ indeed runs in polynomial time.

Exercise 7. Let X be a finite set of $\mathcal{ALC}$ concepts closed under subconcepts and $\mathcal{K}$ (resp. C) be a knowledge base (resp. a concept) such that $\mathsf{sub}(\mathcal{K}) \cup \mathsf{sub}(C) \subseteq X$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation such that

- $\mathcal{I} \models \mathcal{K}$ and $C^{\mathcal{I}} \neq \emptyset$,

- for all role names r occurring in X, $r^{\mathcal{I}}$ is reflexive and transitive.

For all $\mathfrak{a}, \mathfrak{a}' \in \Delta^{\mathcal{I}}$, we write $\mathfrak{a} \sim \mathfrak{a}'$ iff for all concepts $D \in X$, we have $\mathfrak{a} \in D^{\mathcal{I}}$ iff $\mathfrak{a}' \in D^{\mathcal{I}}$. As $\sim$ is an equivalence relation, equivalence classes of $\sim$ are written $[\mathfrak{a}]$ to denote the class of $\mathfrak{a}$. Let us define the interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$:

- $\Delta^{\mathcal{J}} \stackrel{\text{def}}{=} \{[\mathfrak{a}] \mid \mathfrak{a} \in \Delta^{\mathcal{I}}\}$.

- $A^{\mathcal{J}} \stackrel{\text{def}}{=} \{[\mathfrak{a}] \mid \text{ there is } \mathfrak{a}' \in [\mathfrak{a}] \text{ such that } \mathfrak{a}' \in A^{\mathcal{I}}\}$ for all $A \in X$.

- $A^{\mathcal{J}} \stackrel{\text{def}}{=} \emptyset$ for all concept names $A \notin X$ (arbitrary value).

- $r^{\mathcal{J}} \stackrel{\text{def}}{=} \{([\mathfrak{a}], [\mathfrak{b}]) \mid \text{ there are } \mathfrak{a}' \in [\mathfrak{a}], \mathfrak{b}' \in [\mathfrak{b}] \text{ such that for all } \forall r.D \in X,\ \mathfrak{a}' \in (\forall r.D)^{\mathcal{I}} \text{ implies } \mathfrak{b}' \in (\forall r.D)^{\mathcal{I}}\}$ for all role names r occurring in X.

- $r^{\mathcal{J}} \stackrel{\text{def}}{=} \emptyset$ for all role names r not occurring in X (arbitrary value).

- $a^{\mathcal{J}} \stackrel{\text{def}}{=} [\mathfrak{a}]$ with $a^{\mathcal{I}} = \mathfrak{a}$, for all individual names a.

1. Show that for all role names r occurring in X, $r^{\mathcal{J}}$ is reflexive and transitive.

2. Show that $(\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}$ implies $([\mathfrak{a}], [\mathfrak{b}]) \in r^{\mathcal{J}}$, for all role names r occurring in X.

3. Assuming that the concept constructors occurring in X are among $\forall r$ for some r, $\sqcap$ and $\neg$, show that for all $D \in X$ and $\mathfrak{a} \in \Delta^{\mathcal{I}}$, we have $\mathfrak{a} \in D^{\mathcal{I}}$ iff $[\mathfrak{a}] \in D^{\mathcal{J}}$. (This restriction on the concept constructors allows us to reduce the number of cases in the induction step).

4. Conclude that there is a finite interpretation $\mathcal{I}^\star$ such that $\mathcal{I}^\star \models \mathcal{K}$ and $(C)^{\mathcal{I}^\star} \neq \emptyset$ and for all role names r occurring in X, $(r)^{\mathcal{I}^\star}$ is reflexive and transitive.