Exercise 1. Let $K = (T \cup \{A \sqsubseteq C\}, A)$ be a knowledge base such that $A$ is a concept name and $B$ is a concept name that does not occur in $K$ ($B$ is "new"). Show that $K$ is consistent iff $K' = (T \cup \{A \equiv B \sqcap C\}, A)$ is consistent.

Exercise 2. Let $T^* = \{A_1 \equiv C_1, \ldots, A_m \equiv C_m\}$ be an $ALC$ TBox satisfying the following properties.

- Every $A_i$ is a concept name, and $A_i \equiv C_i$ is an abbreviation for $A_i \sqsubseteq C_i$ and $C_i \sqsubseteq A_i$.
- For all $i, j \in [1, m]$, if $A_j$ occurs in $C_i$, then $j > i$.
- If $i \neq j \in [1, m]$, then $A_i$ and $A_j$ are syntactically distinct.

Such a TBox $T^*$ is called acyclic.

1. Briefly define an acyclic graph from $T^*$, which would justify the terminology "$T^*$ is acyclic".

2. Given an interpretation $\mathcal{I}$, show that there exists an interpretation $\mathcal{J}$ such that $\mathcal{J} \models T^*$, the interpretations of the role names and concept names different from $\{A_1, \ldots, A_m\}$ are identical in $\mathcal{I}$ and $\mathcal{J}$.

3. Design an algorithm that takes as input a knowledge base $\mathcal{K} = (T, A)$ with acyclic $T$ and returns an ABox $A'$ such that $\mathcal{K}$ is consistent iff $(\emptyset, A')$ is consistent, and $A'$ contains no $A_i$'s. The proof for the soundness of the algorithm is not requested.

4. Explain why your algorithm terminates and analyse its computational complexity.
Exercise 3. (Exponential-size interpretations) Define a family of concepts \((C_n)_{n \geq 1}\) such that each \(C_n\) is of polynomial size in \(n\) (for a fixed polynomial), \(C_n\) is satisfiable, and the interpretations satisfying \(C_n\) have at least \(2^n\) individuals in its domains.

Exercise 4. (Infinite models) Let \(ALCIN\) be the extension of \(ALC\) with unqualified number restrictions and inverse roles. Let \(C = \neg A \cap \exists r.A\) and \(T = \{ A \sqsubseteq \exists r.A, \top \sqsubseteq (\leq 1 r^-)\}\). Show that for all interpretations \(I = (\Delta^I, \cdot^I)\) such that \(C^I \neq \emptyset\) and \(I \models T\), \(\Delta^I\) is infinite.

**Exercises related to today session**

Exercise 5. Let us consider the translation map \(t\) into first-order logic. Let \(I = (\Delta^I, \cdot^I)\) be an interpretation.

1. Let \(C\) be a complex concept in \(ALC\). Show that for all \(a \in \Delta^I\), we have \(a \in C^I\) iff \(I, \rho[x \leftarrow a] \models t(C, x)\) where \(\rho\) is a first-order assignment.

2. Show that \(I \models K\) iff \(I \models t(K)\).

Exercise 6. (Model-checking in PTIME) Let \(I\) be an interpretation with finite domain and \(C\) be an \(ALC\) concept. Recapitulate the main arguments to show that the algorithm seen in the lecture to compute \(C^I\) indeed runs in polynomial time.

Exercise 7. Let \(X\) be a finite set of \(ALC\) concepts closed under subconcepts and \(K\) (resp. \(C\)) be a knowledge base (resp. a concept) such that \(\text{sub}(K) \cup \text{sub}(C) \subseteq X\). Let \(I = (\Delta^I, \cdot^I)\) be an interpretation such that

- \(I \models K\) and \(C^I \neq \emptyset\),
- for all role names \(r\) occurring in \(X\), \(r^I\) is reflexive and transitive.

For all \(a, a' \in \Delta^I\), we write \(a \sim a'\) iff for all concepts \(D \in X\), we have \(a \in D^I\) iff \(a' \in D^I\). As \(\sim\) is an equivalence relation, equivalence classes of \(\sim\) are written \([a]\) to denote the class of \(a\). Let us define the interpretation \(J = (\Delta^J, \cdot^J)\):
• $\Delta^T \overset{\text{def}}{=} \{ [a] \mid a \in \Delta^I \}$.

• $A^T \overset{\text{def}}{=} \{ [a] \mid \text{there is } a' \in [a] \text{ such that } a' \in A^I \}$ for all $A \in X$.

• $A^J \overset{\text{def}}{=} \emptyset$ for all concept names $A \notin X$ (arbitrary value).

• $r^J \overset{\text{def}}{=} \{ ([a], [b]) \mid \text{there are } a' \in [a], b' \in [b] \text{ such that for all } \forall r.D \in X, a' \in (\forall r.D)^I \implies b' \in (\forall r.D)^J \}$ for all role names $r$ occurring in $X$.

• $r^J \overset{\text{def}}{=} \emptyset$ for all role names $r$ not occurring in $X$ (arbitrary value).

• $a^J \overset{\text{def}}{=} [a]$ with $a^I = a$, for all individual names $a$.

1. Show that for all role names $r$ occurring in $X$, $r^J$ is reflexive and transitive.

2. Show that $(a, b) \in r^I$ implies $([a], [b]) \in r^J$, for all role names $r$ occurring in $X$.

3. Assuming that the concept constructors occurring in $X$ are among $\forall r$ for some $r$, $\cap$ and $\neg$, show that for all $D \in X$ and $a \in \Delta^I$, we have $a \in D^I$ iff $[a] \in D^J$. (This restriction on the concept constructors allows us to reduce the number of cases in the induction step).

4. Conclude that there is a finite interpretation $I^*$ such that $I^* \models K$ and $(C)^{I^*} \neq \emptyset$ and for all role names $r$ occurring in $X$, $(r)^{I^*}$ is reflexive and transitive.