Exercise 1. Let us consider the CGS $\mathcal{M}$ below with two agents.

$$\begin{array}{c}
\text{p} \\
\downarrow \quad \text{(a, a)} \\
\text{s}_1 \\
\downarrow \quad \text{(c, c)} \\
\text{s}_2 \\
\downarrow \quad \text{(b, a)} \\
\text{q} \\
\end{array}$$

Show that $\mathcal{M}, s_2 \models \langle \langle 1 \rangle \rangle (\text{GF } p \land \text{GF } q)$ and $\mathcal{M}, s_2 \not\models \langle \langle 2 \rangle \rangle (\text{GF } p \land \text{GF } q)$.

Exercise 2. Let $\text{Agt}$ be a fixed non-empty finite set of agents with at least two agents, $A \subseteq \text{Agt}$ be a coalition and $\text{PROP} = \{p_1, p_2, p_3, \ldots\}$ be the countably infinite set of propositional variables on which are built $\text{ATL}^+$ formulae and $\text{ATL}$ formulae. Let us define the family of $\text{ATL}^+$ formulae $(\varphi_n)_{n \geq 1}$ such that

$$\varphi_n \overset{\text{def}}{=} \langle \langle A \rangle \rangle (F p_1 \land \cdots \land F p_n).$$

During the lectures, we have seen that the satisfaction of formulae $\varphi_n$ with $n \geq 2$ may require non-positional strategies for the coalition $A$, and the model-checking problem restricted to the formulae $\varphi_n$’s is PSPACE-hard. More generally, given a finite and non-empty set of propositional variables $X \subseteq \text{PROP}$, we write $\varphi(X)$ to denote the formula $\langle \langle A \rangle \rangle (\bigwedge_{p \in X} F p)$. Consequently, $\varphi(\{p_1, \ldots, p_n\})$ is equal to $\varphi_n$ (modulo associativity and commutativity of the conjunction).

Though the model-checking problem for $\text{ATL}$ is in $\text{PTIME}$ and $\text{ATL}$ semantics can be restricted to positional strategies without modifying the satisfaction relation, we would like to define a family of $\text{ATL}$ formulae $(\psi_n)_{n \geq 1}$ such that for all $n \geq 1$,

- the only coalition in $\psi_n$ is $A$ and the only propositional variable in $\psi_n$ are among $\{p_1, \ldots, p_n\}$,
- for all CGS $\mathcal{M}$ with set of agents $\text{Agt}$, for all states $s$ in $\mathcal{M}$,

$$\mathcal{M}, s \models \varphi_n \iff \mathcal{M}, s \models \psi_n.$$  

1. If $A = \emptyset$, then define a formula $\psi_n$ and show that it satisfies the above properties.

2. Define the formula $\psi_1$.

3. Determine whether $\psi_2$ can take the value

$$\langle \langle A \rangle \rangle (F(p_1 \land \langle \langle A \rangle \rangle F p_2)) \lor \langle \langle A \rangle \rangle (F(p_2 \land \langle \langle A \rangle \rangle F p_1))$$

If not, propose an alternative definition. For your choice of $\psi_2$, show that for all CGS $\mathcal{M}$ with set of agents $\text{Agt}$, for all states $s$ in $\mathcal{M}$, $\mathcal{M}, s \models \varphi_2 \iff \mathcal{M}, s \models \psi_2$. 

In the rest of this exercise, we assume that $A \neq \emptyset$. 

4. Define the formula $\psi_3$.
Exercise 3. Show that $\langle \emptyset \rangle G(\psi \Rightarrow (\varphi \land \langle A \rangle X \psi)) \Rightarrow \langle \emptyset \rangle G(\psi \Rightarrow \langle A \rangle G \varphi)$ is valid in ATL.

Exercise 4. Show that $\langle \langle A \rangle G \varphi \rangle \Rightarrow (\varphi \land \langle A \rangle X \langle A \rangle G \varphi)$ is valid for ATL.

Exercise 5. Let $\mathfrak{M} = \langle Agt, S, Act, \alpha, \delta, L \rangle$ be a concurrent game structure with a (finite) set of states $S$, $s \in S$ and $\varphi = \langle A \rangle (F p_1 \land \cdots \land F p_n)$ (the $p_i$’s are propositional variables) be an ATL* (state) formula such that $\mathfrak{M}, s \models \varphi$.

1. Let $\sigma$ be a strategy for the coalition $A$ such that for all the computations $\lambda \in \text{Comp}(s, \sigma)$, we have $\mathfrak{M}, \lambda \models F p_1 \land \cdots \land F p_n$. The set of computations respecting $\sigma$ can be organised as an infinite tree $t_{\sigma}$ such that the label of each infinite branch encodes a computation in $\text{Comp}(s, \sigma)$ and for each computation $\lambda$ in $\text{Comp}(s, \sigma)$, there is an infinite branch with label encoding $\lambda$. The nodes of such a tree $t_{\sigma}$ have their respective labels in $S \times P\{p_1, \ldots, p_n\}$ as we are interested in the path formula $F p_1 \land \cdots \land F p_n$.

Intuitively, a node labelled by $(r, X)$ corresponds to a (finite) history respecting the strategy $\sigma$ ending in the state $r$ and for which it remains to meet a future state satisfying $p$ for each $p \in X$.

Let $t_{\sigma}$ be the smallest labelled tree (‘smallest’ with respect to set inclusion) defined as follows (the finite alphabet $\Sigma$ is $S \times P\{p_1, \ldots, p_n\}$ to define the labelling $h$).

- $\varepsilon \in t_{\sigma}$ and $h(\varepsilon) = (s_0, X_0)$ with $s_0 \overset{\text{def}}{=} s$ and $X_0 \overset{\text{def}}{=} \{p_1, \ldots, p_n\} \setminus L(s)$.

- Assuming that $\text{out}(s, \sigma(s)) = \{r_1, \ldots, r_{\alpha}\}$ for some $\alpha \geq 1$, we have $0, \ldots, \alpha - 1 \in t_{\sigma}$ and for all $i \in \{0, \ldots, \alpha - 1\}$, $h(i) \overset{\text{def}}{=} (r_{i+1}, X_0 \setminus L(r_{i+1}))$.

$0, \ldots, \alpha - 1$ are therefore the only children of $\varepsilon$.

- For the general case, assume that $u \in t_{\sigma}$ with $u = i_1 \cdots i_k$ for some $k \geq 1$, and the label of the finite branch leading to $u$ is $(s_0, X_0) \cdots (s_k, X_k)$. If $\text{out}(s_k, \sigma(s_0 \cdots s_k)) = \{r_1, \ldots, r_{\alpha}\}$ for some $\alpha \geq 1$, then $u \cdot 0, \ldots, u \cdot (\alpha - 1) \in t_{\sigma}$ and for all $i \in \{0, \ldots, \alpha - 1\}$, $h(u \cdot i) \overset{\text{def}}{=} (r_{i+1}, X_k \setminus L(r_{i+1}))$.

$u \cdot 0, \ldots, u \cdot (\alpha - 1)$ are also the only children of $u$.

Let $i_1 i_2 \cdots$ be an infinite branch of $t_{\sigma}$ with label $(s_0, X_0) \cdot (s_1, X_1) \cdot (s_2, X_2) \cdots$. Show the following properties.
• For all $j \leq j'$, $X_j \supseteq X_{j'}$.
• There is $j \geq 0$ such that $\emptyset = X_j = X_{j+1} = X_{j+2} = X_{j+3} \cdots$.
• $\{X_0, X_1, X_2, \ldots\}$ has at most $(n + 1)$ elements.

2. Let $t^*_\sigma$ be the subset of $t_\sigma$ such that

$$t^*_\sigma = \{\varepsilon\} \cup \{u \cdot i \in t_\sigma \mid b(u) \text{ not of the form } (r, \emptyset)\}.$$ 

Show that $t^*_\sigma$ is a finite tree.

3. Given a computation $\lambda$, we say that $\lambda$ witnesses the satisfaction of $Fp_1 \land \cdots \land Fp_n$ before position $K \in \mathbb{N}$ if for all $i \in [1, n]$, there is $pos_i \leq K$ such that $p_i \in L(\lambda(pos_i))$. Show that there is a strategy $\sigma$ for the coalition $A$ such that for all computations $\lambda \in \text{Comp}(s, \sigma)$,

(a) $M, \lambda \models Fp_1 \land \cdots \land Fp_n$ and,
(b) $\lambda$ witnesses the satisfaction of $Fp_1 \land \cdots \land Fp_n$ before position $(n + 1) \times \text{card}(S)$.

4. Let us consider the CGS $M^*$ below (with two agents in $\{1, 2\}$)

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(a, a)  (a, a)  (a, a)
\downarrow   \downarrow   \downarrow
s_1   s_2   s_3
p_1 \rightarrow   \rightarrow   \rightarrow
(a, b)   (b, a)
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(a) Show that $M^*, s_1 \models \langle \langle \{1\} \rangle \rangle(Gp_1 \lor Fp_2)$.

(b) Show that there is no strategy $\sigma$ for the agent 1 such that there is $B \geq 1$ for which for all computations $\lambda \in \text{Comp}(s_1, \sigma)$,

i. $M^*, \lambda \models Gp_1 \lor Fp_2$ and,
ii. if $M^*, \lambda \models Fp_2$ then $\lambda$ witnesses the satisfaction of $Fp_2$ before position $B$.

**Exercise 6.** Let $\models_{pos}$ be the satisfaction relation for ATL formulae when only positional strategies are permitted to witness the satisfaction of formulae whose outermost connective is a strategy modality. Show that $\models_{pos}$ is equal to $\models$ for ATL.