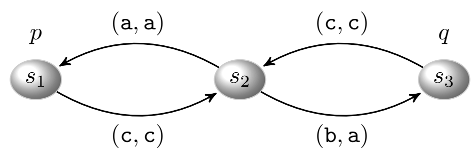
TD: Logical Aspects of Artificial Intelligence
Introduction to ATL-like logics (II) (12/10/2022)

**Exercise 1**. Let us consider the CGS $\mathfrak{M}$ below with two agents.




Show that $\mathfrak{M}, s_2 \models \langle\!\langle 1 \rangle\!\rangle\ (\mathsf{GF}\ p \wedge \mathsf{GF}\ q)$ and $\mathfrak{M}, s_2 \not\models \langle\!\langle 2 \rangle\!\rangle\ (\mathsf{GF}\ p \wedge \mathsf{GF}\ q)$.

**Exercise 2**. Let $Agt$ be a fixed non-empty finite set of agents with at least two agents, $A \subseteq Agt$ be a coalition and $\mathrm{PROP} = \{p_1, p_2, p_3, \ldots\}$ be the countably infinite set of propositional variables on which are built $\mathsf{ATL}^+$ formulae and $\mathsf{ATL}$ formulae. Let us define the family of $\mathsf{ATL}^+$ formulae $(\varphi_n)_{n \geq 1}$ such that

$$\varphi_n \stackrel{\text{def}}{=} \langle\!\langle A \rangle\!\rangle(\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n).$$

During the lectures, we have seen that the satisfaction of formulae $\varphi_n$ with $n \geq 2$ may require non-positional strategies for the coalition $A$, and the model-checking problem restricted to the formulae $\varphi_n$'s is PSPACE-hard. More generally, given a finite and non-empty set of propositional variables $X \subseteq \mathrm{PROP}$, we write $\varphi(X)$ to denote the formula $\langle\!\langle A \rangle\!\rangle(\bigwedge_{p \in X} \mathsf{F}p)$. Consequently, $\varphi(\{p_1, \ldots, p_n\})$ is equal to $\varphi_n$ (modulo associativity and commutativity of the conjunction). Though the model-checking problem for $\mathsf{ATL}$ is in PTIME and $\mathsf{ATL}$ semantics can be restricted to positional strategies without modifying the satisfaction relation, we would like to define a family of $\mathsf{ATL}$ formulae $(\psi_n)_{n \geq 1}$ such that for all $n \geq 1$,

- the only coalition in $\psi_n$ is $A$ and the only propositional variable in $\psi_n$ are among $\{p_1, \ldots, p_n\}$,
- for all CGS $\mathfrak{M}$ with set of agents $Agt$, for all states $s$ in $\mathfrak{M}$,

$$\mathfrak{M}, s \models \varphi_n \text{ iff } \mathfrak{M}, s \models \psi_n.$$

1. If $A = \emptyset$, then define a formula $\psi_n$ and show that it satisfies the above properties.

   ***In the rest of this exercise, we assume that*** $A \neq \emptyset$.

2. Define the formula $\psi_1$.
3. Determine whether $\psi_2$ can take the value

$$\langle\!\langle A \rangle\!\rangle(\mathsf{F}(p_1 \wedge \langle\!\langle A \rangle\!\rangle \mathsf{F}p_2)) \vee \langle\!\langle A \rangle\!\rangle(\mathsf{F}(p_2 \wedge \langle\!\langle A \rangle\!\rangle \mathsf{F}p_1))$$

   If not, propose an alternative definition. For your choice of $\psi_2$, show that for all CGS $\mathfrak{M}$ with set of agents $Agt$, for all states $s$ in $\mathfrak{M}$, $\mathfrak{M}, s \models \varphi_2$ iff $\mathfrak{M}, s \models \psi_2$.

4. Propose a definition for the formulae in the family $(\psi_n)_{n\geq 1}$ (no correctness proof is requested but explanations are welcome) and evaluate the size of $\psi_n$ with respect to $n$.

**Exercise 3**. Show that $\langle\!\langle \emptyset \rangle\!\rangle \mathsf{G}(\psi \Rightarrow (\varphi \wedge \langle\!\langle A \rangle\!\rangle \mathsf{X}\psi)) \Rightarrow \langle\!\langle \emptyset \rangle\!\rangle \mathsf{G}(\psi \Rightarrow \langle\!\langle A \rangle\!\rangle \mathsf{G}\varphi)$ is valid in ATL.

**Exercise 4**. Show that $(\langle\!\langle A \rangle\!\rangle \mathsf{G}\varphi) \Rightarrow (\varphi \wedge \langle\!\langle A \rangle\!\rangle \mathsf{X} \langle\!\langle A \rangle\!\rangle \mathsf{G}\varphi)$ is valid for ATL.

**Exercise 5**. Let $\mathfrak{M} = (Agt, S, Act, \mathtt{act}, \delta, L)$ be a concurrent game structure with a (finite) set of states $S$, $s \in S$ and $\varphi = \langle\!\langle A \rangle\!\rangle (\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n)$ (the $p_i$'s are propositional variables) be an ATL$^*$ (state) formula such that $\mathfrak{M}, s \models \varphi$.

1. Let $\sigma$ be a strategy for the coalition $A$ such that for all the computations $\lambda \in \mathtt{Comp}(s, \sigma)$, we have $\mathfrak{M}, \lambda \models \mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$. The set of computations respecting $\sigma$ can be organised as an infinite tree $\mathbf{t}_\sigma$ such that the label of each infinite branch encodes a computation in $\mathtt{Comp}(s, \sigma)$ and for each computation $\lambda$ in $\mathtt{Comp}(s, \sigma)$, there is an infinite branch with label encoding $\lambda$. The nodes of such a tree $\mathbf{t}_\sigma$ have their respective labels in $S \times \mathcal{P}(\{p_1, \ldots, p_n\})$ as we are interested in the path formula $\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$. Intuitively, a node labelled by $(r, X)$ corresponds to a (finite) history respecting the strategy $\sigma$ ending in the state $r$ and for which it remains to meet a future state satisfying $p$ for each $p \in X$.

   Let $\mathbf{t}_\sigma$ be the smallest labelled tree ('smallest' with respect to set inclusion) defined as follows (the finite alphabet $\Sigma$ is $S \times \mathcal{P}(\{p_1, \ldots, p_n\})$ to define the labelling $\mathfrak{h}$).

   - $\varepsilon \in \mathbf{t}_\sigma$ and $\mathfrak{h}(\varepsilon) = (s_0, X_0)$ with
   $$s_0 \stackrel{\text{def}}{=} s \text{ and } X_0 \stackrel{\text{def}}{=} \{p_1, \ldots, p_n\} \setminus L(s).$$
   - Assuming that $\mathtt{out}(s, \sigma(s)) = \{r_1, \ldots, r_\alpha\}$ for some $\alpha \geq 1$, we have $0, \ldots, \alpha - 1 \in \mathbf{t}_\sigma$ and for all $i \in \{0, \ldots, \alpha - 1\}$,
   $$\mathfrak{h}(i) \stackrel{\text{def}}{=} (r_{i+1}, X_0 \setminus L(r_{i+1})).$$
   $0, \ldots, \alpha - 1$ are therefore the only children of $\varepsilon$.
   - For the general case, assume that $u \in \mathbf{t}_\sigma$ with $u = i_1 \cdots i_k$ for some $k \geq 1$, and the label of the finite branch leading to $u$ is $(s_0, X_0) \cdots (s_k, X_k)$. If $\mathtt{out}(s_k, \sigma(s_0 \cdots s_k)) = \{r_1, \ldots, r_\alpha\}$ for some $\alpha \geq 1$, then $u \cdot 0, \ldots, u \cdot (\alpha - 1) \in \mathbf{t}_\sigma$ and for all $i \in \{0, \ldots, \alpha - 1\}$,
   $$\mathfrak{h}(u \cdot i) \stackrel{\text{def}}{=} (r_{i+1}, X_k \setminus L(r_{i+1})).$$
   $u \cdot 0, \ldots, u \cdot (\alpha - 1)$ are also the only children of $u$.

   Let $i_1 i_2 \cdots$ be an infinite branch of $\mathbf{t}_\sigma$ with label $(s_0, X_0) \cdot (s_1, X_1) \cdot (s_2, X_2) \cdots$. Show the following properties.

- For all $j \leq j'$, $X_j \supseteq X_{j'}$.
- There is $j \geq 0$ such that $\emptyset = X_j = X_{j+1} = X_{j+2} = X_{j+3} \cdots$.
- $\{X_0, X_1, X_2, \ldots\}$ has at most $(n+1)$ elements.

2. Let $\mathbf{t}_\sigma^\star$ be the subset of $\mathbf{t}_\sigma$ such that

$$\mathbf{t}_\sigma^\star = \{\varepsilon\} \cup \{u \cdot i \in \mathbf{t}_\sigma \mid \mathfrak{h}(u) \text{ not of the form } (r, \emptyset)\}.$$

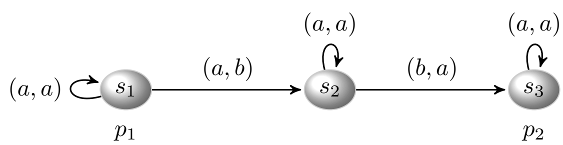
Show that $\mathbf{t}_\sigma^\star$ is a finite tree.

3. Given a computation $\lambda$, we say that $\lambda$ **witnesses the satisfaction of** $\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$ **before position** $K \in \mathbb{N} \overset{\text{def}}{\Leftrightarrow}$ for all $i \in [1, n]$, there is $pos_i \leq K$ such that $p_i \in L(\lambda(pos_i))$. Show that there is a strategy $\sigma$ for the coalition $A$ such that for all computations $\lambda \in \mathtt{Comp}(s, \sigma)$,

   (a) $\mathfrak{M}, \lambda \models \mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$ and,

   (b) $\lambda$ witnesses the satisfaction of $\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$ before position $(n+1) \times \mathsf{card}(S)$.

4. Let us consider the CGS $\mathfrak{M}^\star$ below (with two agents in $\{1, 2\}$)




   (a) Show that $\mathfrak{M}^\star, s_1 \models \langle\!\langle \{1\} \rangle\!\rangle (\mathsf{G}p_1 \vee \mathsf{F}p_2)$.

   (b) Show that there is no strategy $\sigma$ for the agent $1$ such that there is $B \geq 1$ for which for all computations $\lambda \in \mathtt{Comp}(s_1, \sigma)$,

      i. $\mathfrak{M}^\star, \lambda \models \mathsf{G}p_1 \vee \mathsf{F}p_2$ and,

      ii. if $\mathfrak{M}^\star, \lambda \models \mathsf{F}p_2$ then $\lambda$ witnesses the satisfaction of $\mathsf{F}p_2$ before position $B$.

**Exercise 6**. Let $\models_{pos}$ be the satisfaction relation for ATL formulae when only positional strategies are permitted to witness the satisfaction of formulae whose outermost connective is a strategy modality. Show that $\models_{pos}$ is equal to $\models$ for ATL.