Logical Aspects of AI: Knowledge Logics
Exam – November 10th, 2021, 14h00–17h00, Room 1Z53

This exam takes place between 2pm and 5pm. This exam document has five pages and contains four independent exercises. Every student present at the exam in room 1Z53 should complete the timesheet.

Authorised documents. The only authorised documents are the slides, the exercise sheets as well as the correction sheets provided during the sessions. No communication between the students is allowed during the exam time (2pm to 5pm) and no communication is allowed with any other person about the content of the exam.

Format of your copy. As usual, please write down the solutions of the exercises with very much care, provide clearly all the necessary arguments and details. Your copy can be written in French or in English.

Reminder. Please find below material that might become useful.

- Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers, the empty sequence is denoted ε. A **tree** $\mathfrak{T}$ is a subset of $\mathbb{N}^*$ such that ($u \in \mathbb{N}^*$ and $i \in \mathbb{N}$)
  - $u \cdot i \in \mathfrak{T}$ implies $u \in \mathfrak{T}$,
  - $u \cdot (i+1) \in \mathfrak{T}$ implies $u \cdot i \in \mathfrak{T}$.

  Elements of $\mathfrak{T}$ are also called **nodes**. A **labelled tree** is defined as a tree $\mathfrak{T}$ equipped with a map $\mathfrak{h} : \mathfrak{T} \to \Sigma$ where Σ is a (finite) alphabet. Given $u \in \mathfrak{T}$, the **label** of the branch leading to u is $\mathfrak{h}(\varepsilon) \cdot \mathfrak{h}(i_1) \cdot \mathfrak{h}(i_1 i_2) \cdots \mathfrak{h}(i_1 \cdots i_k)$ if $u = i_1 \cdots i_k$. An infinite branch in $\mathfrak{T}$ is an infinite sequence $i_1 i_2 \cdots \in \mathbb{N}^\omega$ in $\mathfrak{T}$ with **label** $\mathfrak{h}(\varepsilon) \cdot \mathfrak{h}(i_1) \cdot \mathfrak{h}(i_1 i_2) \cdots$. Labels for finite branches are defined similarly. A tree $\mathfrak{T}$ is **finite-branching** if for all $u \in \mathfrak{T}$ there is $i \in \mathbb{N}$ such that $u \cdot i \notin \mathfrak{T}$.

- We recall that König's Lemma states that every infinite finite-branching tree has an infinite branch.

Exercise 1. Given a concurrent game structure $\mathfrak{M} = (Agt, S, Act, \texttt{act}, \delta, L)$, coalitions $A \subseteq A' \subseteq Agt$ and a state $s \in S$, show that

$$\mathfrak{M}, s \models (\langle\!\langle \emptyset \rangle\!\rangle\, \mathsf{X} p \wedge \langle\!\langle A \rangle\!\rangle\, \mathsf{X} q) \Rightarrow \langle\!\langle A' \rangle\!\rangle\, \mathsf{X}(p \wedge q).$$

Exercise 2. Let X be a non-empty set with a distinguished element $x_0 \in X$ and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation for $\mathcal{ALC}$. Let $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be the interpretation defined as follows.

- $\Delta^{\mathcal{J}} \stackrel{\text{def}}{=} X \times \Delta^{\mathcal{I}}$.
- $A^{\mathcal{J}} \stackrel{\text{def}}{=} X \times A^{\mathcal{I}}$ for every concept name A.
- $r^{\mathcal{J}} \stackrel{\text{def}}{=} \{((x, \mathfrak{a}), (x, \mathfrak{b})) \mid\ x \in X,\ (\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}\}$ for every role name r.
- $a^{\mathcal{J}} \stackrel{\text{def}}{=} (x_0, \mathfrak{a})$ with $a^{\mathcal{I}} = \mathfrak{a}$, for every individual name a.

1. For all $\mathcal{ALC}$ concepts C, show that $C^{\mathcal{J}} = X \times C^{\mathcal{I}}$.
2. Given a knowledge base $\mathcal{K}$, show that $\mathcal{I} \models \mathcal{K}$ implies $\mathcal{J} \models \mathcal{K}$.
3. Conclude that there is no consistent $\mathcal{ALC}$ knowledge base $\mathcal{K}$ such that for all interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, $\mathcal{I} \models \mathcal{K}$ implies $\Delta^{\mathcal{I}}$ is finite.

Exercise 3. Let X be a finite set of $\mathcal{ALC}$ concepts closed under subconcepts and $\mathcal{K}$ (resp. C) be a knowledge base (resp. a concept) such that $\mathsf{sub}(\mathcal{K}) \cup \mathsf{sub}(C) \subseteq X$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation such that

- $\mathcal{I} \models \mathcal{K}$ and $C^{\mathcal{I}} \neq \emptyset$,
- for all role names r occurring in X, $r^{\mathcal{I}}$ is reflexive and transitive.

For all $\mathfrak{a}, \mathfrak{a}' \in \Delta^{\mathcal{I}}$, we write $\mathfrak{a} \sim \mathfrak{a}'$ iff for all concepts $D \in X$, we have $\mathfrak{a} \in D^{\mathcal{I}}$ iff $\mathfrak{a}' \in D^{\mathcal{I}}$. As $\sim$ is an equivalence relation, equivalence classes of $\sim$ are written $[\mathfrak{a}]$ to denote the class of $\mathfrak{a}$. Let us define the interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$:

- $\Delta^{\mathcal{J}} \stackrel{\text{def}}{=} \{[\mathfrak{a}] \mid \mathfrak{a} \in \Delta^{\mathcal{I}}\}$.
- $A^{\mathcal{J}} \stackrel{\text{def}}{=} \{[\mathfrak{a}] \mid\ \text{there is } \mathfrak{a}' \in [\mathfrak{a}] \text{ such that } \mathfrak{a}' \in A^{\mathcal{I}}\}$ for all $A \in X$.

- $A^{\mathcal{J}} \stackrel{\text{def}}{=} \emptyset$ for all concept names $A \notin X$ (arbitrary value).
- $r^{\mathcal{J}} \stackrel{\text{def}}{=} \{([\mathfrak{a}], [\mathfrak{b}]) \mid \text{ there are } \mathfrak{a}' \in [\mathfrak{a}], \mathfrak{b}' \in [\mathfrak{b}] \text{ such that for all } \forall r.D \in X,\ \mathfrak{a}' \in (\forall r.D)^{\mathcal{I}} \text{ implies } \mathfrak{b}' \in (\forall r.D)^{\mathcal{I}}\}$ for all role names r occurring in X.
- $r^{\mathcal{J}} \stackrel{\text{def}}{=} \emptyset$ for all role names r not occurring in X (arbitrary value).
- $a^{\mathcal{J}} \stackrel{\text{def}}{=} [\mathfrak{a}]$ with $a^{\mathcal{I}} = \mathfrak{a}$, for all individual names a.

1. Show that for all role names r occurring in X, $r^{\mathcal{J}}$ is reflexive and transitive.
2. Show that $(\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}$ implies $([\mathfrak{a}], [\mathfrak{b}]) \in r^{\mathcal{J}}$, for all role names r occurring in X.
3. Assuming that the concept constructors occurring in X are among $\forall r$ for some r, $\sqcap$ and $\neg$, show that for all $D \in X$ and $\mathfrak{a} \in \Delta^{\mathcal{I}}$, we have $\mathfrak{a} \in D^{\mathcal{I}}$ iff $[\mathfrak{a}] \in D^{\mathcal{J}}$. (This restriction on the concept constructors allows us to reduce the number of cases in the induction step).
4. Conclude that there is a finite interpretation $\mathcal{I}^\star$ such that $\mathcal{I}^\star \models \mathcal{K}$ and $(C)^{\mathcal{I}^\star} \neq \emptyset$ and for all role names r occurring in X, $(r)^{\mathcal{I}^\star}$ is reflexive and transitive.

Exercise 4. Let $\mathfrak{M} = (Agt, S, Act, \mathtt{act}, \delta, L)$ be a concurrent game structure with a (finite) set of states S, $s \in S$ and $\varphi = \langle\!\langle A \rangle\!\rangle(\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n)$ (the p_i's are propositional variables) be an ATL^* (state) formula such that $\mathfrak{M}, s \models \varphi$.

1. Let F be a strategy for the coalition A such that for all the computations $\lambda \in \mathtt{Comp}(s, F)$, we have $\mathfrak{M}, \lambda \models \mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$. The set of computations respecting F can be organised as an infinite tree $\mathfrak{T}_F$ such that the label of each infinite branch encodes a computation in $\mathtt{Comp}(s, F)$ and for each computation λ in $\mathtt{Comp}(s, F)$, there is an infinite branch with label encoding λ. The nodes of such a tree $\mathfrak{T}_F$ have their respective labels in $S \times \mathcal{P}(\{p_1, \ldots, p_n\})$ as we are interested in the path formula $\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$. Intuitively, a node labelled by (r, X) corresponds to a (finite) history respecting the strategy F ending in

the state r and for which it remains to meet a future state satisfying p for each $p \in X$.

Let $\mathfrak{T}_F$ be the smallest labelled tree ('smallest' with respect to set inclusion) defined as follows (the finite alphabet Σ is $S \times \mathcal{P}(\{p_1, \ldots, p_n\})$ to define the labelling $\mathfrak{h}$).

- $\varepsilon \in \mathfrak{T}_F$ and $\mathfrak{h}(\varepsilon) = (s_0, X_0)$ with

$$s_0 \stackrel{\text{def}}{=} s \text{ and } X_0 \stackrel{\text{def}}{=} \{p_1, \ldots, p_n\} \setminus L(s).$$

- Assuming that $\mathtt{out}(s, F(s)) = \{r_1, \ldots, r_\alpha\}$ for some $\alpha \geq 1$, we have $0, \ldots, \alpha - 1 \in \mathfrak{T}_F$ and for all $i \in \{0, \ldots, \alpha - 1\}$,

$$\mathfrak{h}(i) \stackrel{\text{def}}{=} (r_{i+1}, X_0 \setminus L(r_{i+1})).$$

  $0, \ldots, \alpha - 1$ are therefore the only children of ε.

- Generally, assume that $u \in \mathfrak{T}_F$ with $u = i_1 \cdots i_k$ for some $k \geq 1$, and the label of the finite branch leading to u is $(s_0, X_0) \cdots (s_k, X_k)$. If $\mathtt{out}(s_k, F(s_0 \cdots s_k)) = \{r_1, \ldots, r_\alpha\}$ for some $\alpha \geq 1$, then $u \cdot 0, \ldots, u \cdot (\alpha - 1) \in \mathfrak{T}_F$ and for all $i \in \{0, \ldots, \alpha - 1\}$,

$$\mathfrak{h}(u \cdot i) \stackrel{\text{def}}{=} (r_{i+1}, X_k \setminus L(r_{i+1})).$$

  $u \cdot 0, \ldots, u \cdot (\alpha - 1)$ are also the only children of u.

Let $i_1 i_2 \cdots$ be an infinite branch of $\mathfrak{T}_F$ with label $(s_0, X_0) \cdot (s_1, X_1) \cdot (s_2, X_2) \cdots$. Show the following properties.

- For all $j \leq j'$, $X_j \supseteq X_{j'}$.
- There is $j \geq 0$ such that $\emptyset = X_j = X_{j+1} = X_{j+2} = X_{j+3} \cdots$.
- $\{X_0, X_1, X_2, \ldots\}$ has at most $(n+1)$ elements.

2. Let $\mathfrak{T}_F^{\star}$ be the subset of $\mathfrak{T}_F$ such that

$$\mathfrak{T}_F^{\star} = \{\varepsilon\} \cup \{u \cdot i \in \mathfrak{T}_F \mid \mathfrak{h}(u) \text{ not of the form } (r, \emptyset)\}.$$

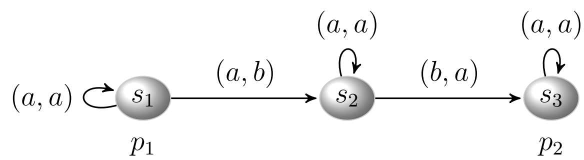
   Show that $\mathfrak{T}_F^{\star}$ is a finite tree.

3. Given a computation λ, we say that λ **witnesses the satisfaction of** $\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$ **before position** $K \stackrel{\text{def}}{\Leftrightarrow}$ for all $i \in [1, n]$, there is $pos_i \leq K$ such that $p_i \in L(\lambda(pos_i))$. Show that there is a strategy F for the coalition A such that for all computations $\lambda \in \mathtt{Comp}(s, F)$,

(a) $\mathfrak{M}, \lambda \models \mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$ and,

(b) λ witnesses the satisfaction of $\mathsf{F}p_1 \wedge \cdots \wedge \mathsf{F}p_n$ before position $(n+1) \times \mathsf{card}(S)$.

4. Let us consider the CGS $\mathfrak{M}^\star$ below (with two agents in $\{1, 2\}$)



(a) Show that $\mathfrak{M}^\star, s_1 \models \langle\!\langle \{1\} \rangle\!\rangle(\mathsf{G}p_1 \vee \mathsf{F}p_2)$.

(b) Show that there is no strategy F for the agent 1 such that there is $B \geq 1$ for which for all computations $\lambda \in \texttt{Comp}(s_1, F)$,

i. $\mathfrak{M}^\star, \lambda \models \mathsf{G}p_1 \vee \mathsf{F}p_2$ and,

ii. if $\mathfrak{M}^\star, \lambda \models \mathsf{F}p_2$ then λ witnesses the satisfaction of $\mathsf{F}p_2$ before position B.