Logical Aspects of AI: Knowledge Logics
Exam – November 10th, 2021, 14h00–17h00, Room 1Z53

This exam takes place between 2pm and 5pm. This exam document has five pages and contains four independent exercises. Every student present at the exam in room 1Z53 should complete the timesheet.

Authorised documents. The only authorised documents are the slides, the exercise sheets as well as the correction sheets provided during the sessions. No communication between the students is allowed during the exam time (2pm to 5pm) and no communication is allowed with any other person about the content of the exam.

Format of your copy. As usual, please write down the solutions of the exercises with very much care, provide clearly all the necessary arguments and details. Your copy can be written in French or in English.

Reminder. Please find below material that might become useful.

- Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers, the empty sequence is denoted $\varepsilon$. A tree $\mathcal{T}$ is a subset of $\mathbb{N}^*$ such that ($u \in \mathbb{N}^*$ and $i \in \mathbb{N}$)
  
  - $u \cdot i \in \mathcal{T}$ implies $u \in \mathcal{T}$,
  
  - $u \cdot (i + 1) \in \mathcal{T}$ implies $u \cdot i \in \mathcal{T}$.

Elements of $\mathcal{T}$ are also called nodes. A labelled tree is defined as a tree $\mathcal{T}$ equipped with a map $h : \mathcal{T} \to \Sigma$ where $\Sigma$ is a (finite) alphabet. Given $u \in \mathcal{T}$, the label of the branch leading to $u$ is $h(\varepsilon) \cdot h(i_1) \cdot h(i_1i_2) \cdots h(i_1 \cdots i_k)$ if $u = i_1 \cdots i_k$. An infinite branch in $\mathcal{T}$ is an infinite sequence $i_1i_2 \cdots \in \mathbb{N}^\omega$ in $\mathcal{T}$ with label $h(\varepsilon) \cdot h(i_1) \cdot h(i_1i_2) \cdots$. Labels for finite branches are defined similarly. A tree $\mathcal{T}$ is finite-branching if for all $u \in \mathcal{T}$ there is $i \in \mathbb{N}$ such that $u \cdot i \notin \mathcal{T}$.

- We recall that König’s Lemma states that every infinite finite-branching tree has an infinite branch.
Exercise 1. Given a concurrent game structure $M = (Agt, S, Act, act, \delta, L)$, coalitions $A \subseteq A' \subseteq Agt$ and a state $s \in S$, show that $M, s \models (\langle \emptyset \rangle Xp \land \langle A \rangle Xq) \Rightarrow \langle A' \rangle X(p \land q)$.

Exercise 2. Let $X$ be a non-empty set with a distinguished element $x_0 \in X$ and $I = (\Delta^I, .^I)$ be an interpretation for $ALC$. Let $J = (\Delta^J, .^J)$ be the interpretation defined as follows.

1. $\Delta^J \overset{\text{def}}{=} X \times \Delta^I$.
2. $A^J \overset{\text{def}}{=} X \times A^I$ for every concept name $A$.
3. $r^J \overset{\text{def}}{=} \{((x, a), (x, b)) \mid x \in X, (a, b) \in r^I\}$ for every role name $r$.
4. $a^J \overset{\text{def}}{=} (x_0, a)$ with $a^I = a$, for every individual name $a$.

1. For all $ALC$ concepts $C$, show that $C^J = X \times C^I$.
2. Given a knowledge base $\mathcal{K}$, show that $I \models \mathcal{K}$ implies $J \models \mathcal{K}$.
3. Conclude that there is no consistent $ALC$ knowledge base $\mathcal{K}$ such that for all interpretations $I = (\Delta^I, .^I)$, $I \models \mathcal{K}$ implies $\Delta^I$ is finite.

Exercise 3. Let $X$ be a finite set of $ALC$ concepts closed under subconcepts and $\mathcal{K}$ (resp. $C$) be a knowledge base (resp. a concept) such that $\text{sub}(\mathcal{K}) \cup \text{sub}(C) \subseteq X$. Let $I = (\Delta^I, .^I)$ be an interpretation such that

- $I \models \mathcal{K}$ and $C^I \neq \emptyset$,
- for all role names $r$ occurring in $X$, $r^I$ is reflexive and transitive.

For all $a, a' \in \Delta^I$, we write $a \sim a'$ iff for all concepts $D \in X$, we have $a \in D^I$ iff $a' \in D^I$. As $\sim$ is an equivalence relation, equivalence classes of $\sim$ are written $[a]$ to denote the class of $a$. Let us define the interpretation $J = (\Delta^J, .^J)$:

- $\Delta^J \overset{\text{def}}{=} \{[a] \mid a \in \Delta^I\}$.
- $A^J \overset{\text{def}}{=} \{[a] \mid \text{there is } a' \in [a] \text{ such that } a' \in A^I\}$ for all $A \in X$. 

2
• $A^\mathcal{J} \overset{\text{def}}{=} \emptyset$ for all concept names $A \not\in X$ (arbitrary value).

• $r^\mathcal{J} \overset{\text{def}}{=} \{( [a], [b]) \mid \text{there are } a' \in [a], b' \in [b] \text{ such that for all } \forall r.D \in X, a' \in (\forall r.D)^I \text{ implies } b' \in (\forall r.D)^I \}$ for all role names $r$ occurring in $X$.

• $r^\mathcal{J} \overset{\text{def}}{=} \emptyset$ for all role names $r$ not occurring in $X$ (arbitrary value).

• $a^\mathcal{J} \overset{\text{def}}{=} [a]$ with $a^I = a$, for all individual names $a$.

1. Show that for all role names $r$ occurring in $X$, $r^\mathcal{J}$ is reflexive and transitive.

2. Show that $(a, b) \in r^I$ implies $([a], [b]) \in r^\mathcal{J}$, for all role names $r$ occurring in $X$.

3. Assuming that the concept constructors occurring in $X$ are among $\forall r$, $\cap$, and $\neg$, show that for all $D \in X$ and $a \in \Delta^I$, we have $a \in D^I$ if $[a] \in D^\mathcal{J}$. (This restriction on the concept constructors allows us to reduce the number of cases in the induction step).

4. Conclude that there is a finite interpretation $\mathcal{I}^*$ such that $\mathcal{I}^* \models \mathcal{K}$ and $(C)^{\mathcal{I}}^* \neq \emptyset$ and for all role names $r$ occurring in $X$, $(r)^{\mathcal{I}}^*$ is reflexive and transitive.

**Exercise 4.** Let $\mathcal{M} = (Agt, S, Act, act, \delta, L)$ be a concurrent game structure with a (finite) set of states $S$, $s \in S$ and $\varphi = \langle A \rangle (F_{p_1} \wedge \cdots \wedge F_{p_n})$ (the $p_i$’s are propositional variables) be an ATL$^*$ (state) formula such that $\mathcal{M}, s \models \varphi$.

1. Let $F$ be a strategy for the coalition $A$ such that for all the computations $\lambda \in \text{Comp}(s, F)$, we have $\mathcal{M}, \lambda \models F_{p_1} \wedge \cdots \wedge F_{p_n}$. The set of computations respecting $F$ can be organised as an infinite tree $\mathcal{T}_F$ such that the label of each infinite branch encodes a computation in $\text{Comp}(s, F)$ and for each computation $\lambda$ in $\text{Comp}(s, F)$, there is an infinite branch with label encoding $\lambda$. The nodes of such a tree $\mathcal{T}_F$ have their respective labels in $S \times \mathcal{P}(\{p_1, \ldots, p_n\})$ as we are interested in the path formula $F_{p_1} \wedge \cdots \wedge F_{p_n}$. Intuitively, a node labelled by $(r, X)$ corresponds to a (finite) history respecting the strategy $F$ ending in
the state \( r \) and for which it remains to meet a future state satisfying \( p \) for each \( p \in X \).

Let \( \mathfrak{T}_F \) be the smallest labelled tree (‘smallest’ with respect to set inclusion) defined as follows (the finite alphabet \( \Sigma \) is \( S \times \mathcal{P}(\{p_1, \ldots, p_n\}) \) to define the labelling \( h \)).

- \( \varepsilon \in \mathfrak{T}_F \) and \( h(\varepsilon) = (s_0, X_0) \) with \( s_0 \stackrel{\text{def}}{=} s \) and \( X_0 \stackrel{\text{def}}{=} \{p_1, \ldots, p_n\} \setminus L(s) \).
- Assuming that \( \text{out}(s, F(s)) = \{r_1, \ldots, r_\alpha\} \) for some \( \alpha \geq 1 \), we have \( 0, \ldots, \alpha - 1 \in \mathfrak{T}_F \) and for all \( i \in \{0, \ldots, \alpha - 1\} \),
  \[ h(i) \stackrel{\text{def}}{=} (r_{i+1}, X_0 \setminus L(r_{i+1})) \]
  \( 0, \ldots, \alpha - 1 \) are therefore the only children of \( \varepsilon \).
- Generally, assume that \( u \in \mathfrak{T}_F \) with \( u = i_1 \cdots i_k \) for some \( k \geq 1 \), and the label of the finite branch leading to \( u \) is \( (s_0, X_0) \cdots (s_k, X_k) \).
  If \( \text{out}(s_k, F(s_0 \cdots s_k)) = \{r_1, \ldots, r_\alpha\} \) for some \( \alpha \geq 1 \), then \( u \cdot 0, \ldots, u \cdot (\alpha - 1) \in \mathfrak{T}_F \) and for all \( i \in \{0, \ldots, \alpha - 1\} \),
  \[ h(u \cdot i) \stackrel{\text{def}}{=} (r_{i+1}, X_k \setminus L(r_{i+1})) \]
  \( u \cdot 0, \ldots, u \cdot (\alpha - 1) \) are also the only children of \( u \).

Let \( i_1 i_2 \cdots \) be an infinite branch of \( \mathfrak{T}_F \) with label \( (s_0, X_0) \cdot (s_1, X_1) \cdot (s_2, X_2) \cdots \). Show the following properties.

- For all \( j \leq j', X_j \supseteq X_{j'} \).
- There is \( j \geq 0 \) such that \( \emptyset = X_j = X_{j+1} = X_{j+2} = X_{j+3} \cdots \).
- \( \{X_0, X_1, X_2, \ldots\} \) has at most \( (n + 1) \) elements.

2. Let \( \mathfrak{T}_F^* \) be the subset of \( \mathfrak{T}_F \) such that

\[ \mathfrak{T}_F^* = \{\varepsilon\} \cup \{u \cdot i \in \mathfrak{T}_F \mid h(u) \text{ not of the form } (r, \emptyset)\} \]

Show that \( \mathfrak{T}_F^* \) is a finite tree.

3. Given a computation \( \lambda \), we say that \( \lambda \) witnesses the satisfaction of \( F_{p_1} \land \cdots \land F_{p_n} \) before position \( K \) \( \Downarrow \) for all \( i \in [1, n] \), there is \( \text{pos}_i \leq K \) such that \( p_i \in L(\lambda(\text{pos}_i)) \). Show that there is a strategy \( F \) for the coalition \( A \) such that for all computations \( \lambda \in \text{Comp}(s, F) \),
4. Let us consider the CGS $\mathcal{M}^*$ below (with two agents in $\{1, 2\}$)

(a) Show that $\mathcal{M}^*, s_1 \models \langle \{1\} \rangle (G p_1 \lor F p_2)$.

(b) Show that there is no strategy $F$ for the agent 1 such that there is $B \geq 1$ for which for all computations $\lambda \in \text{Comp}(s_1, F)$,

i. $\mathcal{M}^*, \lambda \models G p_1 \lor F p_2$ and,

ii. if $\mathcal{M}^*, \lambda \models F p_2$ then $\lambda$ witnesses the satisfaction of $F p_2$ before position $B$. 

\[ M, \lambda \models F_{p_1} \land \cdots \land F_{p_n} \text{ and,} \]

(b) $\lambda$ witnesses the satisfaction of $F_{p_1} \land \cdots \land F_{p_n}$ before position $(n + 1) \times \text{card}(S)$. 

4. Let us consider the CGS $\mathcal{M}^*$ below (with two agents in $\{1, 2\}$)

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i. $\mathcal{M}^*, \lambda \models G p_1 \lor F p_2$ and,

ii. if $\mathcal{M}^*, \lambda \models F p_2$ then $\lambda$ witnesses the satisfaction of $F p_2$ before position $B$. 

\[ M, \lambda \models F_{p_1} \land \cdots \land F_{p_n} \text{ and,} \]
Solution for the exercise 1.

\(M, s \models \langle\emptyset\rangle Xp\) implies that for all transitions \(s \xrightarrow{f} s'\) with \(f \in D_{A^g}(s)\), we have \(p \in L(s')\).

Similarly, \(M, s \models \langle A\rangle Xq\) implies there is \(g \in D_A(s)\) such that for all \(f \in D_{A^g}(s)\) with \(g \sqsubseteq f\) and \(s \xrightarrow{f} s'\), we have \(q \in L(s')\).

So, for all \(f \in D_{A^g}(s)\) with \(g \sqsubseteq f\) and \(s \xrightarrow{f} s'\), we have \(\{p, q\} \subseteq L(s')\).

Consequently, for all \(f \in D_{A^g}(s)\) with \(g' \sqsubseteq f\) and \(s \xrightarrow{f} s'\), we have \(p \in L(s')\) and \(q \in L(s')\). The satisfaction relation for ATL allows us to conclude that \(M, s \models \langle A'\rangle X(p \land q)\) (the witness joint action for this satisfaction is exactly \(g'\)).

Solution for the exercise 2.

1. The proof is by structural induction. For the base case, we have \(A^J = X \times A^I\) by definition. Similarly, \(\top^J = X \times \top^I\) with \(\top^I = \top^I\). For the induction step, we only treat the cases with \(\neg\), \(\sqcap\) and \(\exists r.C\) thanks to the duality between \(\sqcup\) and \(\sqcap\) (resp. between \(\exists r\cdot\) and \(\forall r\cdot\)).

Case \(C = \neg D\).

- \(C^J = (X \times \Delta^I) \setminus D^J\) (\(\mathcal{ALC}\) semantics).
- \(C^J = (X \times \Delta^I) \setminus (X \times D^I)\) (by induction hypothesis).
- \(C^J = X \times (\Delta^I \setminus D^I)\) (by set-theoretical reasoning).
- \(C^J = X \times C^I\) (\(\mathcal{ALC}\) semantics).

Case \(C = D_1 \sqcap D_2\).

- \(C^J = D_1^J \sqcap D_2^J\) (\(\mathcal{ALC}\) semantics).
- \(C^J = X \times D_1^I \sqcap X \times D_2^I\) (by induction hypothesis).
- \(C^J = X \times (D_1^I \sqcap D_2^I)\) (by set-theoretical reasoning).
- \(C^J = X \times (D_1 \sqcap D_2)^I\) (\(\mathcal{ALC}\) semantics).
Case $C = \exists r. D$.

Let $(x, a) \in C^\mathcal{J}$. There is $(x', a')$ such that $((x, a), (x', a')) \in r^\mathcal{J}$ and $(x', a') \in D^\mathcal{J}$ (by $\mathcal{ALC}$ semantics). By definition of $r^\mathcal{J}$, we have $x = x'$ and $(a, a') \in r^\mathcal{I}$ and by induction hypothesis $a' \in D^\mathcal{I}$. By $\mathcal{ALC}$ semantics, $a \in C^\mathcal{I}$.

Conversely, let $x$ be an arbitrary element of $X$ and suppose that $a \in C^\mathcal{I}$. So, there is $a' \in \Delta^\mathcal{I}$ such that $(a, a') \in r^\mathcal{I}$ and $a' \in D^\mathcal{I}$. By definition of $r^\mathcal{J}$, we have $((x, a), (x, a')) \in r^\mathcal{J}$ and $(x, a') \in D^\mathcal{J}$ by induction hypothesis. By $\mathcal{ALC}$ semantics, we can conclude that $(x, a) \in C^\mathcal{J}$.

2. Suppose that $\mathcal{I} \models \mathcal{K}$.

Case $C \subseteq D \in \mathcal{J}$. By assumption, $\mathcal{I} \models C \subseteq D$, i.e. $C^\mathcal{I} \subseteq D^\mathcal{I}$. By set-theoretical reasoning, $X \times C^\mathcal{I} \subseteq X \times D^\mathcal{I}$ and therefore by Question 1, $C^\mathcal{J} \subseteq D^\mathcal{J}$ (equivalent to $\mathcal{J} \models C \subseteq D$).

Case $a : C \in \mathcal{A}$. By assumption, $\mathcal{I} \models a : C$, i.e. $a^\mathcal{I} \in C^\mathcal{I}$. By definition of $\mathcal{J}$, $a^\mathcal{J} = (x_0, a^\mathcal{I})$ and $C^\mathcal{J} = X \times C^\mathcal{I}$ by Question 1. Hence $a^\mathcal{J} \in C^\mathcal{J}$ (equivalent to $\mathcal{J} \models a : C$).

Case $(a, b) : r \in \mathcal{A}$. By assumption, $\mathcal{I} \models (a, b) : r$, i.e. $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$. By definition of $\mathcal{J}$,

- $((x_0, a^\mathcal{I}), (x_0, b^\mathcal{I})) \in r^\mathcal{J}$,
- $a^\mathcal{J} = (x_0, a^\mathcal{I})$ and $b^\mathcal{J} = (x_0, b^\mathcal{I})$.

By $\mathcal{ALC}$ semantics, $\mathcal{J} \models (a, b) : r$.

3. Ad absurdum, suppose that there is a consistent $\mathcal{ALC}$ knowledge base $\mathcal{K}$ such that for all interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, $\mathcal{I} \models \mathcal{K}$ implies $\Delta^\mathcal{I}$ is finite. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \models \mathcal{K}$ ($\mathcal{K}$ is assumed to be consistent). Let $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ be the interpretation defined as follows with $X = \mathbb{N}$:

- $\Delta^\mathcal{J} \overset{\text{def}}{=} \mathbb{N} \times \Delta^\mathcal{I}$ (so $\Delta^\mathcal{J}$ is infinite).
- $A^\mathcal{J} \overset{\text{def}}{=} \mathbb{N} \times A^\mathcal{I}$ for every concept name $A$.
- $r^\mathcal{J} \overset{\text{def}}{=} \{(x, a), (x, b)\} \mid x \in \mathbb{N}, (a, b) \in r^\mathcal{I}\}$ for every role name $r$.
- $a^\mathcal{J} \overset{\text{def}}{=} (0, a)$ with $a^\mathcal{I} = a$, for every individual name $a$.  


Solution for the exercise 3.

1. Let $[a] \in \Delta^\mathcal{J}$ and $r$ be a role name occurring in $X$. Obviously, for all $\forall r.D \in X$, we have $a \in (\forall r.D)^\mathcal{J}$ implies $a \in (\forall r.D)^\mathcal{J}$. Hence, $([a], [a]) \in r^\mathcal{J}$ by definition of $r^\mathcal{J}$. As above $[a]$ is an arbitrary element of $\Delta^\mathcal{J}$, $r^\mathcal{J}$ is reflexive.

Now suppose that $([a], [b]) \in r^\mathcal{J}$ and $([b], [c]) \in r^\mathcal{J}$. By definition of $r^\mathcal{J}$, there is $a' \in [a]$ and $b' \in [b]$ such that for all $\forall r.D \in X$, $a' \in (\forall r.D)^\mathcal{J}$ implies $b' \in (\forall r.D)^\mathcal{J}$. Similarly, there is $b'' \in [b]$ and $c' \in [c]$ such that for all $\forall r.D \in X$, $b'' \in (\forall r.D)^\mathcal{J}$ implies $c' \in (\forall r.D)^\mathcal{J}$.

Let $\forall r.D \in X$ with $a' \in [a]$ and $a' \in (\forall r.D)^\mathcal{J}$. Since $([a], [b]) \in r^\mathcal{J}$, $b' \in (\forall r.D)^\mathcal{J}$. Since $[b] = [b'] = [b'']$, we have also $b'' \in (\forall r.D)^\mathcal{J}$. Since $([b], [c]) \in r^\mathcal{J}$, we get $c' \in (\forall r.D)^\mathcal{J}$. Consequently, $([a], [c]) \in r^\mathcal{J}$ and therefore $r^\mathcal{J}$ is transitive.

2. Assume that $(a, b) \in r^\mathcal{J}$. As $r^\mathcal{J}$ is reflexive and transitive, for all $b' \in r^\mathcal{J}(b)$, we have $b' \in r^\mathcal{J}(a)$.

Now suppose that $a \in (\forall r.D)^\mathcal{J}$ with $\forall r.D \in X$. By $\mathcal{ALC}$ semantics, for all $a' \in r^\mathcal{J}(a)$, we have $a' \in D^\mathcal{J}$. A fortiori (by the above remark), for all $b' \in r^\mathcal{J}(b)$, we have $b' \in D^\mathcal{J}$, i.e. $b \in (\forall r.D)^\mathcal{J}$. As $\forall r.D$ is arbitrary, we conclude that for all $\forall r.D \in X$, $a \in (\forall r.D)^\mathcal{J}$ implies $b \in \forall r.D^\mathcal{J}$. By definition of $r^\mathcal{J}$, we get $([a], [b]) \in r^\mathcal{J}$ (take $a' = a$ and $b' = b$).

3. The proof is by structural induction. For the base $D = A$, by definition of $\mathcal{J}$, we have $A^\mathcal{J} = \{[a] \mid \text{there is } a' \in [a] \text{ such that } a' \in A^\mathcal{J}\}$. This is equivalent to $A^\mathcal{J} = \{[a] \mid a \in A^\mathcal{J}\}$ since all the elements in $[a]$ agree on the concepts in $X$. Similarly, $\top^\mathcal{J} = \Delta^\mathcal{J}$, which is precisely $\{[a] \mid a \in \top^\mathcal{J} = \Delta^\mathcal{J}\}$. Let us consider now the induction step with a case analysis depending on the outermost concept constructor.

Case $D = \neg D'$
- $D^\mathcal{J} = \Delta^\mathcal{J} \setminus (D')^\mathcal{J}$ (by $\mathcal{ALC}$ semantics).
Let us show that $D^J = \{ [a] \mid a \in \Delta^J \} \setminus \{ [a] \mid a \in (D')^J \}$ (by definition of $\Delta^J$, $X$ is closed under subconcepts and by induction hypothesis).

- $D^J = \{ [a] \mid a \notin (D')^J \}$ (by set-theoretical reasoning).
- $D^J = \{ [a] \mid a \in (\neg D')^J \}$ (by ALC semantics).

**Case $D = D_1 \cap D_2$**

- $D^J = D_1^J \cap D_2^J$ (by ALC semantics).
- $D^J = \{ [a] \mid a \in D_1^J \} \cap \{ [a] \mid a \in D_2^J \}$ ($X$ is closed under subconcepts and by induction hypothesis).
- $D^J = \{ [a] \mid a \in (D_1 \cap D_2)^J \}$ (by set-theoretical reasoning).
- $D^J = \{ [a] \mid a \in (D_1 \cap D_2)^J \}$ (by ALC semantics).

**Case $D = \forall r.D'$**

First, suppose that $a \in (\forall r.D')^J$. *Ad absurdum*, suppose that $[a] \notin (\forall r.D')^J$. By ALC semantics, there is $[b]$ such that $([a], [b]) \in r^J$ and $[b] \notin (D')^J$. By definition of $r^J$, there is $a' \in [a]$ and $b' \in [b]$ such that for all $\forall r.D'' \in X$, $a' \in (\forall r.D'')^J$ implies $b' \in (\forall r.D'')^J$. As $[a] = [a']$, $a' \in (\forall r.D')^J$ and therefore $b' \in (\forall r.D')^J$. As $r^J$ is reflexive, $b' \in (D')^J$ and by the induction hypothesis $[b'] \in (D')^J$ (the set $X$ is closed under subconcepts, so we can use the induction hypothesis). However $[b] = [b']$ and therefore $[b] \in (D')^J$, which leads to contradiction.

Second, suppose that $[a] \in (\forall r.D')^J$. *Ad absurdum*, suppose that there is $a' \in [a]$ such that $a' \notin (\forall r.D')^J$. By ALC semantics, there is $[b']$ such that $([a'], [b']) \in r^J$ and $[b'] \notin (D')^J$.

By Question 2, $([a'], [b']) \in r^J$ and therefore $([a], [b']) \in r^J$. Since $X$ is closed under subconcepts, by the induction hypothesis, we obtain $[b'] \notin (D')^J$ too. Hence, $[a] \notin (\forall r.D')^J$, which leads to contradiction.

4. Let us show that $\mathcal{I}^* = J$ does the job with $X = \text{sub}(K) \cup \text{sub}(C)$.

- $\Delta^J$ is finite and $\text{card}(\Delta^J) \leq 2^{\text{card}(X)}$ with finite $X$.
- By Question 3, for all $D \in X$, we have $D^J = \{ [a] \mid a \in D^J \}$.
- Consequently, $C^J \neq \emptyset$ implies $C^J \neq \emptyset$. 

9
Furthermore, \( D^I \subseteq (D')^I \) implies \( D^J \subseteq (D')^J \) and therefore for all GCIs \( D \subseteq D' \in \mathcal{T} \), we have \( \mathcal{I} \models D \subseteq D' \) implies \( \mathcal{J} \models D \subseteq D' \). As \( \mathcal{I} \models \mathcal{T} \) we get \( \mathcal{J} \models \mathcal{T} \).

Let \( a : D \in A \). By assumption, we have \( \mathcal{I} \models a : D \) and therefore \( a^I \in D^I \). By Question 3, \( [a^I] \in D^J \). By definition of \( J \), \( a^J = [a^I] \) and therefore \( \mathcal{J} \models a : D \).

Let \( (a, b) : r \in A \). By assumption, we have \( \mathcal{I} \models (a, b) : r \) and therefore \( (a^I, b^I) \in r^I \). By Question 2, \( ([a], [b]) \in r^J \). As \( a^J = [a] \) and \( b^J = [b] \) by definition of \( J \), we get \( (a^J, b^J) \in r^J \). Therefore \( \mathcal{J} \models (a, b) : r \) (by ALC semantics).

For all role names \( r \) occurring \( X \), \( r^J \) is reflexive and transitive by Question 1.

Solution for the exercise 4.

1 In order to show the first property. Let \( j \in \mathbb{N} \). By definition of \( h \), we have \( X_{j+1} = X_j \setminus L(s_{j+1}) \). Obviously \( X_j \supseteq X_{j+1} \) by a simple set-theoretical reasoning. Consequently, for all \( j' \geq j \) we have \( X_j \supseteq X_{j'} \) by transitivity of \( \supseteq \).

In order to show the second property, by construction of \( \mathfrak{T}_F \), \( \lambda = s_0 s_1 \cdots \) is a (maximal) computation in \( \text{Comp}(s, F) \) and therefore \( \mathfrak{M}, \lambda \models F p_1 \land \cdots \land F p_n \). This means that there are \( j_1, \ldots, j_n \) such that for all \( k \in [1, n] \), we have \( p_k \in L(s_{j_k}) \). Hence, for all \( j > \max(j_1, \ldots, j_n) \), we have \( X_{j_k} = \emptyset \).

The third property is a consequence of the two first ones. Indeed \( X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \) with \( \text{card}(X_0) \leq n \) and \( \text{card}(X_0) \geq \text{card}(X_1) \geq \text{card}(X_2) \geq \cdots \). So, if \( \text{card}(X_j) = \text{card}(X_{j'}) \) for some \( j, j' \), then \( X_j = X_{j'} \). Consequently, \( \{X_0, X_1, X_2, \ldots\} \) has at most \( (n + 1) \) elements.

2 Let \( \mathfrak{T}_F^* \) be the subset of \( \mathfrak{T}_F \) such that

\[
\mathfrak{T}_F^* = \{\varepsilon\} \cup \{u \cdot i \in \mathfrak{T}_F \mid h(u) \text{ not of the form } (r, \emptyset)\}.
\]

First, let us check that \( \mathfrak{T}_F^* \) is a tree.
• If \( u \cdot i \in \mathcal{I}_F^* \), then \( h(u) \) is not of the form \((r, \emptyset)\). If \( u = u' \cdot i' \), then a fortiori \( h(u') \) is not of the form \((r, \emptyset)\) by monotonicity (see question 1). Hence, \( u = u' \cdot i' \) belongs to \( \mathcal{I}_F^* \) too.

• If \( u \cdot (i + 1) \in \mathcal{I}_F^* \), then again \( h(u) \) is not of the form \((r, \emptyset)\). As \( \mathcal{I}_F^* \subseteq \mathcal{I}_F \) and \( \mathcal{I}_F \) is a tree, we get \( u \cdot i \in \mathcal{I}_F^* \) and \( h(u) \) is not of the form \((r, \emptyset)\). By definition of \( \mathcal{I}_F^* \) this implies \( u \cdot i \in \mathcal{I}_F^* \).

It remains to prove that \( \mathcal{I}_F^* \) is finite. By König’s Lemma (every infinite finite-branching tree has an infinite branch), we can conclude that \( \mathcal{I}_F^* \) is a finite tree and we write \( h^* \) to denote the restriction of \( h \) to \( \mathcal{I}_F^* \). Indeed, \textit{ad absurdum}, suppose that \( \mathcal{I}_F^* \) is infinite. Since \( \mathcal{I}_F^* \) is a subset of \( \mathcal{I}_F \) and \( \mathcal{I}_F \) is finite-branching (since \( S \) is a finite set), then \( \mathcal{I}_F^* \) is finite-branching too. By König’s Lemma, \( \mathcal{I}_F^* \) has an infinite branch \( i_1i_2\cdots \), say with label \((s_0, X_0) \cdot (s_1, X_1) \cdot (s_2, X_2) \cdots \). However, by Question 1, there is (a minimal) \( j \geq 1 \) such that \( X_j = \emptyset \) which leads to a contradiction as \( i_1i_2\cdots i_j \) has no children in \( \mathcal{I}_F^* \) by definition. To be completely precise, \( X_0 \) might be already equal to the empty set, in which case \( \mathcal{I}_F^* \) is made of the root node only, which also leads to a contradiction.

3 To answer to the question, it remains to define \( F \) such that the height of \( \mathcal{I}_F^* \) is bounded by \((n + 1) \times \text{card}(S)\), i.e. the length of the longest strings in \( \mathcal{I}_F^* \) is bounded by \((n + 1) \times \text{card}(S)\).

Let \( F \) be a strategy for the coalition \( A \) such that for all the computations \( \lambda \in \text{Comp}(s, F) \), we have \( \mathcal{M}, \lambda \models Fp_1 \land \cdots \land Fp_n \). Suppose that a branch of \( \mathcal{I}_F^* \) contains a finite branch \( i_1i_2\cdots i_K \) with label \((s_0, X_0) \cdots (s_K, X_K) \) such that \( K \geq (n + 1) \times \text{card}(S) \). So, there are \( j < j' \) such that \((s_j, X_j) = (s_{j'}, X_{j'}) \) (by Question 1, the set of \( X_i \)'s can have at most \((n + 1) \) elements). Let us define the strategy \( F' \) as a slight modification of the strategy \( F \) such that \( F' \) still witnesses the satisfaction of \( \mathcal{M}, s \models \langle A \rangle (Fp_1 \land \cdots \land Fp_n) \) and \( \text{card}(\mathcal{I}_F^* \setminus \mathcal{I}_F^* F) < \text{card}(\mathcal{I}_F) \). By repeating this process a finite amount of times, we can reach a strategy satisfying the requirements of the question. The strategy \( F' \) is defined from \( F \) by only modifying the value returned for the histories of the form \( s_0 \cdots s_j \cdot \lambda_f \) where \( \lambda_f \) is a history in \( S^* \) so that \( s_0 \cdots s_j \cdot \lambda_f \) history is an authorised history.

\[
F'(s_0 \cdots s_j \cdot \lambda_f) \overset{\text{def}}{=} F(s_0 \cdots s_j \cdots s_{j'} \cdot \lambda_f).
\]
We insist: $F'$ is defined as $F$ for the other forms of histories from $s = s_0$ and $\Sigma_F^*$ is obtained from $\Sigma_F$ by replacing the (infinite) subtree with root $i_1 i_2 \cdots i_j$ by the subtree with root $i'_1 i'_2 \cdots i'_j$ in $\Sigma_F$. As stated above, this transformation can be lifted to the truncated and finite relevant part of $\Sigma_F^*$: as a result, $\Sigma_F^*$ can be obtained from $\Sigma_F^*$ by replacing the subtree with root $i_1 i_2 \cdots i_j$ by the subtree with root $i'_1 i'_2 \cdots i'_j$ in $\Sigma_F^*$ but has at least one node less.

Below, we illustrate schematically how $\Sigma_F^*$ is obtained from $\Sigma_F^*$, typically by replacing the subtree $\Sigma_1$ by the subtree $\Sigma_2$.

Note that $F'$ is still a strategy such that for all the computations $\lambda \in \text{Comp}(s, F')$, we have $M, \lambda \models F_{p_1} \land \cdots \land F_{p_n}$ as all the leaves of $\Sigma_F^*$ have still labels of the form $(r, \emptyset)$.

4(a) Let us define a positional strategy $F$ for the agent 1 such that $F(s_1) = a$ (no choice for the agent 1 for these states) and $F(s_2) = b$. The set of maximal computations starting from $s_1$ and respecting the strategy $F$ can be characterised by the $\omega$-regular expression below:

$$s_1^\omega \cup s_1^s s_2 s_3^\omega$$

Along every state in $\lambda = s_1^\omega$, $p_1$ holds true, so $\lambda \models G p_1$. Moreover, for all $i \geq 1$, the computation $\lambda = s_1^i s_2 s_3^\omega$ satisfies $\lambda \models F p_2$ since $p_2$ holds at the state $s_3$. Consequently, for all the computations $\lambda \in \text{Comp}(s_1, F)$, we have $\lambda \models G p_1 \lor F p_2$. Hence, $M^*, s_1 \models \langle \langle \{1\} \rangle \rangle (G p_1 \lor F p_2)$. 12
4(b) Let $F$ for a strategy for the agent 1, not necessarily a positional one, such that for all $\lambda \in \text{Comp}(s_1, F)$, we have $\mathcal{M}^*, s_1 \models \mathcal{L}\{1\}(G p_1 \lor F p_2)$.

Let $B \geq 1$. As at the state $s_1$, the agent 1 has no choice, $s_1^B s_2$ is a finite history respecting $F$. Moreover, by assumption on $F$, there is a computation $\lambda \in \text{Comp}(s_1, F)$ with prefix $s_1^B s_2$ such that $\mathcal{M}^*, \lambda \models G p_1 \lor F p_2$. Necessarily, $\mathcal{M}^*, \lambda \models F p_2$ since $s_2$ does not satisfy $p_1$.

However, as only the state $s_3$ satisfies $p_2$, $\lambda$ witnesses the satisfaction of $F p_2$ necessarily strictly after the position $B$. 