The correction below for the exercises 1-5 is more detailed than what was expected in the copies. Moreover, the subject (exercises 1 to 4, bonus exercise 5) was a bit too long for a two-hour exam but this has been taken into account in the grading.

Solution for the exercise 1.

1. We have $\mathcal{T} \not\models A_0 \sqsubseteq \exists r.A_1$ iff there is some interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ such that $\mathcal{I} \models \mathcal{T}$ and there is $a \in \Delta^\mathcal{I}$ such that $a \in A_0^\mathcal{I}$ and $a \notin (\exists r.A_1)^\mathcal{I}$. Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ with $\Delta^\mathcal{I} = \{0\}$, $r^\mathcal{I} = \emptyset$ for all role names $r$, $A_0^\mathcal{I} \overset{\text{def}}{=} \{0\}$, $A_3^\mathcal{I} \overset{\text{def}}{=} \{0\}$, $A_5^\mathcal{I} \overset{\text{def}}{=} \{0\}$, and $B^\mathcal{I} = \emptyset$ for all other concept names $B$.

   - We have $0 \in A_0^\mathcal{I}$ and $0 \notin \exists r.A_1^\mathcal{I}$.
   - $\mathcal{I} \models A_0 \sqsubseteq \forall r.A_1$ because $A_0^\mathcal{I} = (\forall r.A_1)^\mathcal{I} = \{0\}$.
   - $\mathcal{I} \models A_1 \sqsubseteq \neg A_4$ because $A_1^\mathcal{I} = \emptyset$.
   - $\mathcal{I} \models A_0 \sqsubseteq A_2 \cup A_3$ because $A_0^\mathcal{I} = (A_2 \cup A_3)^\mathcal{I} = \{0\}$.
   - $\mathcal{I} \models A_2 \sqsubseteq \exists r.A_4$ because $A_2^\mathcal{I} = \emptyset$.
   - $\mathcal{I} \models \exists r.\neg A_1 \sqsubseteq A_5$ because $(\exists r.\neg A_1)^\mathcal{I} = \emptyset$ and $A_5^\mathcal{I} = \{0\}$.
   - $\mathcal{I} \models A_3 \sqsubseteq A_5$ because $A_3^\mathcal{I} = A_5^\mathcal{I} = \{0\}$.

   Consequently, $\mathcal{I} \models \mathcal{T}$ and therefore $\mathcal{T} \models A_0 \sqsubseteq \exists r.A_1$ does not hold.

2. *Ad absurdum*, suppose that $\mathcal{K}$ is consistent. So, there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models A$ and in particular

   (a) $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$,
   (b) $a^\mathcal{I} \in A_0^\mathcal{I}$,
   (c) $b^\mathcal{I} \in A_4^\mathcal{I}$,
   (d) $\mathcal{I} \models \{A_0 \sqsubseteq \forall r.A_1, A_1 \sqsubseteq \neg A_4\}$.

As (a) and (b), $\mathcal{I} \models A_0 \sqsubseteq \forall r.A_1$ entails $b^\mathcal{I} \in A_1^\mathcal{I}$. As $\mathcal{I} \models A_1 \sqsubseteq \neg A_4$, we conclude $b^\mathcal{I} \notin A_4^\mathcal{I}$, which is in contradiction with (c). Consequently, $\mathcal{K}$ is not consistent.
Solution for the exercise 2.

1. In order to use the tableaux calculus for $ALC$ introduced in the course, the concepts in $K$ need to be in NNF and, the GCI$s should be of the form $\top \sqsubseteq C$. The knowledge base $K$ can be transformed into the logically equivalent $K' = (T', A')$ with

$$T' = \{ \top \sqsubseteq \neg A \sqcup \exists r. A \} \quad A' = \{ a: (A \sqcap (\exists r. B)) \sqcup \exists r. \neg B \}$$

In order to show that $K$ is consistent, it is sufficient to derive from $A'$ an ABox $A''$ (i.e. $A' \Rightarrow A''$) using the tableaux rule for $ALC$ with the $\sqsubseteq$-rule parameterised by $T'$ such that $A''$ is clash-free and complete when the blocking technique is used. We provide a derivation of $A''$ from the leftmost branch of the tableaux in Figure 1 such that $A''$ is made of all the concept assertions and role assertions from that leftmost branch. Note that $A''$ is clash-free and complete. Other branches leading to clashes are shown on the tableau but are not strictly speaking needed.

2. $(T, A)$ and $(T', A')$ have exactly the same interpretations because transforming a concept into an equivalent concept in NNF preserves the semantics and similarly $C \sqsubseteq C'$ and $\top \sqsubseteq \neg C \cup C'$ are satisfied by exactly the same interpretations.

According to the soundness proof with blocking, an interpretation $I$ such that $I \models (T', A')$ can be built from $A''$ such that

- $\Delta^I$ is the set of individual names occurring in $A''$ except the blocked ones,
- $(A^*)^I$ is equal to the set of individual names $a$ in $\Delta^I$ such that $a : A^* \in A''$,
- For all individual names $a, b \in \Delta^I$, $(a, b) \in r^I$ iff either $(a, b) : r \in A''$ or there is a blocked individual names $b''$ in $A''$ such that $(a, b'') : r \in A''$ and $b''$ is blocked by $b$.

Consequently, an interpretation $I$ satisfying $(T', A')$ (and therefore satisfying also $(T, A)$) is defined as follows.

- $\Delta^I \equiv \{ a, b_1, b_2 \}$. 

2
Figure 1: A “tableaux” for deriving the complete and clash-free $A$
• $A^I \overset{\text{def}}{=} \{ a \}$, $B^I \overset{\text{def}}{=} \{ b_2 \}$, and the interpretation of the other concept names is arbitrary.

• $r^I \overset{\text{def}}{=} \{(a, b_1), (a, b_2), (b_1, b_1)\}$ and the interpretation of the other role names is arbitrary.

The soundness proof guarantees that $\mathcal{I} \models \mathcal{K}$ but this could be also checked directly.

**Solution for the exercise 3.** Let us show that $\langle A \rangle^\mathcal{G} \varphi \Rightarrow (\varphi \land \langle A \rangle^\mathcal{X} \langle A \rangle^\mathcal{G} \varphi)$ is valid for ATL, that is for all CGS $\mathcal{M}$ and states $s$, we have $\mathcal{M}, s \models \langle A \rangle^\mathcal{G} \varphi$ implies $\mathcal{M}, s \models \varphi \land \langle A \rangle^\mathcal{X} \langle A \rangle^\mathcal{G} \varphi$.

Let $\mathcal{M}$ and $s$ be such that $\mathcal{M}, s \models \langle A \rangle^\mathcal{G} \varphi$. Below, we shall show that $\mathcal{M}, s \models \varphi \land \langle A \rangle^\mathcal{X} \langle A \rangle^\mathcal{G} \varphi$. By definition of the semantics for ATL, there is a strategy $F$ for the coalition $A$ such that for all computations $\lambda \in \text{Comp}(s, F)$, for all positions $i \in \mathbb{N}$, we have $\mathcal{M}, \lambda(i) \models \varphi$. As $\text{Comp}(s, F)$ is non-empty (the action manager always returns a non-empty set of actions for each pair $(a, s)$), there is a computation $\lambda \in \text{Comp}(s, F)$ and therefore $\mathcal{M}, \lambda(0) \models \varphi$. But $\lambda(0)$ is precisely $s$, whence $\mathcal{M}, s \models \varphi$. It remains to show that $\mathcal{M}, s \models \langle A \rangle^\mathcal{X} \langle A \rangle^\mathcal{G} \varphi$.

Let $f = F(s)$ be the joint action for the coalition $A$ and we know that $\text{out}(s, f) \subseteq [\varphi]^\mathcal{M}$ since $\mathcal{M}, s \models \langle A \rangle^\mathcal{G} \varphi$. Suppose that $\text{out}(s, f) = \{ s_1^*, \ldots , s_n^* \}$ for some $\alpha \geq 1$.

For each $i \in [1, \alpha]$, let $F_i^*$ be a strategy such that for each history $s_0' \cdots s_n'$ with $s_0' = s_i^*$, we have $F_i^*(s_0' \cdots s_n') \overset{\text{def}}{=} F(s_0' \cdots s_n')$. Consequently, for all $\lambda \in \text{Comp}(s_i^*, F_i^*)$, $s \cdots \lambda \in \text{Comp}(s, F)$ (here we use the fact that $f = F(s)$ and $s_i^* \in \text{out}(s, f)$). Hence, for all positions $j \in \mathbb{N}$, we have $\mathcal{M}, \lambda(j) \models \varphi$ and therefore $\mathcal{M}, s_i^* \models \langle A \rangle^\mathcal{G} \varphi$. This means that $\text{out}(s, f) \subseteq [\langle A \rangle^\mathcal{G} \varphi]^\mathcal{M}$. However, $\mathcal{M}, s \models \langle A \rangle^\mathcal{X} \langle A \rangle^\mathcal{G} \varphi$ precisely when there is joint action $f'$ such that $\text{out}(s, f') \subseteq [\langle A \rangle^\mathcal{G} \varphi]^\mathcal{M}$. Hence, $\mathcal{M}, s \models \langle A \rangle^\mathcal{X} \langle A \rangle^\mathcal{G} \varphi$.

**Solution for the exercise 4.**

1. Here is the derivation of $A_1 \subseteq A_2$.

$$
\frac{
A_1 \subseteq B_2 
\quad B_1 \cap B_2 \subseteq A_2
}{
A_1 \subseteq A_2
} \quad \frac{
A_1 \subseteq \exists r.A_1 
\quad A_1 \subseteq B
}{
A_1 \subseteq \exists r.B
} \quad \frac{
\exists r.A_1 \subseteq \exists r.B 
\quad A_1 \subseteq B
}{
\exists r.B \subseteq B_1
} \quad \frac{
\exists r.B \subseteq B_1
\quad A_1 \subseteq B_1
}{
A_1 \subseteq A_2
}
$$
2. First, suppose that $S(T) \subseteq \{A_1^*, \ldots, A_n^*\} \cup \{r_1, \ldots, r_\beta\}$ for some $\alpha, \beta \geq 1$ where each $A_i^*$ is a concept name and each $r_i$ is a role name. As $T$ is finite, $\alpha$ and $\beta$ always exist. Let us define $T^c$ as $\{C \sqsubseteq D \mid T \vdash C \sqsubseteq D\}$. Thanks to the $\epsilon$-rule, $T \subseteq T^c$ and $T^c$ is simple and complete. The fact that $T^c$ is simple is only due to the property that all the conclusions in inference rules are authorised GCIs in simple TBoxes. Note that completeness is a consequence of the property: $T^c \vdash C \sqsubseteq D$ implies $T \vdash C \sqsubseteq D$. Furthermore, observe that the set of GCIs of the form

$$A \sqsubseteq B \quad A \sqcap A_2 \sqsubseteq B \quad A \sqsubseteq \exists r.B \quad \exists r.A \sqsubseteq B$$

built from $S(T)$ is finite and actually cubic in $\alpha + \beta$. Consequently, $T^c$ is finite. Hence, $T^c$ is a simple and complete TBox with $T \subseteq T^c$. In order to establish that $T^c$ is the smallest such a set, ad absurdum, suppose that $T \subseteq T' \subset T^c$, and $T'$ is simple and complete. So, there is $C \sqsubseteq D$ such that $T \vdash C \sqsubseteq D$ and $C \sqsubseteq D \notin T'$. As $T \subseteq T'$, we can also conclude that $T' \vdash C \sqsubseteq D$. However, we have just seen that $C \sqsubseteq D \notin T'$, which is in contradiction with the completeness of $T'$. In order to compute $T^c$, we proceed as follows (saturation algorithm). Given a simple TBox $T$, we write $C(T)$ to denote the set of GCIs obtained from $T$ by applying one inference rule from premisses in $T$. As each rule involves at most three premisses, and each rule inference can be checked in linear time in the size of its premisses, computing $C(T)$ requires cubic time in the size of $T$. Here is the saturation algorithm.

- $X = T$;
- While $C(X) \neq X$ do $X := C(X)$;
- return $X$.

The while loop is visited a number of times at most cubic in $\alpha + \beta$ and the size of $X$ is also at most cubic in $\alpha + \beta$. Hence, the returned value $X$ is computed in cubic time in $T$ and is equal to $T^c$.

3. The proof is on the length of the derivation to establish $T \vdash C \sqsubseteq D$. In order to provide a complete formal treatment, let us introduce the notion $T$-derivation. A $T$-derivation is a sequence $C_1 \sqsubseteq
$D_1, \ldots, C_K \subseteq D_K$ ($K \geq 1$) such that for all $i \in [1, K]$, at least one of the conditions below holds.

(a) $C_i \subseteq D_i \in \mathcal{T}$ (use of the $\in$-rule).
(b) $C_i \subseteq D_i$ is of the form $A \subseteq A$ for some concept name $A$ in $S(T)$ (use of the id-rule).
(c) $C_i \subseteq D_i$ is of the form $A \subseteq \top$ for some concept name $A$ in $S(T)$ (use of the $\top$-rule).
(d) there are $i_1, i_2 < i$ such that $C_{i_1} = C_i$, $D_{i_1} = C_{i_2}$, $D_{i_2} = D_i$ (use of the trans-rule).
(e) Similarly (and we omit the very details herein), there are GCIs that occur strictly before $C_i \subseteq D_i$ in the sequence that can be used as premisses for either the $\exists$-rule or the $\sqcap$-rule leading exactly to the conclusion $C_i \subseteq D_i$.

So, $\mathcal{T} \vdash C \subseteq D$ iff there is a $\mathcal{T}$-derivation $(C_1 \subseteq D_1, \ldots, C_K \subseteq D_K)$ such that $C_K \subseteq D_K$ is equal to $C \subseteq D$. It remains to show that for all the $\mathcal{T}$-derivations $(C_1 \subseteq D_1, \ldots, C_K \subseteq D_K)$, we have $\mathcal{T} \vdash C_K \subseteq D_K$ (which guarantees that for all interpretations $\mathcal{I}$, we have $\mathcal{I} \vdash \mathcal{T}$ iff $\mathcal{I} \vdash \mathcal{T} \cup \{C_K \subseteq D_K\}$).

The proof is by induction on $i$. For the base case, $C_1 \subseteq D_1$ satisfies one condition among (a), (b), (c). As $A \subseteq A$ and $A \subseteq \top$ hold in all interpretations, the cases for (b) and (c) are immediate. Similarly, obviously $\mathcal{T} \vdash C_1 \subseteq D_1$. Indeed, $\mathcal{I} \vdash \mathcal{T}$ implies for all $C \subseteq D \in \mathcal{T}$, we have $\mathcal{I} \vdash C \subseteq D$. In particular, this leads to $\mathcal{I} \vdash C_1 \subseteq D_1$ as $C_1 \subseteq D_1 \in \mathcal{T}$ in the case (a).

For the induction step, we only treat the cases with the trans-rule and the $\exists$-rule, the case with the $\forall$-rule is omitted but very similar.

Suppose that

$$
\frac{\mathcal{T} \vdash C_i \subseteq D_{i_1}, \mathcal{T} \vdash D_{i_2} \subseteq D_i}{\mathcal{T} \vdash C_i \subseteq D_i}
$$

with $i_1, i_2 < i$ and $C_{i_1} = C_i$, $D_{i_1} = C_{i_2}$, $D_{i_2} = D_i$. By (IH), $\mathcal{T} \vdash C_i \subseteq D_{i_1}$ and $\mathcal{T} \vdash D_{i_2} \subseteq D_i$. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \vdash \mathcal{T}$. So $\mathcal{I} \vdash \{C_i \subseteq D_{i_1}, D_{i_1} \subseteq D_i\}$, which leads to $C_i^\mathcal{I} \subseteq D_i^\mathcal{I}$ by transitivity of
set-inclusion and therefore $I \models C_i \subseteq D_i$. Consequently $T \models C_i \subseteq D_i$ as $I$ above were arbitrary.

Consider now the case of the $\exists$-rule.

\[
\frac{T \vdash A \subseteq \exists r.A_1, T \vdash A_1 \subseteq B_1}{T \vdash A \subseteq \exists r.B_1} \quad \text{\exists-rule}
\]

Let $I$ be an interpretation such that $I \models T$. By (IH), $I \models \{A \subseteq \exists r.A_1, A_1 \subseteq B_1\}$. In particular, this entails that $I \models \exists r.A_1 \subseteq \exists r.B_1$ as $I \models A_1 \subseteq B_1$. Indeed, $(a, b) \in r^I$ and $b \in A_1^I$ imply $(a, b) \in r^I$ and $b \in B_1^I$ as $A_1^I \subseteq B_1^I$. Now $I \models \{A \subseteq \exists r.A_1, \exists r.A_1 \subseteq \exists r.B_1\}$ implies $I \models A \subseteq \exists r.B_1$ by transitivity of set-inclusion. Consequently $T \models C_i \subseteq D_i$ as $I$ above were arbitrary.

To conclude, suppose that $C \subseteq D \in T^c$. By definition of $T^c$, we have $T \vdash C \subseteq D$. According to the developments above, we get $T \models C \subseteq D$.

4. First, let us check that $T^I = \Delta^I$. By definition, $T^I$ is equal to $\{A \in \Delta^I \mid A \subseteq T \in T^c\}$. Since $T^c$ is complete, by application of the $\top$-rule, for all concept names in $S(T)$, we have $A \subseteq T \in T^c$. Hence $T^I$ is equal to the set of all concept names in $S(T)$ (including $\top$), which is precisely $\Delta^I$ by definition.

Let us show that $I \models T^c$. We make a case analysis and we use the fact that $T^c$ is complete.

$A \subseteq B \in T^c$. Suppose that $A' \in A^I$. By definition, $A' \subseteq A \in T^c$. As $T^c$ is complete, by application of the trans-rule, we get $A' \subseteq B \in T^c$. By definition of $I$, $A' \in B^I$. In conclusion, $I \models A \subseteq B$.

$A \subseteq \exists r.B \in T^c$. Suppose that $A' \in A^I$. By definition, $A' \subseteq A \in T^c$. As $T^c$ is complete, by application of the trans-rule, we get $A' \subseteq \exists r.B \in T^c$. By definition of $r^I$, we have $(A', B) \in r^I$. Moreover, $B \in B^I$ as $B \subseteq B \in T^c$ by completeness of $T^c$ and thanks to the $\text{id}$-rule. Hence, $A' \in (\exists r.B)^I$. In conclusion, $I \models A \subseteq \exists r.B$.

$\exists r.A \subseteq B \in T^c$. Suppose that $A' \in (\exists r.A)^I$. So, there is $A''$ such that $(A', A'') \in r^I$ and $A'' \in A^I$. By definition of $r^I$, $A' \subseteq \exists r'.A'' \in T^c$. Hence, $A'' \subseteq B \in T^c$ by completeness of $T^c$. Consequently, $I \models A' \subseteq B$.
By definition of $A^2$, $A'' \subseteq A \in T^c$. By completeness of $T^c$ and the $\exists$-rule, we have $A' \subseteq \exists r.A \in T^c$. By completeness of $T^c$ and the trans-rule, $A' \subseteq B \in T^c$. By definition of $B^\perp$, we get $A' \in B^\perp$.

In conclusion, $I \models A \subseteq B$.

5. Suppose that $A \subseteq B \not\in T^c$. By definition of $B^\perp$, $A \not\in B^\perp$. Moreover, $A \in A^2$ as $A \subseteq A \in T^c$ thanks to the id-rule. Consequently, $I \models T$ (since $I \models T^c$ and $T \subseteq T^c$), and $I \not\models A \subseteq B$. So, $T \not\models A \subseteq B$.

6. By combining the answers for the questions 3. and 5., we get that $A \subseteq B \in T^c$ iff $T \models A \subseteq B$. By the question 2., $T^c$ can be computed in polynomial time in the size of $T$. Here is the simple polynomial-time algorithm to check whether for all interpretations $I$, $(I \models T$ implies $I \models A \subseteq B)$.

(a) compute $T^c$ from $T$;
(b) check whether $A \subseteq B$ belongs to $T^c$.

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Solution for the exercise 5.

1. Figure contains a graphical representation of the CGS $M_{T,n}$ with $T = \{t_0, t_1\}$ and $n = 3$. Note that for any $T$ and $n$, for all infinite computations $\lambda$ starting from $(0, 0, t_0)$, for all $i, j \in [0, n - 1]$, there is a unique position $I$ such that $\lambda(I)$ is of the form $(i, j, t)$.

2. The path formula $\text{Error}^H(i, j)$ is defined as

$$\bigvee_{t \in T} \left( \left( F((i, j) \mapsto t) \land \bigvee_{t', (t', t) \in H} F((i - 1, j) \mapsto t') \right) \right)$$

3. The path formula $\text{Error}^V(i, j)$ is defined as

$$\bigvee_{t \in T} \left( \left( F((i, j) \mapsto t) \land \bigvee_{t', (t', t) \not\in V} F((i, j - 1) \mapsto t') \right) \right)$$
Figure 2: CGS $\mathcal{M}_{T,n}$ with $T = \{t_0, t_1\}$ and $n = 3$
4. The formula $\varphi_{T,n}$ is defined as follows, taking advantage of the property (that would need to be proved) that strategies for Player 2 in the $(n \times n)$-tiling game problem correspond to strategies for agent 2 in $\mathcal{M}_{T,n}$ as far as the infinite computations from $(0, 0, t_0)$ are concerned.

\[
\begin{align*}
\langle\langle 2 \rangle\rangle \left( \bigwedge_{i \in [1,n-1]} \neg \text{Error}^H(i,0) \right) & \land \\
\left( \bigwedge_{j \in [1,n-1]} \left( \bigvee_{i \in [1,n-1]} (\text{Error}^H(i,j) \lor \text{Error}^V(i,j)) \Rightarrow \bigvee_{j' \in [1,j]} \text{Error}^V(0,j') \right) \right)
\end{align*}
\]

As $\mathcal{M}_{T,n}$ is of polynomial size in the size of $T$ and $n$, and $\varphi_{T,n}$ is of polynomial size in the size of $T$ and $n$, the above developments correspond to key steps to establish that the model-checking problem for ATL is PSPACE-hard.