Logical Aspects of AI: Knowledge Logics
Correction of the exam – January 13th, 2021, 2pm-6pm, online

The correction below for the exercices 1-5 is more detailed than what was expected in the copies. Moreover, the subject (exercises 1 to 4, bonus exercise 5) was a bit too long for a two-hour exam but this has been taken into account in the grading.

Solution for the exercise 1.

1. We have $\mathcal{T} \not\models A_0 \sqsubseteq \exists r.A_1$ iff there is some interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ such that $\mathcal{I} \models \mathcal{T}$ and there is $\mathfrak{a} \in \Delta^{\mathcal{I}}$ such that $\mathfrak{a} \in A_0^{\mathcal{I}}$ and $\mathfrak{a} \not\in (\exists r.A_1)^{\mathcal{I}}$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{0\}$, $r^{\mathcal{I}} = \emptyset$ for all role names r, $A_0^{\mathcal{I}} \stackrel{\text{def}}{=} \{0\}$, $A_3^{\mathcal{I}} \stackrel{\text{def}}{=} \{0\}$, $A_5^{\mathcal{I}} \stackrel{\text{def}}{=} \{0\}$, and $B^{\mathcal{I}} = \emptyset$ for all other concept names B.

   - We have $0 \in A_0^{\mathcal{I}}$ and $0 \not\in \exists r.A_1^{\mathcal{I}}$.
   - $\mathcal{I} \models A_0 \sqsubseteq \forall r.A_1$ because $A_0^{\mathcal{I}} = (\forall r.A_1)^{\mathcal{I}} = \{0\}$.
   - $\mathcal{I} \models A_1 \sqsubseteq \neg A_4$ because $A_1^{\mathcal{I}} = \emptyset$.
   - $\mathcal{I} \models A_0 \sqsubseteq A_2 \sqcup A_3$ because $A_0^{\mathcal{I}} = (A_2 \sqcup A_3)^{\mathcal{I}} = \{0\}$.
   - $\mathcal{I} \models A_2 \sqsubseteq \exists r.A_4$ because $A_2^{\mathcal{I}} = \emptyset$.
   - $\mathcal{I} \models \exists r.\neg A_1 \sqsubseteq A_5$ because $(\exists r.\neg A_1)^{\mathcal{I}} = \emptyset$ and $A_5^{\mathcal{I}} = \{0\}$.
   - $\mathcal{I} \models A_3 \sqsubseteq A_5$ because $A_3^{\mathcal{I}} = A_5^{\mathcal{I}} = \{0\}$.

   Consequently, $\mathcal{I} \models \mathcal{T}$ and therefore $\mathcal{T} \models A_0 \sqsubseteq \exists r.A_1$ does not hold.

2. *Ad absurdum*, suppose that $\mathcal{K}$ is consistent. So, there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$ and in particular

   (a) $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$,

   (b) $a^{\mathcal{I}} \in A_0^{\mathcal{I}}$,

   (c) $b^{\mathcal{I}} \in A_4^{\mathcal{I}}$,

   (d) $\mathcal{I} \models \{A_0 \sqsubseteq \forall r.A_1,\ A_1 \sqsubseteq \neg A_4\}$.

   As (a) and (b), $\mathcal{I} \models A_0 \sqsubseteq \forall r.A_1$ entails $b^{\mathcal{I}} \in A_1^{\mathcal{I}}$. As $\mathcal{I} \models A_1 \sqsubseteq \neg A_4$, we conclude $b^{\mathcal{I}} \not\in A_4^{\mathcal{I}}$, which is in contradiction with (c). Consequently, $\mathcal{K}$ is not consistent.

Solution for the exercise 2.

1. In order to use the tableaux calculus for $\mathcal{ALC}$ introduced in the course, the concepts in $\mathcal{K}$ need to be in NNF and, the GCIs should be of the form $\top \sqsubseteq C$. The knowledge base $\mathcal{K}$ can be transformed into the logically equivalent $\mathcal{K}' = (\mathcal{T}', \mathcal{A}')$ with

   $$\mathcal{T}' = \{\top \sqsubseteq \neg A \sqcup \exists r.A\} \quad \mathcal{A}' = \{a : (A \sqcap (\exists r.B)) \sqcap \exists r.\neg B\}$$

   In order to show that $\mathcal{K}$ is consistent, it is sufficient to derive from $\mathcal{A}'$ an ABox $\mathcal{A}''$ (i.e. $\mathcal{A}' \xrightarrow{*} \mathcal{A}''$) using the tableaux rule for $\mathcal{ALC}$ with the $\sqsubseteq$-rule parameterised by $\mathcal{T}'$ such that $\mathcal{A}''$ is clash-free and complete when the blocking technique is used. We provide a derivation of $\mathcal{A}''$ from the leftmost branch of the tableaux in Figure 1 such that $\mathcal{A}''$ is made of all the concept assertions and role assertions from that leftmost branch. Note that $\mathcal{A}''$ is clash-free and complete. Other branches leading to clashes are shown on the tableau but are not strictly speaking needed.

2. $(\mathcal{T}, \mathcal{A})$ and $(\mathcal{T}', \mathcal{A}')$ have exactly the same interpretations because transforming a concept into an equivalent concept in NNF preserves the semantics and similarly $C \sqsubseteq C'$ and $\top \sqsubseteq \neg C \sqcup C'$ are satisfied by exactly the same interpretations.

   According to the soundness proof with blocking, an interpretation $\mathcal{I}$ such that $\mathcal{I} \models (\mathcal{T}', \mathcal{A}')$ can be built from $\mathcal{A}''$ such that

   - $\Delta^{\mathcal{I}}$ is the set of individual names occurring in $\mathcal{A}''$ except the blocked ones,
   - $(A^*)^{\mathcal{I}}$ is equal to the set of individual names a in $\Delta^{\mathcal{I}}$ such that $a : A^* \in \mathcal{A}''$,
   - For all individual names $a, b \in \Delta^{\mathcal{I}}$, $(a, b) \in r^{\mathcal{I}}$ iff either $(a, b) : r \in \mathcal{A}''$ or there is a blocked individual names b'' in $\mathcal{A}''$ such that $(a, b'') : r \in \mathcal{A}''$ and b'' is blocked by b.

   Consequently, an interpretation $\mathcal{I}$ satisfying $(\mathcal{T}', \mathcal{A}')$ (and therefore satisfying also $(\mathcal{T}, \mathcal{A})$) is defined as follows.

   - $\Delta^{\mathcal{I}} \stackrel{\text{def}}{=} \{a, b_1, b_2\}$.

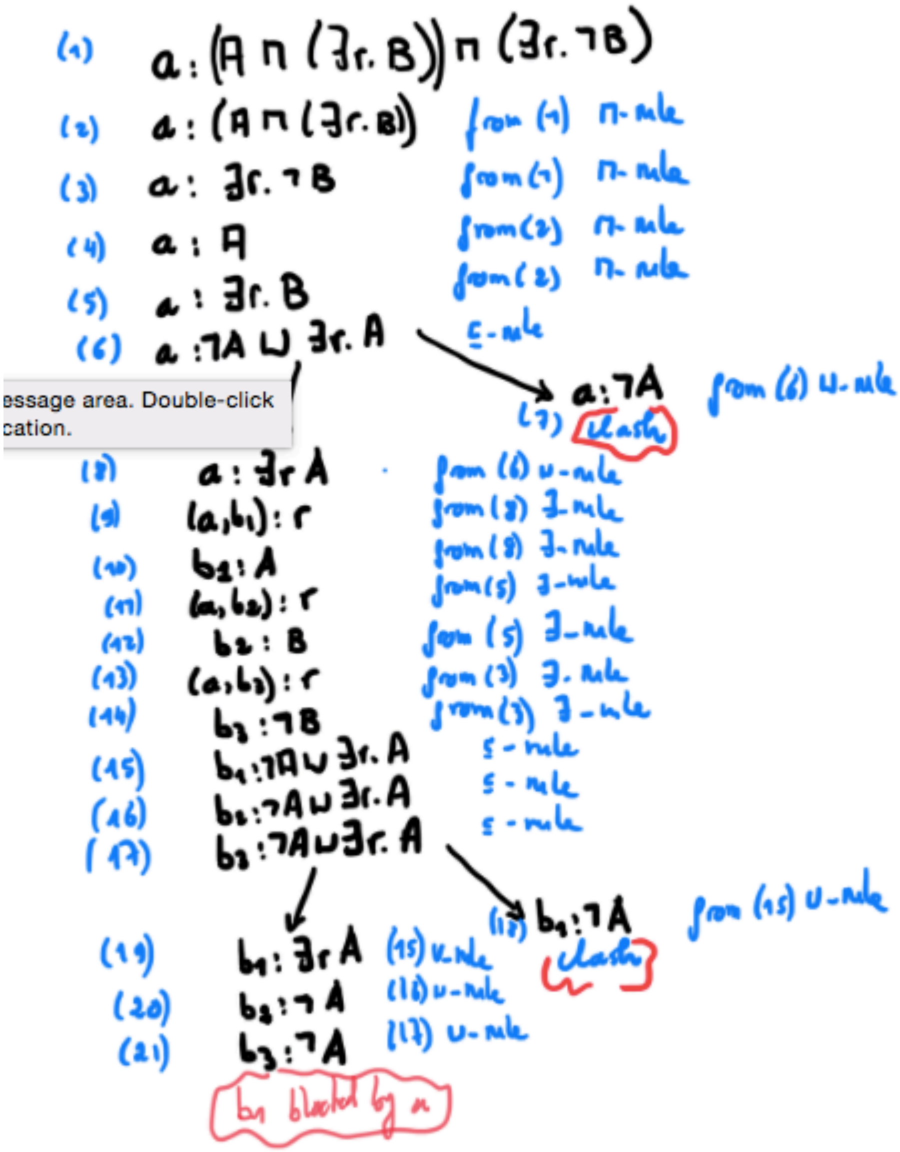

Figure 1: A "tableaux" for deriving the complete and clash-free $\mathcal{A}''$

- $A^{\mathcal{I}} \stackrel{\text{def}}{=} \{a\}$, $B^{\mathcal{I}} \stackrel{\text{def}}{=} \{b_2\}$, and the interpretation of the other concept names is arbitrary.
- $r^{\mathcal{I}} \stackrel{\text{def}}{=} \{(a, b_1), (a, b_2), (b_1, b_1)\}$ and the interpretation of the other role names is arbitrary.

The soundness proof guarantees that $\mathcal{I} \models \mathcal{K}$ but this could be also checked directly.

Solution for the exercise 3. Let us show that $\langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi \Rightarrow (\varphi \wedge \langle\!\langle A\rangle\!\rangle \mathsf{X} \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi)$ is valid for ATL, that is for all CGS $\mathfrak{M}$ and states s, we have $\mathfrak{M}, s \models \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$ implies $\mathfrak{M}, s \models \varphi \wedge \langle\!\langle A\rangle\!\rangle \mathsf{X} \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$.

Let $\mathfrak{M}$ and s be such that $\mathfrak{M}, s \models \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$. Below, we shall show that $\mathfrak{M}, s \models \varphi \wedge \langle\!\langle A\rangle\!\rangle \mathsf{X} \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$. By definition of the semantics for ATL, there is a strategy F for the coalition A such that for all computations $\lambda \in \mathtt{Comp}(s, F)$, for all positions $i \in \mathbb{N}$, we have $\mathfrak{M}, \lambda(i) \models \varphi$. As $\mathtt{Comp}(s, F)$ is non-empty (the action manager always returns a non-empty set of actions for each pair (a, s)), there is a computation $\lambda \in \mathtt{Comp}(s, F)$ and therefore $\mathfrak{M}, \lambda(0) \models \varphi$. But $\lambda(0)$ is precisely s, whence $\mathfrak{M}, s \models \varphi$. It remains to show that $\mathfrak{M}, s \models \langle\!\langle A\rangle\!\rangle \mathsf{X} \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$.

Let $\mathfrak{f} = F(s)$ be the joint action for the coalition A and we know that $\mathtt{out}(s, \mathfrak{f}) \subseteq [\![\varphi]\!]^{\mathfrak{M}}$ since $\mathfrak{M}, s \models \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$. Suppose that $\mathtt{out}(s, \mathfrak{f}) = \{s^\star_1, \ldots, s^\star_\alpha\}$ for some $\alpha \geq 1$.

For each $i \in [1, \alpha]$, let $F^\star_i$ be a strategy such that for each history $s'_0 \cdots s'_n$ with $s'_0 = s^\star_i$, we have $F^\star_i(s'_0 \cdots s'_n) \stackrel{\text{def}}{=} F(s s'_0 \cdots s'_n)$. Consequently, for all $\lambda \in \mathtt{Comp}(s^\star_i, F^\star_i)$, $s \cdot \lambda \in \mathtt{Comp}(s, F)$ (here we use the fact that $\mathfrak{f} = F(s)$ and $s^\star_i \in \mathtt{out}(s, \mathfrak{f})$). Hence, for all positions $j \in \mathbb{N}$, we have $\mathfrak{M}, \lambda(j) \models \varphi$ and therefore $\mathfrak{M}, s^\star_i \models \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$. This means that $\mathtt{out}(s, \mathfrak{f}) \subseteq [\![\langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi]\!]^{\mathfrak{M}}$. However, $\mathfrak{M}, s \models \langle\!\langle A\rangle\!\rangle \mathsf{X} \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$ precisely when there is joint action $\mathfrak{f}'$ such that $\mathtt{out}(s, \mathfrak{f}') \subseteq [\![\langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi]\!]^{\mathfrak{M}}$. Hence, $\mathfrak{M}, s \models \langle\!\langle A\rangle\!\rangle \mathsf{X} \langle\!\langle A\rangle\!\rangle \mathsf{G}\varphi$.

Solution for the exercise 4.

1. Here is the derivation of $A_1 \sqsubseteq A_2$.

$$
\dfrac{
\dfrac{}{A_1 \sqsubseteq B_2}\,\in
\qquad
\dfrac{}{B_1 \sqcap B_2 \sqsubseteq A_2}\,\in
\qquad
\dfrac{
\dfrac{
\dfrac{}{A_1 \sqsubseteq \exists r.A_1}\,\in
\qquad
\dfrac{
\dfrac{}{A_1 \sqsubseteq \top}\,\top
\qquad
\dfrac{}{\top \sqsubseteq B}\,\in
}{A_1 \sqsubseteq B}\,\text{trans}
}{A_1 \sqsubseteq \exists r.B}\,\exists
\qquad
\dfrac{}{\exists r.B \sqsubseteq B_1}\,\in
}{A_1 \sqsubseteq B_1}\,\text{trans}
}{A_1 \sqsubseteq A_2}\,\sqcap
$$

2. First, suppose that $\mathcal{S}(\mathcal{T}) \subseteq \{A_1^*, \ldots, A_\alpha^*\} \cup \{r_1, \ldots, r_\beta\}$ for some $\alpha, \beta \geq 1$ where each A_i^* is a concept name and each r_i is a role name. As $\mathcal{T}$ is finite, α and β always exist. Let us define $\mathcal{T}^c$ as $\{C \sqsubseteq D \mid \mathcal{T} \vdash C \sqsubseteq D\}$. Thanks to the $\in$-rule, $\mathcal{T} \subseteq \mathcal{T}^c$ and $\mathcal{T}^c$ is simple and complete. The fact that $\mathcal{T}^c$ is simple is only due to the property that all the conclusions in inference rules are authorised GCIs in simple TBoxes. Note that completeness is a consequence of the property: $\mathcal{T}^c \vdash C \sqsubseteq D$ implies $\mathcal{T} \vdash C \sqsubseteq D$. Furthermore, observe that the set of GCIs of the form

   $$A \sqsubseteq B \quad A_1 \sqcap A_2 \sqsubseteq B \quad A \sqsubseteq \exists r.B \quad \exists r.A \sqsubseteq B$$

   built from $\mathcal{S}(\mathcal{T})$ is finite and actually cubic in $\alpha + \beta$. Consequently, $\mathcal{T}^c$ is finite. Hence, $\mathcal{T}^c$ is a simple and complete TBox with $\mathcal{T} \subseteq \mathcal{T}^c$. In order to establish that $\mathcal{T}^c$ is the smallest such a set, *ad absurdum*, suppose that $\mathcal{T} \subseteq \mathcal{T}' \subset \mathcal{T}^c$, and $\mathcal{T}'$ is simple and complete. So, there is $C \sqsubseteq D$ such that $\mathcal{T} \vdash C \sqsubseteq D$ and $C \sqsubseteq D \notin \mathcal{T}'$. As $\mathcal{T} \subseteq \mathcal{T}'$, we can also conclude that $\mathcal{T}' \vdash C \sqsubseteq D$. However, we have just seen that $C \sqsubseteq D \notin \mathcal{T}'$, which is in contradiction with the completeness of $\mathcal{T}'$.

   In order to compute $\mathcal{T}^c$, we proceed as follows (saturation algorithm). Given a simple TBox $\mathcal{T}$, we write $C(\mathcal{T})$ to denote the set of GCIs obtained from $\mathcal{T}$ by applying one inference rule from premisses in $\mathcal{T}$. As each rule involves at most three premisses, and each rule inference can be checked in linear time in the size of its premisses, computing $C(\mathcal{T})$ requires cubic time in the size of $\mathcal{T}$. Here is the saturation algorithm.

   - $X = \mathcal{T}$;
   - While $C(X) \neq X$ do $X := C(X)$;
   - return X.

   The while loop is visited a number of times at most cubic in $\alpha + \beta$ and the size of X is also at most cubic in $\alpha + \beta$. Hence, the returned value X is computed in cubic time in $\mathcal{T}$ and is equal to $\mathcal{T}^c$.

3. The proof is on the length of the derivation to establish $\mathcal{T} \vdash C \sqsubseteq D$. In order to provide a complete formal treatment, let us introduce the notion $\mathcal{T}$-derivation. A $\mathcal{T}$-**derivation** is a sequence $(C_1 \sqsubseteq$

$D_1, \ldots, C_K \sqsubseteq D_K)$ $(K \geq 1)$ such that for all $i \in [1, K]$, at least one of the conditions below holds.

(a) $C_i \sqsubseteq D_i \in \mathcal{T}$ (use of the $\in$-rule).

(b) $C_i \sqsubseteq D_i$ is of the form $A \sqsubseteq A$ for some concept name A in $\mathcal{S}(\mathcal{T})$ (use of the id-rule).

(c) $C_i \sqsubseteq D_i$ is of the form $A \sqsubseteq \top$ for some concept name A in $\mathcal{S}(\mathcal{T})$ (use of the $\top$-rule).

(d) there are $i_1, i_2 < i$ such that $C_{i_1} = C_i$, $D_{i_1} = C_{i_2}$, $D_{i_2} = D_i$ (use of the trans-rule).

(e) Similarly (and we omit the very details herein), there are GCIs that occur strictly before $C_i \sqsubseteq D_i$ in the sequence that can be used as premisses for either the $\exists$-rule or the $\sqcap$-rule leading exactly to the conclusion $C_i \sqsubseteq D_i$.

So, $\mathcal{T} \vdash C \sqsubseteq D$ iff there is a $\mathcal{T}$-derivation $(C_1 \sqsubseteq D_1, \ldots, C_K \sqsubseteq D_K)$ such that $C_K \sqsubseteq D_K$ is equal to $C \sqsubseteq D$. It remains to show that for all the $\mathcal{T}$-derivations $(C_1 \sqsubseteq D_1, \ldots, C_K \sqsubseteq D_K)$, we have $\mathcal{T} \models C_K \sqsubseteq D_K$ (which guarantees that for all interpretations $\mathcal{I}$, we have $\mathcal{I} \models \mathcal{T}$ iff $\mathcal{I} \models \mathcal{T} \cup \{C_K \sqsubseteq D_K\}$).

The proof is by induction on i. For the base case, $C_1 \sqsubseteq D_1$ satisfies one condition among (a), (b), (c). As $A \sqsubseteq A$ and $A \sqsubseteq \top$ hold in all interpretations, the cases for (b) and (c) are immediate. Similarly, obviously $\mathcal{T} \models C_1 \sqsubseteq D_1$. Indeed, $\mathcal{I} \models \mathcal{T}$ implies for all $C \sqsubseteq D \in \mathcal{T}$, we have $\mathcal{I} \models C \sqsubseteq D$. In particular, this leads to $\mathcal{I} \models C_1 \sqsubseteq D_1$ as $C_1 \sqsubseteq D_1 \in \mathcal{T}$ in the case (a).

For the induction step, we only treat the cases with the trans-rule and the $\exists$-rule, the case with the $\sqcap$-rule is omitted but very similar.

Suppose that

$$\frac{\mathcal{T} \vdash C_i \sqsubseteq D_{i_1}, \ \ \mathcal{T} \vdash D_{i_1} \sqsubseteq D_i}{\mathcal{T} \vdash C_i \sqsubseteq D_i}$$

with $i_1, i_2 < i$ and $C_{i_1} = C_i$, $D_{i_1} = C_{i_2}$, $D_{i_2} = D_i$. By (IH), $\mathcal{T} \models C_i \sqsubseteq D_{i_1}$ and $\mathcal{T} \models D_{i_1} \sqsubseteq D_i$. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \models \mathcal{T}$. So $\mathcal{I} \models \{C_i \sqsubseteq D_{i_1}, D_{i_1} \sqsubseteq D_i\}$, which leads to $C_i^{\mathcal{I}} \subseteq D_i^{\mathcal{I}}$ by transitivity of set-inclusion and therefore $\mathcal{I} \models C_i \sqsubseteq D_i$. Consequently $\mathcal{T} \models C_i \sqsubseteq D_i$ as $\mathcal{I}$ above were arbitrary.

Consider now the case of the $\exists$-rule.

$$\frac{\mathcal{T} \vdash \overbrace{A \sqsubseteq \exists r.A_1}^{= C_{i_1} \sqsubseteq D_{i_1}}, \quad \mathcal{T} \vdash \overbrace{A_1 \sqsubseteq B_1}^{= C_{i_2} \sqsubseteq D_{i_2}}}{\mathcal{T} \vdash \underbrace{A \sqsubseteq \exists r.B_1}_{= C_i \sqsubseteq D_i}} \; \exists\text{-rule}$$

Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \models \mathcal{T}$. By (IH), $\mathcal{I} \models \{A \sqsubseteq \exists r.A_1, A_1 \sqsubseteq B_1\}$. In particular, this entails that $\mathcal{I} \models \exists r.A_1 \sqsubseteq \exists r.B_1$ as $\mathcal{I} \models A_1 \sqsubseteq B_1$. Indeed, $(\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}$ and $\mathfrak{b} \in A_1^{\mathcal{I}}$ imply $(\mathfrak{a}, \mathfrak{b}) \in r^{\mathcal{I}}$ and $\mathfrak{b} \in B_1^{\mathcal{I}}$ as $A_1^{\mathcal{I}} \subseteq B_1^{\mathcal{I}}$. Now $\mathcal{I} \models \{A \sqsubseteq \exists r.A_1, \exists r.A_1 \sqsubseteq \exists r.B_1\}$ implies $\mathcal{I} \models A \sqsubseteq \exists r.B_1$ by transitivity of set-inclusion. Consequently $\mathcal{T} \models C_i \sqsubseteq D_i$ as $\mathcal{I}$ above were arbitrary.

To conclude, suppose that $C \sqsubseteq D \in \mathcal{T}^c$. By definition of $\mathcal{T}^c$, we have $\mathcal{T} \vdash C \sqsubseteq D$. According to the developments above, we get $\mathcal{T} \models C \sqsubseteq D$.

4. First, let us check that $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$. By definition, $\top^{\mathcal{I}}$ is equal to $\{A \in \Delta^{\mathcal{I}} \mid A \sqsubseteq \top \in \mathcal{T}^c\}$. Since $\mathcal{T}^c$ is complete, by application of the $\top$-rule, for all concept names in $\mathcal{S}(\mathcal{T})$, we have $A \sqsubseteq \top \in \mathcal{T}^c$. Hence $\top^{\mathcal{I}}$ is equal to the set of all concept names in $\mathcal{S}(\mathcal{T})$ (including $\top$), which is precisely $\Delta^{\mathcal{I}}$ by definition.

   Let us show that $\mathcal{I} \models \mathcal{T}^c$. We make a case analysis and we use the fact that $\mathcal{T}^c$ is complete.

   $A \sqsubseteq B \in \mathcal{T}^c$**.** Suppose that $A' \in A^{\mathcal{I}}$. By definition, $A' \sqsubseteq A \in \mathcal{T}^c$. As $\mathcal{T}^c$ is complete, by application of the trans-rule, we get $A' \sqsubseteq B \in \mathcal{T}^c$. By definition of $\mathcal{I}$, $A' \in B^{\mathcal{I}}$. In conclusion, $\mathcal{I} \models A \sqsubseteq B$.

   $A \sqsubseteq \exists r.B \in \mathcal{T}^c$**.** Suppose that $A' \in A^{\mathcal{I}}$. By definition, $A' \sqsubseteq A \in \mathcal{T}^c$. As $\mathcal{T}^c$ is complete, by application of the trans-rule, we get $A' \sqsubseteq \exists r.B \in \mathcal{T}^c$. By definition of $r^{\mathcal{I}}$, we have $(A', B) \in r^{\mathcal{I}}$. Moreover, $B \in B^{\mathcal{I}}$ as $B \sqsubseteq B \in \mathcal{T}^c$ by completeness of $\mathcal{T}^c$ and thanks to the id-rule. Hence, $A' \in (\exists r.B)^{\mathcal{I}}$. In conclusion, $\mathcal{I} \models A \sqsubseteq \exists r.B$.

   $\exists r.A \sqsubseteq B \in \mathcal{T}^c$**.** Suppose that $A' \in (\exists r.A)^{\mathcal{I}}$. So, there is A'' such that $(A', A'') \in r^{\mathcal{I}}$ and $A'' \in A^{\mathcal{I}}$. By definition of $r^{\mathcal{I}}$, $A' \sqsubseteq \exists r'.A'' \in \mathcal{T}^c$.

By definition of $A^{\mathcal{I}}$, $A'' \sqsubseteq A \in \mathcal{T}^c$. By completeness of $\mathcal{T}^c$ and the $\exists$-rule, we have $A' \sqsubseteq \exists r.A \in \mathcal{T}^c$. By completeness of $\mathcal{T}^c$ and the trans-rule, $A' \sqsubseteq B \in \mathcal{T}^c$. By definition of $B^{\mathcal{I}}$, we get $A' \in B^{\mathcal{I}}$. In conclusion, $\mathcal{I} \models \exists r.A \sqsubseteq B$.

$A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{T}^c$. Suppose that $A' \in (A_1 \sqcap A_2)^{\mathcal{I}}$. So $A' \in A_1^{\mathcal{I}}$ and $A' \in A_2^{\mathcal{I}}$ and by definition, $A' \sqsubseteq A_1 \in \mathcal{T}^c$ and $A' \sqsubseteq A_2 \in \mathcal{T}^c$ As $\mathcal{T}^c$ is complete, by application of the $\sqcap$-rule, we get $A' \sqsubseteq B \in \mathcal{T}^c$. By definition of $\mathcal{I}$, $A' \in B^{\mathcal{I}}$. In conclusion, $\mathcal{I} \models A \sqsubseteq B$.

5. Suppose that $A \sqsubseteq B \notin \mathcal{T}^c$. By definition of $B^{\mathcal{I}}$, $A \notin B^{\mathcal{I}}$. Moreover, $A \in A^{\mathcal{I}}$ as $A \sqsubseteq A \in \mathcal{T}^c$ thanks to the id-rule. Consequently, $\mathcal{I} \models \mathcal{T}$ (since $\mathcal{I} \models \mathcal{T}^c$ and $\mathcal{T} \subseteq \mathcal{T}^c$), and $\mathcal{I} \not\models A \sqsubseteq B$. So, $\mathcal{T} \not\models A \sqsubseteq B$.

6. By combining the answers for the questions 3. and 5., we get that $A \sqsubseteq B \in \mathcal{T}^c$ iff $\mathcal{T} \models A \sqsubseteq B$. By the question 2., $\mathcal{T}^c$ can be computed in polynomial time in the size of $\mathcal{T}$. Here is the simple polynomial-time algorithm to check whether for all interpretations $\mathcal{I}$, ($\mathcal{I} \models \mathcal{T}$ implies $\mathcal{I} \models A \sqsubseteq B$).
   (a) compute $\mathcal{T}^c$ from $\mathcal{T}$;
   (b) check whether $A \sqsubseteq B$ belongs to $\mathcal{T}^c$.

⋆⋆⋆⋆⋆

Solution for the exercise 5.

1. Figure 2 contains a graphical representation of the CGS $\mathfrak{M}_{T,n}$ with $T = \{t_0, t_1\}$ and $n = 3$. Note that for any T and n, for all infinite computations λ starting from $(0, 0, t_0)$, for all $i, j \in [0, n-1]$, there is a unique position I such that $\lambda(I)$ is of the form (i, j, t).

2. The path formula $Error^H(i, j)$ is defined as

$$\bigvee_{t \in T} \big((\mathsf{F}((i,j) \mapsto t) \wedge \bigvee_{t', (t',t) \notin H} \mathsf{F}((i-1,j) \mapsto t')\big)$$

3. The path formula $Error^V(i, j)$ is defined as

$$\bigvee_{t \in T} \big((\mathsf{F}((i,j) \mapsto t) \wedge \bigvee_{t', (t',t) \notin V} \mathsf{F}((i,j-1) \mapsto t')\big)$$

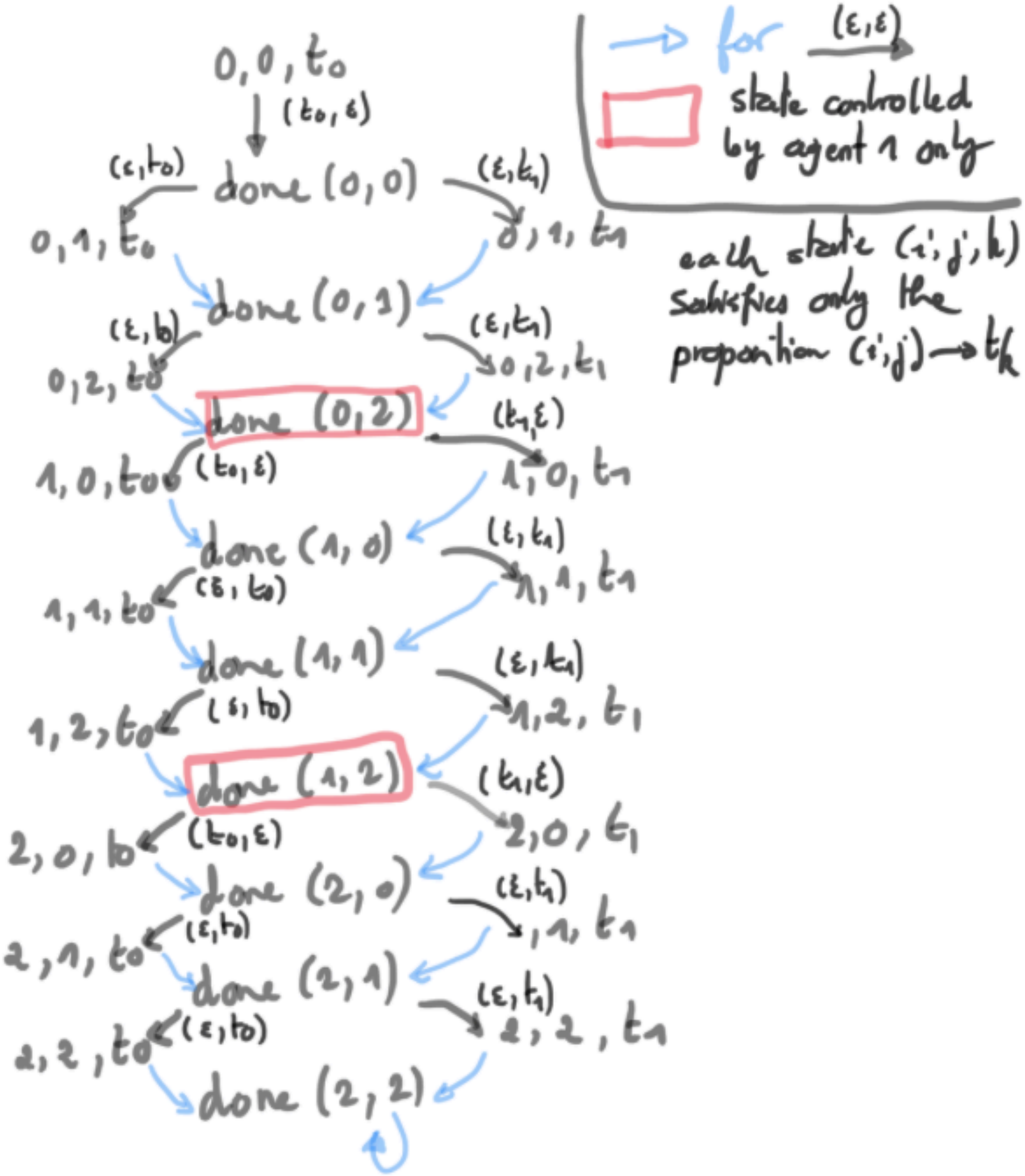

Figure 2: CGS $\mathfrak{M}_{T,n}$ with $T = \{t_0, t_1\}$ and $n = 3$

4. The formula $\varphi_{\mathtt{T},n}$ is defined as follows, taking advantage of the property (that would need to be proved) that strategies for Player 2 in the $(n \times n)$-tiling game problem correspond to strategies for agent 2 in $\mathfrak{M}_{T,n}$ as far as the infinite computations from $(0, 0, t_0)$ are concerned.

$$\langle\!\langle 2 \rangle\!\rangle \Big(\overbrace{\bigwedge_{i \in [1,n-1]} \neg Error^H(i,0)}^{Player\ 2\ does\ not\ lose\ immediately\ on\ row\ 0} \quad \wedge$$

$$\bigwedge_{j \in [1,n-1]} \Big(\overbrace{\bigvee_{i \in [1,n-1]} (Error^H(i,j) \vee Error^V(i,j))}^{if\ Player\ 2\ loses\ immediately\ on\ row\ j} \Rightarrow \underbrace{\bigvee_{j' \in [1,j]} Error^V(0,j')}_{then\ Player\ 1\ loses\ immediately\ before} \Big) \Big)$$

As $\mathfrak{M}_{T,n}$ is of polynomial size in the size of T and n, and $\varphi_{\mathtt{T},n}$ is of polynomial size in the size of $\mathtt{T}$ and n, the above developments correspond to key steps to establish that the model-checking problem for ATL[†] is PSPACE-hard.