Exercise 1. Let us consider the CGS $\mathcal{M}$ below with two agents.

![Graphical representation of an interpretation $I$ with agents $A_g$, concepts $S$, actions $A_t$, and an interpretation $\mathcal{M}$ over two agents $s_1, s_2$.]

Show that $\mathcal{M}, s_1 \models \langle 1 \rangle (\text{GF } p \land \text{GF } q)$ and $\mathcal{M}, s_2 \not\models \langle 2 \rangle (\text{GF } p \land \text{GF } q)$.

Exercise 2. Show that $\langle 0 \rangle \mathcal{G}(\psi \Rightarrow (\varphi \land \langle 1 \rangle X \psi)) \Rightarrow \langle 0 \rangle \mathcal{G}(\psi \Rightarrow \langle 1 \rangle \mathcal{G} \varphi)$ is valid in ATL.

Exercise 3. Given a finite interpretation $\mathcal{I}$, an individual $a \in \Delta^I$ and an $\mathcal{ALC}$ concept $C$, we have seen that checking whether $a \in C^I$ can be checked in polynomial time. Below, we aim at getting this result by using the decision procedure dedicated to the model-checking problem for ATL (written MC(ATL)), known to be in PTIME too. In this exercise, the role names are among $r_1, \ldots, r_\alpha$, the concept names are among $A_1, \ldots, A_\beta$ with fixed $\alpha, \beta \geq 1$. For the sake of simplicity, we exclude $\top$ and $\bot$ and the only concept constructors are restricted to $\neg$, $\sqcap$ and $\exists r_i$, unless otherwise stated. Similarly for ATL, we restrict ourselves to the propositional variables $p_1, \ldots, p_\beta$ and $q_1, \ldots, q_\alpha$.

Given an interpretation $\mathcal{I} = (\Delta^I, \cdot)$ with finite domain $\Delta^I$, we associate the finite CGS $\mathcal{M}_\mathcal{I} = (\text{Agt}, S, \text{Act}, \text{act}, \delta, L)$ as follows:

- $\text{Agt} \overset{\text{def}}{=} \{1\}$,
- $S \overset{\text{def}}{=} \Delta^I \cup \{(a, b, i) \mid i \in [1, \alpha], a, b \in \Delta^I, (a, b) \in r_i^I\}$,
- $\text{Act} \overset{\text{def}}{=} \{(a, b, i) \mid i \in [1, \alpha], a, b \in \Delta^I, (a, b) \in r_i^I\} \cup \{\varepsilon\}$.
- For all $s \in S$ such that $s = (a, b, i)$, we have $\text{act}(1, s) \overset{\text{def}}{=} \{\varepsilon\}$.
- For all $s \in S$ such that $s = a \in \Delta^I$, we have $\text{act}(1, s) \overset{\text{def}}{=} \{(a, b, i) \mid i \in [1, \alpha], a, b \in \Delta^I, (a, b) \in r_i^I\}$.
- As there is a unique agent, we can assume that $\delta$ is defined for a subset of $S \times \text{Act}$, $\delta(a, b, i) \overset{\text{def}}{=} (a, b, i)\), where $\delta(a, b, i) \overset{\text{def}}{=} (a, b, i)$, $\delta((a, b, i), \varepsilon) \overset{\text{def}}{=} b$.
- for all other pairs in $S \times \text{Act}$, $\delta$ is undefined.

- For all $a \in S$, $L(a) \overset{\text{def}}{=} \{p_i \mid i \in [1, \beta], a \in A_i^I\}$; for all $(a, b, i) \in S$, $L((a, b, i)) \overset{\text{def}}{=} \{q_i\}$.

Here is the graphical representation of an interpretation $\mathcal{I}$ (left) and its associated CGS $\mathcal{M}_\mathcal{I}$ (right) for $\alpha = 2$ and $\beta = 1$. 


1. Assume that the size of \( \mathcal{I} \) is defined as 
\[ \text{card}(\Delta \mathcal{I}) \times \beta + \sum_{i=1}^{\alpha} \text{card}(r_i^2) \] 
(written \(|\mathcal{I}|\)) and the size of \( \mathcal{M}_I \) is defined as 
\[ \text{card}(S) \times \beta + \text{card}(S)^2 \times \text{card}(\text{Act}) \] 
(written \(|\mathcal{M}_I|\)), show that \(|\mathcal{M}_I|\) is polynomial in \(|\mathcal{I}|\).

2. Let us define the translation map \( t \) from ALC concepts to ATL formulae:
- \( t(A_i) \equiv p_i \)
- \( t(\neg D) \equiv \neg t(D) \)
- \( t(D_1 \sqcap D_2) \equiv t(D_1) \land t(D_2) \)
- \( t(\exists r_i.D) \equiv \langle\{1\}\rangle X (q_i \land \langle\{1\}\rangle X t(D)) \).

Show that for all \( a \in \Delta \mathcal{I} \) and for all ALC concepts \( C \), we have \( a \in C \mathcal{I} \) (in ALC) iff \( \mathcal{M}_I, a \models t(C) \) (in ATL). *(an additional hypothesis is needed for the above statement to make sense, see the options discussed during the session)*

3. Using the known results about MC(\( \text{ATL} \)), conclude that checking whether \( a \in C \mathcal{I} \) (for ALC) can be done in PTIME.

**Exercise 4.** Show that \( (\langle A\rangle G \varphi) \Rightarrow (\varphi \land \langle A\rangle X (\langle A\rangle G \varphi)) \) is valid for ATL.

**Exercise 5.** Let \( \text{ATL}^\dagger \) be a fragment of \( \text{ATL}^* \) whose state formulae and path formulae are defined as follows.
- State formulae: \( \varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid \langle A\rangle \Phi \)
- Path formulae: \( \Phi ::= \neg \Phi \mid (\Phi \land \Phi) \mid F \varphi \)

Below, we recall the satisfaction relation for state formulae and for path formulae (we omit a few clauses for Boolean connectives).

\[
\begin{align*}
\mathcal{M}, s \models p & \iff p \in L(s) \\
\mathcal{M}, s \models \langle A\rangle \Phi & \iff \text{there is a strategy } F_A \text{ such that for all } \\
& \lambda = s_0s_1 \ldots \in \text{Comp}(s,F_A), \text{ we have } \mathcal{M}, \lambda \models \Phi \\
\mathcal{M}, \lambda \models \neg \Psi & \iff \mathcal{M}, \lambda \not\models \Psi \\
\mathcal{M}, \lambda \models \Psi_1 \land \Psi_2 & \iff \mathcal{M}, \lambda \models \Psi_1 \text{ and } \mathcal{M}, \lambda \models \Psi_2 \\
\mathcal{M}, \lambda \models F \varphi & \iff \mathcal{M}, \lambda(i) \models \varphi \text{ holds for some } i \geq 0
\end{align*}
\]

Let \( T = (T,H,V, t_0) \) be a tiling system with \( T = \{t_0, \ldots, t_K\} \) \((K \geq 0)\) and \( n \geq 2 \). Let us consider the CGS \( \mathcal{M}_{T,n} = (\text{Agt}, S, \text{Act}, \text{act}, \delta, L) \) defined as follows.

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• $Agt \overset{\text{def}}{=} \{1, 2\}, Act \overset{\text{def}}{=} \{\varepsilon\} \cup T$.

• $S \overset{\text{def}}{=} \{(0, 0, t_0)\} \cup \{\text{done}(i, j) \mid i, j \in [0, n - 1]\} \cup \{(i, j, t_k) \mid i, j \in [0, n - 1], k \in [0, K], i + j \neq 0\}$.

• The propositional variables are of the form \((i, j) \mapsto t_k\) with \(i, j \in [0, n - 1], k \in [0, K]\). By definition, the only states \(s\) such that \(L(s)\) is non-empty are those of the form \((i, j, t_k)\) and

\[
L((i, j, t_k)) \overset{\text{def}}{=} \{(i, j) \mapsto t_k\}.
\]

• The definition of the transition function \(\delta\) is performed as follows.

  - The only transition from \((0, 0, t_0)\) is \((0, 0, t_0) \xrightarrow{(0, t_0, \varepsilon)} \text{done}(0, 0)\).
  
  - The only transition from \(\text{done}(n - 1, n - 1)\) (self-loop) is

\[
\text{done}(n - 1, n - 1) \xrightarrow{(\varepsilon, \varepsilon)} \text{done}(n - 1, n - 1)
\]

  - For all \(j \in [0, n - 2]\), the only transitions from \(\text{done}(n - 1, j)\) are

\[
\text{done}(n - 1, j) \xrightarrow{(0, j + 1, t_0)} \ldots, \text{done}(n - 1, j) \xrightarrow{(0, j + 1, t_K)}
\]

  - For all \(i \in [0, n - 2]\) and \(j \in [0, n - 1]\), the only transitions from \(\text{done}(i, j)\) are

\[
\text{done}(i, j) \xrightarrow{(\varepsilon, t_0)} (i + 1, j, t_0), \ldots, \text{done}(i, j) \xrightarrow{(\varepsilon, t_K)} (i + 1, j, t_K)
\]

  - For all \(i, j \in [0, n - 1]\) and \(k \in [0, K]\) with \(i + j \neq 0\), the only transition from \((i, j, t_k)\) is \((i, j, t_k) \xrightarrow{(\varepsilon, \varepsilon)} \text{done}(i, j)\).  

• The definition of act should be clear from the definition of \(\delta\). For instance, if \(i < n - 1\), then \(\text{act}(2, \text{done}(i, j)) \overset{\text{def}}{=} T\) and \(\text{act}(1, \text{done}(i, j)) \overset{\text{def}}{=} \{\varepsilon\}\). We omit the other cases.

1. Represent graphically the CGS $\mathcal{M}_{T, n}$ for $T = \{t_0, t_1\}$ and $n = 3$.

   In the following questions, no correctness proof is required. We expect only the definition of formulae, possibly accompanied with a few hints.

2. For all $i \in [1, n - 1]$ and $j \in [0, n - 1]$, design an ATL$^\dagger$ path formula $\text{Error}^H(i, j)$ such that for every infinite computation $\lambda$ from $(0, 0, t_0)$, $\mathcal{M}_{T, n, \lambda} \models \text{Error}^H(i, j)$ iff there are $I, J \in \mathbb{N}$ such that $\lambda(I) = (i, j, t)$ and $\lambda(J) = (i - 1, j, t')$ for some tiles $t, t'$ such that $(t', t) \not\in H$.

3. For all $i \in [0, n - 1]$ and $j \in [1, n - 1]$, define an ATL$^\dagger$ path formula $\text{Error}^V(i, j)$ such that for every infinite computation $\lambda$ from $(0, 0, t_0)$, $\mathcal{M}_{T, n, \lambda} \models \text{Error}^V(i, j)$ iff there are $I, J \in \mathbb{N}$ such that $\lambda(I) = (i, j, t)$ and $\lambda(J) = (i, j - 1, t')$ for some tiles $t, t'$ such that $(t', t) \not\in V$.
4. Define a state formula $\varphi_{T,n}$ such that $\mathcal{M}_{T,n, (0, 0, t_0)} \models \varphi_{T,n}$ iff Player 2 has a winning strategy with the tiling system $T$ and on the arena $[0, n - 1] \times [0, n - 1]$ for the $(n \times n)$-tiling game problem.