Logical Aspects of AI: Knowledge Logics
Exam – January 13th, 2021, 14h00–18h00, Online

This exam takes place between 2pm and 6pm but it is intended to be an exam that requires between two and three hours. The extra time is provided to the students in order to attenuate the effects of an exam online. This exam document has five pages and contains five independent exercises. Copies should be sent by email to `demri@lsv.fr` *strictly before 6pm on January 13th as well as by a second means freely choosen by each student (dropbox for instance).*
Bonus exercise. *The students are expected to solve the exercises 1 to 4 within that time frame. Exercise 5 is optional and it is intended for those who wish to collect extra points during the extra time. Nevertheless, the number of points (added to the mark on 20 for the first four exercices) is limited (at most 3 points).*
Authorised documents. *The only authorised documents are the slides, the exercise sheets as well as the correction sheets provided during the sessions. No communication between the students is allowed during the exam time (2pm to 6pm) and no communication is allowed with any other person about the content of the exam.*
Contact. *During the exam, you can contact S. Demri through the collaborate plateform at the usual address or by email to* `demri@lsv.fr`.
Format of your copy. *As usual, please write down the solutions of the exercises with very much care. Your copy can be written in French or in English.*

Reminder. Please find below notations and definitions that might be useful.

- In $\mathcal{ALC}$, A and B denote concept names, whereas C and D denote arbitrary concepts, unless otherwise said.
- As usual, in ATL, we write $\varphi \Rightarrow \psi$ as a shortcut for $\neg(\varphi \wedge \neg\psi)$.

Exercise 1. Consider the $\mathcal{ALC}$ knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with

- $\mathcal{T} = \{A_0 \sqsubseteq \forall r.A_1,\ A_1 \sqsubseteq \neg A_4,\ A_0 \sqsubseteq A_2 \sqcup A_3,\ A_2 \sqsubseteq \exists r.A_4,\ \exists r.\neg A_1 \sqsubseteq A_5,\ A_3 \sqsubseteq A_5\}$,
- $\mathcal{A} = \{a : A_0, (a, b) : r, b : A_4\}$.

(the A_i's are concept names)

1. Do we have $\mathcal{T} \models A_0 \sqsubseteq \exists r.A_1$?
2. Is $\mathcal{K}$ consistent?

Exercise 2. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base with

$$\mathcal{T} = \{A \sqsubseteq \neg\forall r.\neg A\} \quad \mathcal{A} = \{a : (A \sqcap (\exists r.B)) \sqcap (\neg\forall r.B)\}$$

1. Using the tableaux calculus for $\mathcal{ALC}$, show that $\mathcal{K}$ is consistent.
2. Based on the derivation from Question 1., define an interpretation satisfying $\mathcal{K}$.

Exercise 3. Show that $(\langle\!\langle A \rangle\!\rangle \mathsf{G}\varphi) \Rightarrow (\varphi \wedge \langle\!\langle A \rangle\!\rangle \mathsf{X} \langle\!\langle A \rangle\!\rangle \mathsf{G}\varphi)$ is valid for ATL.

Exercise 4. An $\mathcal{ALC}$ TBox $\mathcal{T}$ is said to be **simple** if it contains GCIs of the form

$$A \sqsubseteq B \quad A_1 \sqcap A_2 \sqsubseteq B \quad A \sqsubseteq \exists r.B \quad \exists r.A \sqsubseteq B$$

where A, the A_i's and B are arbitrary concept names or $\top$ and r is an arbitrary role name. In the sequel, by convention, we consider that $\top$ is a special concept name (instead of a truth constant) whose interpretation is always the full interpretation domain $\Delta^{\mathcal{I}}$ (assuming that the interpretation is $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$). Given a simple TBox $\mathcal{T}$, we write $\mathcal{S}(\mathcal{T})$ to denote the set of concept and role names occurring in $\mathcal{T}$ with the addition of $\top$ (when $\top$ is not already in $\mathcal{T}$). Below, let us consider a fixed simple TBox $\mathcal{T}$.

1. We introduce rules to deduce new GCIs from $\mathcal{T}$. In the rules below, A, the A_i's and B are concept names in $\mathcal{S}(\mathcal{T})$, the C_i's use only concept names and role names from $\mathcal{S}(\mathcal{T})$ too. We write $\mathcal{T} \vdash C \sqsubseteq D$ when $C \sqsubseteq D$ can be derived from $\mathcal{T}$ by applying the rules below.

$$\frac{C \sqsubseteq D \in \mathcal{T}}{\mathcal{T} \vdash C \sqsubseteq D} \in\text{-rule} \qquad \frac{}{\mathcal{T} \vdash A \sqsubseteq A} \text{id-rule} \qquad \frac{}{\mathcal{T} \vdash A \sqsubseteq \top} \top\text{-rule}$$

$$\frac{\mathcal{T} \vdash A \sqsubseteq A_1, \;\; \mathcal{T} \vdash A \sqsubseteq A_2, \;\; \mathcal{T} \vdash A_1 \sqcap A_2 \sqsubseteq B}{\mathcal{T} \vdash A \sqsubseteq B} \sqcap\text{-rule}$$

$$\frac{\mathcal{T} \vdash A \sqsubseteq \exists r.A_1, \;\; \mathcal{T} \vdash A_1 \sqsubseteq B}{\mathcal{T} \vdash A \sqsubseteq \exists r.B} \; \exists\text{-rule}$$

$$\frac{\mathcal{T} \vdash C_1 \sqsubseteq C_2, \;\; \mathcal{T} \vdash C_2 \sqsubseteq C_3}{\mathcal{T} \vdash C_1 \sqsubseteq C_3} \; \text{trans-rule}$$

under the condition: $\{C_1 \sqsubseteq C_3\}$ is simple.

Show that

$$\{\exists r.B \sqsubseteq B_1, A_1 \sqsubseteq B_2, B_1 \sqcap B_2 \sqsubseteq A_2, A_1 \sqsubseteq \exists r.A_1, \top \sqsubseteq B\} \;\vdash\; A_1 \sqsubseteq A_2.$$

2. A simple TBox $\mathcal{T}^c$ is **complete** $\stackrel{\text{def}}{\Leftrightarrow}$ for all $C \sqsubseteq D$, $\mathcal{T}^c \vdash C \sqsubseteq D$ implies $C \sqsubseteq D \in \mathcal{T}^c$. Given a simple TBox $\mathcal{T}$, show that there is a smallest complete and simple $\mathcal{T}^c$ such that $\mathcal{T} \subseteq \mathcal{T}^c$ ('smallest' refers to set-inclusion). Moreover, evaluate the time required to compute $\mathcal{T}^c$ from $\mathcal{T}$.

3. Prove that if $\mathcal{T} \vdash C \sqsubseteq D$, then $\mathcal{T}$ and $\mathcal{T} \cup \{C \sqsubseteq D\}$ are satisfied by exactly the same interpretations. Conclude $C \sqsubseteq D \in \mathcal{T}^c$ (with $\mathcal{T}^c$ from Question 2.) implies $\mathcal{T} \models C \sqsubseteq D$.

4. Let $\mathcal{I}$ be the interpretation defined as follows (depending on $\mathcal{T}$ via $\mathcal{T}^c$).

   - $\Delta^{\mathcal{I}}$ is the set of concept names from $\mathcal{S}(\mathcal{T})$ (including $\top$).
   - $A^{\mathcal{I}} \stackrel{\text{def}}{=} \{B \in \Delta^{\mathcal{I}} \mid B \sqsubseteq A \in \mathcal{T}^c\}$ for all concept names A in $\mathcal{S}(\mathcal{T})$.
   - $r^{\mathcal{I}} \stackrel{\text{def}}{=} \{(A, B) \mid A \sqsubseteq \exists r.B \in \mathcal{T}^c\}$ for all role names r in $\mathcal{S}(\mathcal{T})$.

   Verify that $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$. Show that $\mathcal{I} \models \mathcal{T}^c$.

5. Conclude that for all $A, B \in \mathcal{S}(\mathcal{T})$, we have $A \sqsubseteq B \notin \mathcal{T}^c$ implies $\mathcal{T} \not\models A \sqsubseteq B$.

6. Show that given an arbitrary simple TBox $\mathcal{T}$ and $A, B \in \mathcal{S}(\mathcal{T})$, checking whether $\mathcal{T} \models A \sqsubseteq B$ can be done in polynomial time in the size of $\mathcal{T}$.

⋆⋆⋆⋆⋆

Exercise 5. (bonus) Let $\mathsf{ATL}^{\dagger}$ be a fragment of ATL^{*} whose state formulae and path formulae are defined as follows.

$$\begin{array}{llll}\text{State formulae:} & \varphi & ::= & p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\!\langle A \rangle\!\rangle \Phi \\ \text{Path formulae:} & \Phi & ::= & \neg\Phi \mid (\Phi \wedge \Phi) \mid \mathsf{F}\varphi\end{array}$$

Below, we recall the satisfaction relation for state formulae and for path formulae (we omit a few clauses for Boolean connectives).

$$\begin{array}{lll}\mathfrak{M}, s \models p & \overset{\text{def}}{\Leftrightarrow} & p \in L(s) \\ \mathfrak{M}, s \models \langle\!\langle A \rangle\!\rangle \Phi & \overset{\text{def}}{\Leftrightarrow} & \text{there is a strategy } F_A \text{ such that for all} \\ & & \lambda = s_0 s_1 \ldots \in \mathtt{Comp}(s, F_A), \text{ we have } \mathfrak{M}, \lambda \models \Phi \\ \mathfrak{M}, \lambda \models \neg\Psi & \text{iff} & \mathfrak{M}, \lambda \not\models \Psi \\ \mathfrak{M}, \lambda \models \Psi_1 \wedge \Psi_2 & \text{iff} & \mathfrak{M}, \lambda \models \Psi_1 \text{ and } \mathfrak{M}, \lambda \models \Psi_2 \\ \mathfrak{M}, \lambda \models \mathsf{F}\varphi & \text{iff} & \mathfrak{M}, \lambda(i) \models \varphi \text{ holds for some } i \geq 0\end{array}$$

Let $\mathtt{T} = (T, H, V, t_0)$ be a tiling system with $T = \{t_0, \ldots, t_K\}$ ($K \geq 0$) and $n \geq 2$. Let us consider the CGS $\mathfrak{M}_{T,n} = (Agt, S, Act, \mathtt{act}, \delta, L)$ defined as follows.

- $Agt \overset{\text{def}}{=} \{1, 2\}$, $Act \overset{\text{def}}{=} \{\varepsilon\} \cup T$.
- $S \overset{\text{def}}{=} \{(0, 0, t_0)\} \cup \{done(i, j) \mid i, j \in [0, n-1]\} \cup \{(i, j, t_k) \mid i, j \in [0, n-1],\ k \in [0, K], i + j \neq 0\}$.
- The propositional variables are of the form "$(i, j) \mapsto t_k$" with $i, j \in [0, n-1]$, $k \in [0, K]$. By definition, the only states s such that $L(s)$ is non-empty are those of the form (i, j, t_k) and

$$L((i, j, t_k)) \overset{\text{def}}{=} \{(i, j) \mapsto t_k\}.$$

- The definition of the transition function δ is performed as follows.
  - The only transition from $(0, 0, t_0)$ is $(0, 0, t_0) \xrightarrow{(t_0, \varepsilon)} done(0, 0)$.
  - The only transition from $done(n-1, n-1)$ (self-loop) is

$$done(n-1, n-1) \xrightarrow{(\varepsilon, \varepsilon)} done(n-1, n-1)$$

  - For all $j \in [0, n-2]$, the only transitions from $done(n-1, j)$ are

$$done(n-1, j) \xrightarrow{(t_0, \varepsilon)} (0, j+1, t_0), \ldots, done(n-1, j) \xrightarrow{(t_K, \varepsilon)} (0, j+1, t_K)$$

– For all $i \in [0, n-2]$ and $j \in [0, n-1]$, the only transitions from $done(i,j)$ are

$$done(i,j) \xrightarrow{(\varepsilon,t_0)} (i+1,j,t_0), \ldots, done(i,j) \xrightarrow{(\varepsilon,t_K)} (i+1,j,t_K)$$

– For all $i, j \in [0, n-1]$ and $k \in [0, K]$ with $i + j \neq 0$, the only transition from (i,j,t_k) is $(i,j,t_k) \xrightarrow{(\varepsilon,\varepsilon)} done(i,j)$.

- The definition of `act` should be clear from the definition of δ. For instance, if $i < n-1$, then $\texttt{act}(2, done(i,j)) \stackrel{\text{def}}{=} T$ and $\texttt{act}(1, done(i,j)) \stackrel{\text{def}}{=} \{\varepsilon\}$. We omit the other cases.

1. (0.5 point) Represent graphically the CGS $\mathfrak{M}_{T,n}$ for $T = \{t_0, t_1\}$ and $n = 3$.

   In the following questions, no correctness proof is required. We expect only the definition of formulae, possibly accompanied with a few hints to facilitate the correction.

2. (0.5 point) For all $i \in [1, n-1]$ and $j \in [0, n-1]$, design an $\mathsf{ATL}^\dagger$ path formula $Error^H(i,j)$ such that for every infinite computation λ from $(0,0,t_0)$, $\mathfrak{M}_{T,n}, \lambda \models Error^H(i,j)$ iff there are $I, J \in \mathbb{N}$ such that $\lambda(I) = (i,j,t)$ and $\lambda(J) = (i-1,j,t')$ for some tiles t, t' such that $(t',t) \notin H$.

3. (0.5 point) For all $i \in [0, n-1]$ and $j \in [1, n-1]$, define an $\mathsf{ATL}^\dagger$ path formula $Error^V(i,j)$ such that for every infinite computation λ from $(0,0,t_0)$, $\mathfrak{M}_{T,n}, \lambda \models Error^V(i,j)$ iff there are $I, J \in \mathbb{N}$ such that $\lambda(I) = (i,j,t)$ and $\lambda(J) = (i,j-1,t')$ for some tiles t, t' such that $(t',t) \notin V$.

4. (1.5 points) Define a state formula $\varphi_{\mathtt{T},n}$ such that $\mathfrak{M}_{T,n}, (0,0,t_0) \models \varphi_{\mathtt{T},n}$ iff Player 2 has a winning strategy with the tiling system `T` and on the arena $[0, n-1] \times [0, n-1]$ for the $(n \times n)$-tiling game problem.