Theorem 3 (Kruskal) *If $\geq_{\mathcal{F}}$ is a wqo, then $\trianglelefteq$ is a wqo on $T(\mathcal{F}$.*

Proof:
By contradiction, assume that the set of counter-example sequences

$$\mathcal{E} = \{\{u_i\}_{i\in\mathbb{N}} \mid \forall i < j.u_i \not\trianglelefteq u_j\}$$

is not empty.

We construct a minimal sequence as follows: $\mathcal{E}_0 = \mathcal{E}$, u_i is a minimal (w.r.t. size) term such that there is a sequence $u_0, \ldots, u_i, \ldots$ in $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$ is the set of sequences in $\mathcal{E}_i$, that start with $u_0, \ldots, u_i$.

The sequence $\{u_i\}_{i\in\mathbb{N}}$ belongs to $\mathcal{E}$: for every $i < j$, there is a sequence $u_0, \ldots, u_j, v_{j+1}, \ldots$ in $\mathcal{E}_{i+1} \subseteq \mathcal{E}$ hence $u_i \not\trianglelefteq u_j$.

Next, we extract from $\{u_i\}_{i\in\mathbb{N}}$ a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ such that the root symbols f_{n_i} of u_{n_i} are increasing; this is possible since $\geq_{\mathcal{F}}$ is a wqo and thanks to the proposition 5.

Now, consider the set $\mathcal{D}$ of strict subterms of $\{u_{n_i}\}_{i\in\mathbb{N}}$. We claim that $\trianglelefteq$ is a wqo on $\mathcal{D}$. Indeed, consider an infinite sequence $\{v_j\}_{j\in\mathbb{N}}$ of terms in $\mathcal{D}$. If there are two identical terms in the sequence, then there is $i < j$ such that $v_i \trianglelefteq v_j$. Otherwise, v_0 is a strict subterm of some u_{m_0}. We extract from v_j the subsequence v_{k_j} of terms that are not subterms of $u_0, \ldots, u_{m_0}$. This is possible since there are only finitely many subterms of $u_0, \ldots, u_{m_0}$, while there are infinitely many distinct terms in v_j. The sequence $u_0, \ldots, u_{m_0-1}, v_{k_0}, v_{k_1} \ldots$ is not in $\mathcal{E}$, since $|v_{k_0}| < |u_{m_0}|$ and by minimality of the counter-example $\{u_i\}_{i\in\mathbb{N}}$. Then either there are two indices $k_i < k_j$ such that $v_{k_i} \trianglelefteq v_{k_j}$ or else there is an index $j < m_0$ and an index ℓ such that $u_j \trianglelefteq v_{k_\ell}$. But, in the latter case, since v_{k_ℓ} is a subterm of some $u_m, m > m_0$, we would have $u_j \trianglelefteq u_m$ with $j < m$, which is not possible.

To summarize, in any case, there are $j < k$ such that $v_j \trianglelefteq v_k$: $\trianglelefteq$ is a wqo on $\mathcal{D}$.

By Higman's lemma (lemma 3), $\trianglelefteq^w_\trianglelefteq$ is a wqo on $\mathcal{D}^*$. Consider now the sequence of words $\{w_i\}_{i\in\mathbb{N}}$ in $\mathcal{D}^*$ defined by $w_i = s_1 \cdots s_{m_i}$ if $u_{n_i} = f_{n_i}(s_1, \ldots, s_{m_i})$ (i.e., the concatenation of immediate subterms of u_{n_i}). Since $\trianglelefteq^w_\trianglelefteq$ is a wqo, there are two indices $i < j$ such that $w_i \trianglelefteq^w_\trianglelefteq w_j$. Then, by definition of $\trianglelefteq$ and since the sequence f_{n_i} is increasing, this implies $u_{n_i} \trianglelefteq u_{n_j}$. This is a contradiction since $\{u_i\}_{i\in\mathbb{N}}$ is supposed to belong to $\mathcal{E}$.

Hence $\mathcal{E}$ is empty.

$\square$

In many applications below, we consider a finite set $\mathcal{F}$, in which case the wqo on $\mathcal{F}$ does not matter (any reflexive and transitive relation on $\mathcal{F}$ is a wqo, for instance the equality) and is therefore not precised.

3.4 Simplification (quasi)-orderings

Definition 11 *A* simplification (quasi-)ordering *is a (quasi-)ordering $\leq$ on $T(\mathcal{F}, X)$ such that*

1. *If s is a strict subterm of t, then $s < t$.*
2. *(**Stability**) for every terms t, u and every substitution σ, if $t < u$ then $t\sigma < u\sigma$ (and if $t \simeq u$, then $t\sigma \simeq u\sigma$)*
3. *(**Monotonicity**) For every $t_1, \ldots, t_n, u_1, \ldots, u_m$, if $t_1 \leq u_1, \ldots, t_n \leq u_n$, then $f(t_1, \ldots, t_n) \leq f(u_1, \ldots, u_n)$ and, if $t_i < u_i$ for some i, then $f(t_1, \ldots, t_n) < f(u_1, \ldots, u_n)$.*

Proposition 11 *If $\mathcal{F}$ is finite, then simplification orderings are well-founded on $T(\mathcal{F}, X)$.*

Proof:
Let $\leq$ be a simplification ordering on $T(\mathcal{F}, X)$ where $\mathcal{F}$ is finite.

Let $x_0 \in \mathcal{X}$ and $\mathcal{T} = T(\mathcal{F}, x_0)$. Let $s_0 \in \mathcal{T}$.

First observe that, thanks to the first and last properties of simplification orderings, any simplification ordering contains the embedding (that extends the equality on $\mathcal{F}$). In particular $\leq$ contains $\trianglelefteq$

Now, if $t_0 > t_1 > \ldots$ is an infinite strictly decreasing sequence in $T(\mathcal{F}, X)$, let u_i be the term obtained from t_i by replacing every variable of t_i with s_0. By stability, $u_0 > u_1 > \ldots$ is a strictly decreasing sequence in $\mathcal{T}$.

On the other hand, $\mathcal{F} \cup \{x_0\}$ is finite, hence, thanks to Kruskal theorem, $\trianglelefteq$ is a wqo on $\mathcal{T}$. Therefore there are two indices $i < j$ such that $u_i \trianglelefteq u_j$. This contradicts the fact that $\leq$ contains $\trianglelefteq$. □

3.5 Recursive path orderings

Definition 12 *Let $\mathcal{F}$ be a set of function symbols, $\geq_{\mathcal{F}}$ be a wqo on $\mathcal{F}$ and* status *is a mapping from $\mathcal{F}$ into $\{\mathsf{lex}, \mathsf{mul}\}$. The recursive path (quasi-)ordering $\geq_{rpo}$ that extends $\geq_{\mathcal{F}}$ and* status *is defined on $T(\mathcal{F}, X)$ as follows:*

$$s \equiv f(s_1, \ldots, s_n) \geq g(t_1, \ldots, t_m) \equiv t$$

iff one of the following conditions is satisfied:

1. *(subterm):* $\exists i.\ s_i \geq_{rpo} t$
2. *(precedence):* $f >_{\mathcal{F}} g$ *and* $\forall i.\ s >_{rpo} t_i$
3. *(multiset):* $f \simeq_{\mathcal{F}} g$ *and* $\mathsf{status}(f) = \mathsf{status}(g) = \mathsf{mul}$ *and*

$$\{\{s_1, \ldots, s_n\}\} \geqq^{mul}_{rpo} \{\{t_1, \ldots, t_m\}\}$$

4. *(lexicographic):* $f \simeq_{\mathcal{F}} g$ *and* $\mathsf{status}(f) = \mathsf{status}(g) = \mathsf{lex}$ *and*

$$\forall i.\ s >_{rpo} t_i$$

and

$$(s_1, \ldots, s_n) \geqq^{lex}_{rpo} (t_1, \ldots, t_m)$$

When s *is a variable,* $s \geq_{rpo} t$ *iff* $s = t$.
When t *is a variable,* $s \geq_{rpo} t$ *iff* $t \in Var(s)$.

In this definition $\geqq^{mul}_{rpo}$ and $\geqq^{lex}_{rpo}$ are respectively the multiset and the lexicographic extension of the recursive path ordering.

This definition is effective: all recursive calls to $\geq_{rpo}$ (or its multiset/lexicographic extensions) are on pairs of terms whose total size is strictly smaller.

Also note that we considered in this definition a lexicographic comparison from left to right. It is also possible to add other status, comparing lexicographically a permutation of the subterms (for instance from right to left). We did not include this possibility, for simplicity.

Lemma 4 *If* $s \geq_{rpo} g(t_1, \ldots, t_n)$*, then, for every* i*,* $s >_{rpo} t_i$.

Proof:
We proceed by induction on the sum of the sizes of s, t, distinguishing between the cases in the proof of $s \geq_{rpo} t$:

Subterm: If $s_j \geq_{rpo} t$ for some j, then, by induction hypothesis, $s_j >_{rpo} t_i$ for all i, hence $s \geq_{rpo} t_i$ for all i. Suppose $t_i \geq_{rpo} s$. Then $t_i > s_j$ by induction hypothesis, which is a contradiction. Hence $s >_{rpo} t_i$.

Precedence or Lexicographic: $s >_{rpo} t_i$ by definition

Multiset: by definition of the multiset extension, for every i there is a j such that $s_j \geq_{rpo} t_i$, hence $s \geq_{rpo} t_i$. Assume by contradiction that $t_i \geq_{rpo} s$. By induction hypothesis, for every j, $t_i >_{rpo} s_j$. A contradiction.

□

Lemma 5 *If $s \geq_{rpo} t$ by Subterm or Precedence, then $s >_{rpo} t$.*

Proof:
(Sketch): by contradiction, using lemma 4. □

Let $=_{mul}$ be the least symmetric and reflexive relation such that, if $f \simeq_{calF} g$ and there is a permulation π such that $s_1 =_{mul} t_{\pi(1)}, \ldots, s_n =_{mul} t_{\pi(n)}$, then $f(s_1, \ldots, s_n) =_{mul} g(t_1, \ldots, t_n)$.

Lemma 6 *$s \geq_{rpo} t$ and $t \geq_{rpo} s$ iff $s =_{mul} t$.*

Proof:
(Sketch): by induction, using lemma 5. □

Lemma 7 *$\geq_{rpo}$ is reflexive.*

Lemma 8 *If t is a strict subterm of s, then $s >_{rpo} t$.*

Proof:
(Sketch): use lemmas 4 and 6. □

Lemma 9 *$\geq_{rpo}$ is transitive.*

Proof:
(Sketch): We use an induction on the sum of the sizes of the three terms and rely on lemma 4 for instance. □

Lemma 10 *$\geq_{rpo}$ is a quasi-ordering. If $\geq_{\mathcal{F}}$ is a total ordering, then $\geq_{rpo}$ is a total ordering on $T(\mathcal{F})$.*

Proof:
(Sketch). For the first part, we use lemma 10 and lemma 7. For the second part, we reason by contradiction, considering a minimal (w.r.t. size) pair of incomparable terms. □

Lemma 11 $\geq_{rpo}$ *is monotonic (in the sense of definition 11).*

Proof:
(Sketch): use the cases 3 and 4 in the definition of $\geq_{rpo}$. □

Lemma 12 $\geq_{rpo}$ *is stable by substitution.*

Proof:
(Sketch): by induction on the sum of the sizes of s, t, we prove $s >_{rpo} t \Rightarrow s\sigma >_{rpo} t\sigma$. □

Theorem 4 $\geq_{rpo}$ *is a simplification ordering. In particular it is well-founded.*