Chapter 3

Termination

3.1 Wqos

A quasi-ordering is a transitive and reflexive relation. The equivalence relation \( \simeq \) associated with a quasi-ordering \( \geq \) is defined as \( x \simeq y \) iff \( x \geq y \) and \( y \geq x \). The strict ordering associated with a quasi-ordering \( \geq \) is the relation \( \geq \setminus \simeq \).

Definition 4 A quasi-ordering \( \geq \) is well-founded if there is no infinite sequence \( \{s_i\}_{i \in \mathbb{N}} \) such that, for every \( i \), \( s_i > s_{i+1} \).

Definition 5 A well quasi ordering (wqo in short) is a quasi ordering \( \geq \) such that, for every infinite sequence \( \{s_i\}_{i \in \mathbb{N}} \), there are two indices \( i < j \) such that \( s_j \geq s_i \).

Proposition 1 If \( \geq \) is a wqo on the set \( D \), then every infinite subset of \( D \) contains finitely many minimal elements, up to \( \simeq \).

Proof: By contradiction: if there was infinitely many minimal elements, we could construct an infinite sequence of pairwise incomparable elements. \( \square \)

Proposition 2 Any wqo is well-founded.

Proposition 3 Any quasi-ordering that contains a wqo is well-founded.

Proof: Any infinite decreasing sequence does not contain two elements \( i < j \) such
that $s_j \geq s_i$. 

\begin{lemma}
If $\geq$ is well-founded, then for every $d$ there is a $d'$ such that $d \geq d'$ and, for every $d''$, $d' \not> d''$ ($d'$ is minimal).
\end{lemma}

\begin{proof}
let us construct a strictly decreasing sequence as follows: $d_0 = d$ and, if $d_n$ is not minimal, then $d_n > d_{n+1}$. By well-foundedness this sequence is finite, hence there is a $n$ such that $d_n$ is minimal. Furthermore, $d_0 \geq d_n$ by transitivity.
\end{proof}

\begin{proposition}
A quasi-ordering $\geq$ is a wqo iff
\begin{enumerate}
\item It is well-founded
\item Every infinite sequence contains two comparable elements
\end{enumerate}
\end{proposition}

\begin{proof}
The only if direction is a consequence of the two previous propositions.
Consider now a well founded quasi-ordering such that any infinite sequence contains two comparable elements. Let $\{s_i\}_{i \in \mathbb{N}}$ be an infinite sequence. If there are two indices $i, j$ such that $s_i \simeq s_j$, then the proof is completed. Assume now it is not the case.
Let $M = \{s_i | i \in \mathbb{N}, \forall j, s_i \not> s_j\}$ (minimal elements). By well-foundedness and lemma 1, 
\begin{equation}
\{s_i | i \in \mathbb{N}\} = \bigcup_{m \in M} \{s_j | j \in \mathbb{N}, s_j \geq m\}
\end{equation}

Since every infinite sequence contains two comparable elements, $M$ is finite, hence there is a $s_{i_0} = m \in M$ such that $\{s_j | j \in \mathbb{N}, s_j \geq m\}$ is infinite. In particular it contains a $s_{j_0}$ with $j_0 > i_0$. This shows that there are two indices $i_0 < j_0$ such that $s_{i_0} \leq s_{j_0}$. $\geq$ is therefore a wqo.
\end{proof}

\begin{proposition}
If $\geq$ is a wqo, then from every infinite sequence $\{s_i\}_{i \in \mathbb{N}}$ it is possible to extract a subsequence $\{s_{i_j}\}_{j \in \mathbb{N}}$ such that, for every $j$, $s_{i_{j+1}} \geq s_{i_j}$.
\end{proposition}

\begin{proof}
We construct by induction on $j$ an increasing subsequence $s_{i_j}$ such that the sets $E_j = \{s_k | k \geq i_j, s_k \geq s_{i_j}\}$ is infinite. $E_0 = \mathbb{N}$ and, for every $j$, we let
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$M_j$ be the set of minimal elements of $E_j$, up to $\simeq$: $\forall e \in E_j, (\forall e' \in E_j.e \not\simeq e') \Rightarrow (\exists e'' \in M_j.e \simeq e'')$ and two elements in $M_j$ are incomparable.

By proposition 3 and lemma 1,

$$E_j = \bigcup_{m \in M_j} \{ s_k \in E_j \mid s_k \geq m \}$$

Since $E_j$ is infinite, there is a $m = s_{i_j+1} \in M$ such that $E_{j+1} = \{ s_k \in E_j \mid s_k \geq m \}$ is infinite.

By construction, $s_{i_{j+1}} \geq s_{i_j}$ for every $j$. □

3.2 Construction of orderings

Definition 6 If $(D_1, \geq_1), \ldots, (D_n, \geq_n)$ are quasi-ordered sets, then the product quasi-ordering $\geq_{\times} = (\geq_1, \ldots, \geq_n)$ is defined on $D_1 \times \cdots \times D_n$ by

$$(d_1, \ldots, d_n) \geq_{\times} (d'_1, \ldots, d'_n) \text{ iff } \forall i. d_i \simeq_i d'_i$$

Proposition 6 A product quasi-ordering is a wqo (resp. is well-founded) iff each of its components is a wqo (resp. well-founded).

Example: the product ordering on $\mathbb{N}^k$ is a wqo.

Definition 7 If $(D_1, \geq_1), \ldots, (D_n, \geq_n)$ are quasi-ordered sets, then the lexicographic composition $\geq_{\text{lex}} = (\geq_1, \ldots, \geq_n)_{\text{lex}}$ is defined on $D_1 \times \cdots \times D_n$ by

$$(d_1, \ldots, d_n) >_{\text{lex}} (d'_1, \ldots, d'_n) \text{ iff } \exists j. (\forall i < j. d_i \simeq_i d'_i) \wedge d_j >_j d'_j$$

Proposition 7 The lexicographic composition of quasi-orderings is a wqo (resp. is well-founded) iff each of its components is a wqo (resp. well-founded).

A (finite) multiset on $D$ is a mapping from $D$ to $\mathbb{N}$, which is 0, except on a finite subset of $D$. $M + N$ is defined by $(M + N)(k) = M(k) + N(k)$. $\emptyset$ is the multiset mapping every element to 0. $\{|x_1, \ldots, x_n\}$ is the multiset mapping $x_i$ to $|\{j \in \{1, \ldots, n\} \mid x_j = x_i\}|$ and 0 otherwise.

Definition 8 The multiset extension of a quasi-ordering $\geq$ on $D$ is the least quasi-ordering $\geq_{\text{mul}}$ on the multisets such that:
1. $M \geq_{\text{mul}} \emptyset$

2. for every $M, M, N$,

$$M \geq_{\text{mul}} M' \Rightarrow M + N \geq_{\text{mul}} M' + N$$

3. for every $n \in \mathbb{N}$, for every $M, x, x_1, \ldots, x_n$,

$$(\forall i. x >_D x_i) \Rightarrow M + \{x\} \geq_{\text{mul}} M + \{x_1, \ldots, x_n\}$$

**Proposition 8** The multiset extension of a quasi-ordering $\geq$ is well-founded (resp. is a wqo) iff $\geq$ is well-founded (resp. is a wqo).

### 3.3 Embedding

**Definition 9** Let $(D, \leq)$ be a quasi-ordered set. The embedding extension $\sqsubseteq_{\leq}w$ of $\leq$ on $D^*$ is the least relation on $D^*$ such that

1. $\epsilon \sqsubseteq_{\leq}w \epsilon$

2. for every $u, v \in D^*$, for every $a \in D$, $u \sqsubseteq_{\leq}w v \Rightarrow u \sqsubseteq_{\leq}w a \cdot v$

3. for every $a, b \in D$ and every $u, v \in D^*$,

$$u \sqsubseteq_{\leq}w v \land a \leq b \Rightarrow au \sqsubseteq_{\leq}w bv$$

**Lemma 2** $\sqsubseteq_{\leq}w$ is a well-founded quasi-ordering if $\leq$ is a well-founded quasi-ordering.

**Lemma 3 (Higman)** $\sqsubseteq_{\leq}w$ is a wqo iff $\leq$ is a wqo.

**Proof:**

By contradiction: assume there is an infinite sequence $\{w_i\}_{i \in \mathbb{N}}$ such that, for every $i < j$, $w_i \not\sqsubseteq_{\leq}w_j$. Then the set $\mathcal{E} = \{(w_i)_{i \in \mathbb{N}} \mid \forall i < j. w_i \not\sqsubseteq_{\leq}w_j\}$ is not empty. We construct by induction a minimal counter-example $(v_i)_{i \in \mathbb{N}}$ and non-empty sets of counter-examples $\mathcal{E}_i$ as follows: $\mathcal{E}_0 = \mathcal{E}$. Let $(w_i)_{i \in \mathbb{N}} \in \mathcal{E}_j$ be such that $w_0 = v_0, \ldots, w_{j-1} = v_{j-1}$ and $|w_j|$ is minimal. We let then $v_j = w_j$ and $E_{j+1} = \{(w_i)_{i \in \mathbb{N}} \in E_j \mid w_0 = v_0, \ldots, w_j = v_j\}$. $E_{j+1}$ is non-empty by construction.

Consider then the sequence $a_i$ of the first letters of $v_i$. Since $\leq$ is a wqo, thanks to proposition 5 there is an infinite increasing subsequence $\{a_i\}_{i \in \mathbb{N}}$. 
Consider then the sequence \( \{x_i\}_{i \in \mathbb{N}}: v_1, \ldots, v_{i_0-1}, v'_{i_0}, v'_{i_1}, \ldots, v'_{i_n}, \ldots \) where \( v'_{i_j} \) is obtained from \( v_{i_j} \) by removing the first letter \( a_{i_j} \). By minimality assumption on the counter example, there are two indices \( j < k \) such that \( x_j \leq_{\leq} x_k \). By construction of the sequence \( v_i, j \geq i_0 \) (otherwise \( v_i \) is not a counter-example sequence): there are two indices \( m < n \) such that \( v'_{i_m} \leq_{\leq} v'_{i_n} \). But, since \( a_{i_m} \leq a_{i_n} \), thanks to the last point of the definiton, \( v_{i_m} = a_{i_m} \cdot v'_{i_m} \leq_{\leq} a_{i_n} \cdot v'_{i_n} = v_{i_n} \). A contradiction.

\( \square \)

**Definition 10** Assuming a quasi-ordering on \( F \), embedding \( \subseteq \) is the least relation on \( T(F) \) such that
1. for every \( u \in T(|\text{calF}|) \), \( u \preceq u \)

2. for every \( f \in \mathcal{F} \), \( i \in [1..a(f)] \), \( u_1, \ldots, u_{a(f)}, v \in T(\mathcal{F}) \), \( v \preceq u_i \Rightarrow v \preceq f(u_1, \ldots, u_{a(f)}) \)

3. for every \( f, g \in \mathcal{F} \) such that \( a(f) = m \) and \( a(g) = n \geq m \), for every increasing index sequence \( j_1 < \ldots < j_m \), if, for every \( k \), \( v_k \preceq u_{j_k} \), then \( f(v_1, \ldots, v_k) \preceq g(u_1, \ldots, u_n) \).

An example of embedding is displayed in the figure 3.1, when \( \geq_{\mathcal{F}} \) is the equality.

**Proposition 9** The tree embedding \( \preceq \) is well-founded iff \( \geq_{\mathcal{F}} \) is well-founded.

**Proposition 10** Assume that \( \mathcal{F} \) is finite and \( \geq_{\mathcal{F}} \) is the equality. Then the tree embedding is simply the rewrite relation on \( T(\mathcal{F}) \), associated with the rewriting system

\[
f(x_1, \ldots, x_n) \rightarrow x_i \quad f \in \mathcal{F}, i \in [1..n]
\]