Verification and Game Theory

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Thanks to:

my co-authors Nicolas Markey, Romain Brenguier,
Michael Ummels, Nathan Thomasset
Stéphane Le Roux for recent discussions on the subject
Thomas Brihaye for some of the slides

Outline

1. Verification and game theory
2. What is a game?
3. A glimpse on strategic games
4. Games on graphs
   - The general model
   - Focus on a simple scenario
   - Adding probabilities to the setting?
   - Concurrent games
5. Conclusion

Computer programming

Computer programming is a difficult task

- understand deeply the initial problem;
- find a solution;
- write the program correctly.

Software bugs

- It is an error, a failure in a computer program or system that induces an incorrect result.
- It may have catastrophic consequences.

Software bugs

Bug example

In August 2005, a Malaysian Airlines MH124 (Boeing 777) that was on autopilot suddenly ascended 2,000 feet.

Bug consequences

- loss of confidence from users' point of view,
- loss of credibility from institutions' point of view,
- large financial loss,
- human loss,. . .

$\Rightarrow$ Real need to **verify** the correctness of a program!

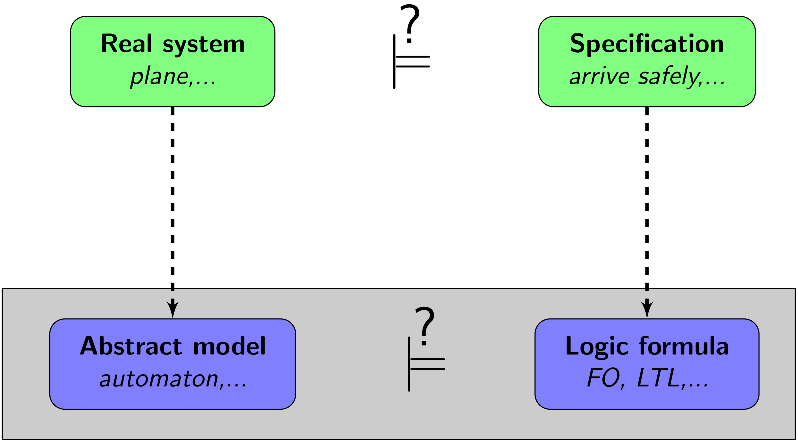
The model-checking approach to verification

Real system
plane,...

$$\stackrel{?}{\models}$$

Specification
arrive safely,...

The model-checking approach to verification



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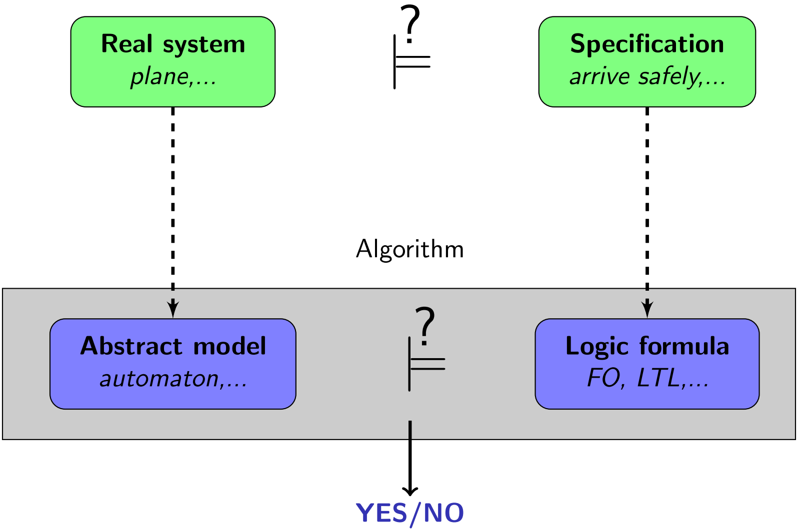


The autopilot case

Requirement: to arrive safely **in every weather condition**,

The autopilot case

Requirement: to arrive safely **in every weather condition**,
while minimising the fuel consumption.

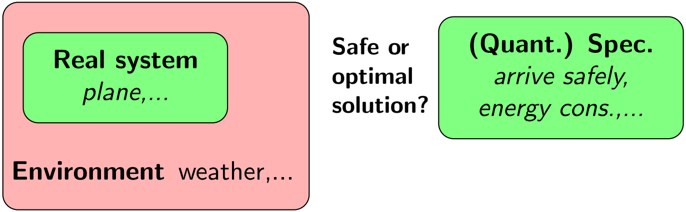
Controlling computer systems

Real system
plane,...

Environment weather,...

(Quant.) Spec.
arrive safely,
energy cons.,...

Controlling computer systems



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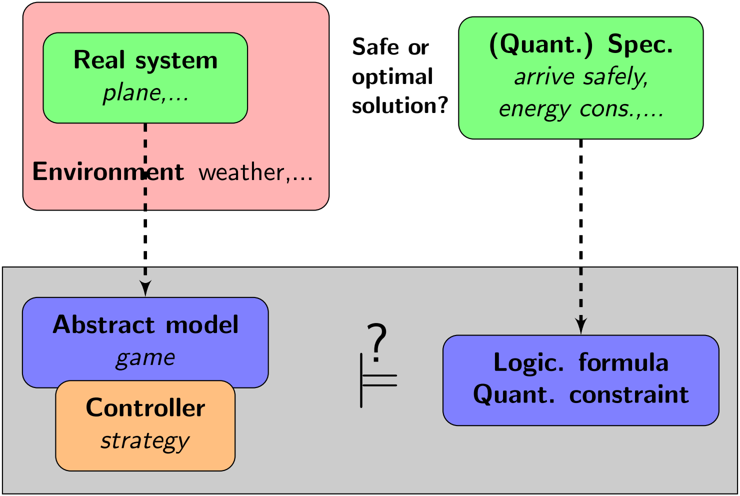

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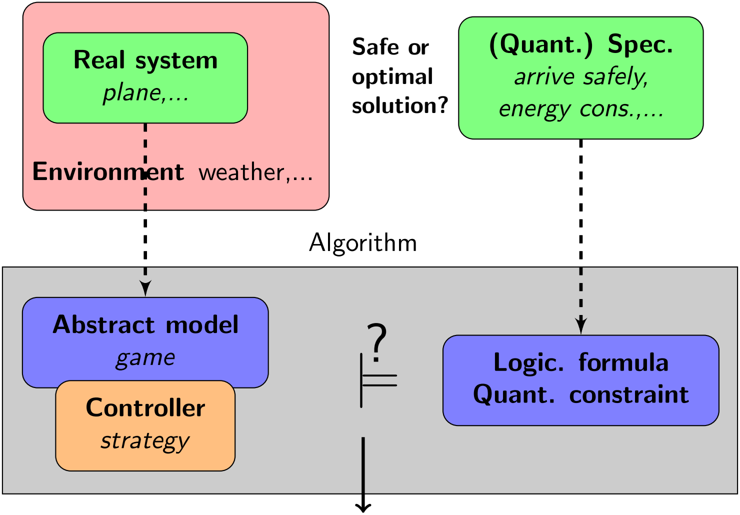

NO/YES + A controller

The autopilot case

Requirement: to arrive safely **in every weather condition**,

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Requirement: to arrive safely **in every weather condition**, **taking into account the other planes**,

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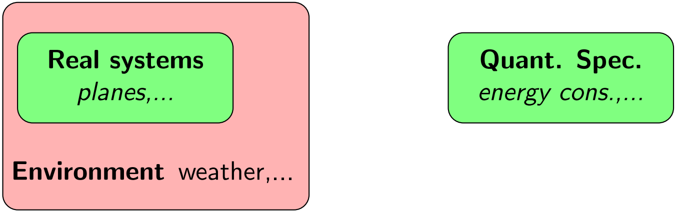
Controlling complex interactive computer systems



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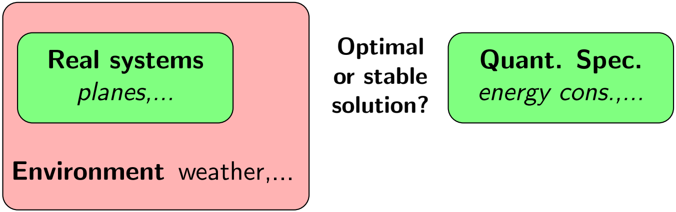

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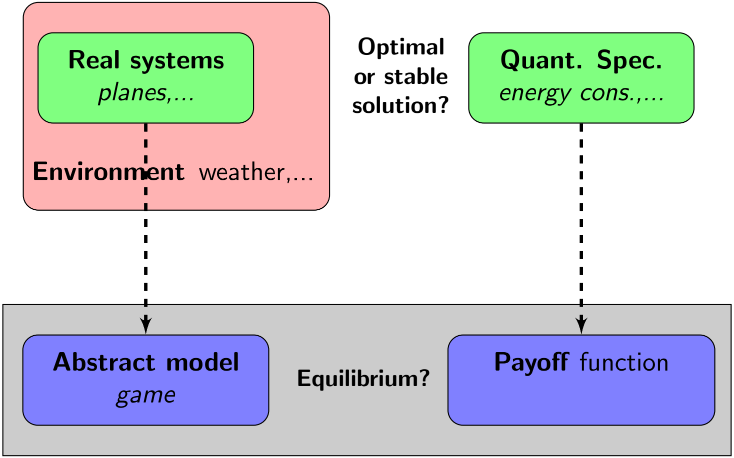

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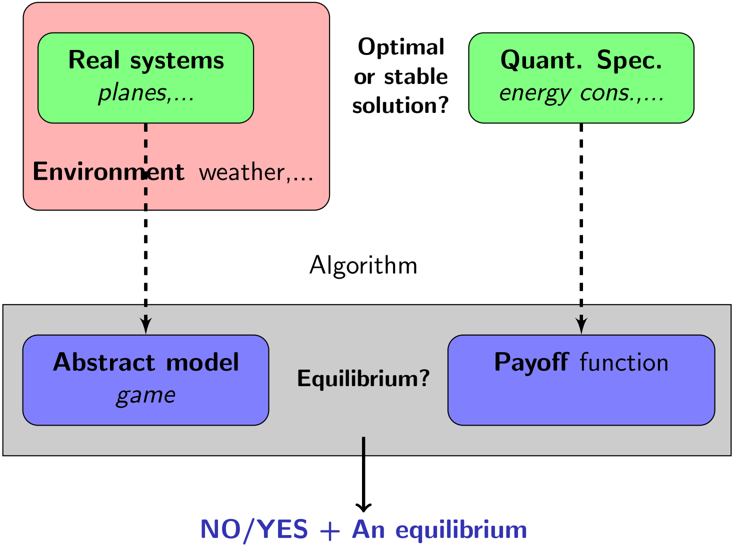

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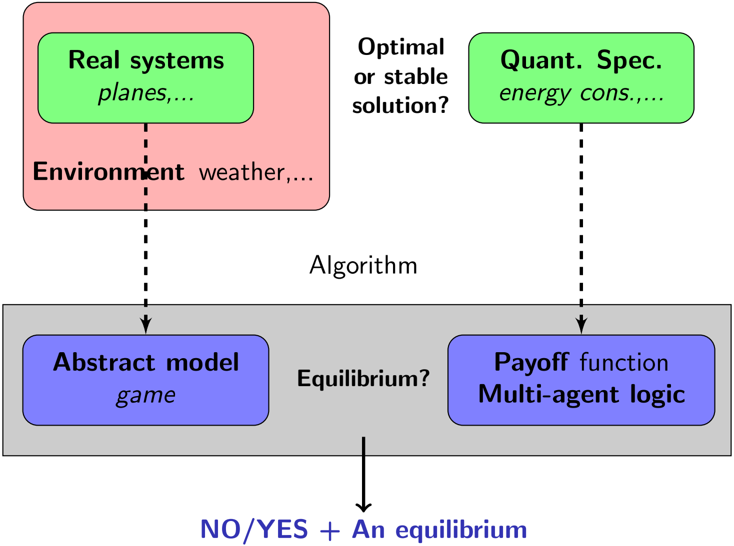

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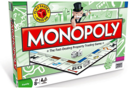
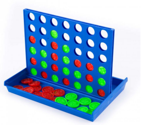
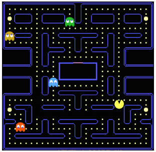
Games we play for fun







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Goal: Model and analyze (using mathematical tools) situations of interactive decision making

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- Several decision makers (called players)

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Wide range of applicability

"[...] it is a context-free mathematical toolbox"

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Interactivity!

Wide range of applicability

"[...] it is a context-free mathematical toolbox"

- Social science: e.g. social choice theory
- Theoretical economics: e.g. models of markets, auctions
- Political science: e.g. fair division
- Biology: e.g. evolutionary biology
- ...

[MSZ13] Maschler, Solan, Zamir. Game theory *(Cambridge University Press)*

The prisoner dilemma

Two suspects are arrested by the police. The police, having separated both prisoners, visit each of them to offer the same deal.

- If one testifies (`Defects`) for the prosecution against the other and the other remains silent (`Cooperates`), the betrayer goes free and the silent accomplice receives the full 10-year sentence.
- If both remain silent, both are sentenced to only 3 years in jail.
- If each betrays the other, each receives a 5-year sentence.

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How should the prisoners act?

Modelled as a matrix game

	C	D
C	$(-3,-3)$	$(-10,0)$
D	$(0,-10)$	$(-5,-5)$

The Nim game

The rules (simplified version)

- Two players, turn-based games
- Initially, there are 8 matches
- On each turn, a player must remove 1 or 2 matches
- The player removing the last match wins the game

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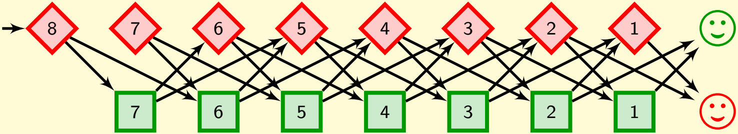
Modelled as a game played on a graph



Various models of games

Many models of games

- Strategic games
- Repeated games
- Games played on graphs
- Games played using equations
- ...

Many features

- imperfect information
- presence of randomness
- continuous time
- ...

Let us suppose that:

- we have fixed a game,
- we have identified an adequate model for this game.

The next natural question is:

What is a **solution** for this game?

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Strategic games (aka matrix games, or one-shot games)

Strategic game

A strategic game G is a triple $\left(\mathsf{Agt}, \Sigma, (g_A)_{A \in \mathsf{Agt}}\right)$ where:

- Agt is the finite and non empty set of players,
- Σ is a non empty set of actions,
- $g_A : \Sigma^{\mathsf{Agt}} \to \mathbb{R}$ is the payoff function of player $A \in \mathsf{Agt}$.

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Example: Prisoner dilemma

- $\mathsf{Agt} = \{A_1, A_2\}$,
- $\Sigma = \{\mathtt{C}, \mathtt{D}\}$
- (g_{A_1}, g_{A_2}) is given by

	C	D
C	$(-3, -3)$	$(-10, 0)$
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Hypotheses made in classical game theory

Hypotheses

- The players are **intelligent** (i.e. they reason perfectly and quickly)
- The players are **rational** (i.e. they want to maximise their payoff)
- The players are **selfish** (i.e. they only care for their own payoff)

Optimality

Dominating profile

A profile $\mathbf{b} \in \Sigma^{\mathsf{Agt}}$ is dominating if

$$\forall \mathbf{c} \in \Sigma^{\mathsf{Agt}}\ \forall A \in \mathsf{Agt} \qquad g_A(\mathbf{c}) \leq g_A(\mathbf{b})$$

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	L	R
T	$(0, 0)$	$(2, 1)$
B	$(3, 2)$	$(1, 2)$

- $(\mathtt{B}, \mathtt{L})$ is optimal!

Strict domination

Stricly dominated action (or strategy)

An action $b_A \in \Sigma$ is strictly dominated by $c_A \in \Sigma$ for player $A \in \mathrm{Agt}$ if

$$\forall \mathbf{a}_{-A} \in \Sigma^{\mathrm{Agt} \setminus \{A\}} \qquad g_A(b_A, \mathbf{a}_{-A}) < g_A(c_A, \mathbf{a}_{-A})$$

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The only rational issue of the game is (D, D) whose payoff is $(-5, -5)$.

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The only rational issue of the game is (D, D) whose payoff is $(-5, -5)$.

(Even though this is sub-optimal)

Stability: the concept of Nash equilibria

Nash equilibrium

Let $\left(\mathsf{Agt}, \Sigma, (g_A)_{A \in \mathsf{Agt}}\right)$ be a strategic game and $\mathbf{b} \in \Sigma^{\mathsf{Agt}}$ be a strategy profile. We say that $\mathbf{b}$ is a Nash equilibrium iff

$$\forall A \in \mathsf{Agt},\ \forall d_A \in \Sigma \text{ s.t. } g_A(\mathbf{b}_{-A}, d_A) \leq g_A(\mathbf{b})$$

A rational player should not deviate from the Nash equilibrium.

[Nash50] Equilibrium Points in n-Person Games (1950).

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- $(\mathtt{D}, \mathtt{D})$ is the unique Nash equilibrium...

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- ... even if (C, C) would be better for both prisoners

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	L	R
T	$(0, 0)$	$(2, 1)$
B	$(3, 2)$	$(1, 2)$

- R dominates L (but not strictly)
- $(\mathtt{B}, \mathtt{R})$ is not a Nash equilibrium, but $(\mathtt{T}, \mathtt{R})$ is a Nash equilibrium
- $(\mathtt{B}, \mathtt{L})$ is optimal, hence a Nash equilibrium

[Nash50] Equilibrium Points in *n*-Person Games (1950).

Do all the finite matrix games have a Nash equilibrium?

Do all the finite matrix games have a Nash equilibrium?

No!

Do all the finite matrix games have a Nash equilibrium?

No!

The matching penny game

	a	b
a	$(1, 0)$	$(0, 1)$
b	$(0, 1)$	$(1, 0)$

Mixed strategies

Given E, we denote $\Delta(E)$ the set of probability distributions over E.

Mixed strategies

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If Σ is the of actions (or strategies), $\Delta(\Sigma)$ is the set of mixed strategies.

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Expected payoff

Let $\sigma = (\sigma_{A_1}, \ldots, \sigma_{A_n})$ be a mixed strategy profile. Let $A \in \mathsf{Agt}$:

$$\widetilde{g}_A(\sigma) = \sum_{\mathbf{b}=(b_A)_{A \in \mathsf{Agt}} \in \Sigma^{\mathsf{Agt}}} \underbrace{\left(\prod_{A \in \mathsf{Agt}} \sigma_A(b_A)\right)}_{\text{probability of } \mathbf{b}} g_A(\mathbf{b})$$

is the expected payoff of player A.

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Mixed extension of game G

$\widetilde{G} \stackrel{\text{def}}{=} \Big(\mathrm{Agt}, \Delta(\Sigma), (\widetilde{g}_A)_{A\in\mathrm{Agt}}\Big)$ is a game.

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$\widetilde{G} \stackrel{\text{def}}{=} \left(\mathrm{Agt}, \Delta(\Sigma), (\widetilde{g}_A)_{A \in \mathrm{Agt}}\right)$ is a game.

G has a mixed Nash equilibrium iff $\widetilde{G}$ has a Nash equilibrium.

Nash equilibria in mixed strategies

	a	b
a	$(1, 0)$	$(0, 1)$
b	$(0, 1)$	$(1, 0)$

The following profile is a Nash equilibrium in mixed strategies:

$$\sigma_{A_1} = \frac{1}{2} \cdot \mathtt{a} + \frac{1}{2} \cdot \mathtt{b} \quad \text{and} \quad \sigma_{A_2} = \frac{1}{2} \cdot \mathtt{a} + \frac{1}{2} \cdot \mathtt{b}$$

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Nash Theorem [Nash50]

Any finite game admits mixed Nash equilibria.

[Nash50] Equilibrium Points in *n*-Person Games (1950).

Best response

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Let $A \in \mathrm{Agt}$ and $\mathbf{a}_{-A} \in \Sigma^{\mathrm{Agt}\setminus\{A\}}$ be a strategy profile for A's opponents.

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- Best response correspondence of Player A

$$\begin{aligned} \mathrm{BR}_A : \Sigma^{\mathrm{Agt}\setminus\{A\}} &\rightarrow \mathcal{P}(\Sigma) \\ \mathbf{a}_{-A} &\rightarrow \{b_A \mid b_A \text{ is a best response to } \mathbf{a}_{-A}\} \end{aligned}$$

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- Best response correspondence of the game

$$\begin{aligned} \mathsf{BR} : \Sigma^{\mathsf{Agt}} &\to \mathcal{P}\left(\Sigma^{\mathsf{Agt}}\right) \\ \mathbf{a} &\to \prod_{A \in \mathsf{Agt}} \mathsf{BR}_A(\mathbf{a}_{-A}) \end{aligned}$$

Best response and Nash equilibrium

Proposition

Let $\mathbf{a}$ be a strategy profile.

$$\mathbf{a} \text{ is a Nash equilibrium} \quad \text{if and only if} \quad \mathbf{a} \in \mathrm{BR}(\mathbf{a})$$

An example

	L	R
T	$(1, -1)$	$(0, 0)$
B	$(0, 0)$	$(2, -2)$

An example

	L	R
T	$(1, -1)$	$(0, 0)$
B	$(0, 0)$	$(2, -2)$

A strategy consists in giving a probability distribution over $\{\mathtt{T}, \mathtt{B}\}$ (resp. $\{\mathtt{L}, \mathtt{R}\}$), that is, it consists in fixing the probability to play T (resp. L).

Assume

$$\sigma_{A_1} = \frac{1}{4} \cdot \mathtt{T} + \frac{3}{4} \cdot \mathtt{B} \qquad \text{and} \qquad \sigma_{A_2} = \frac{1}{2} \cdot \mathtt{L} + \frac{1}{2} \cdot \mathtt{R}$$

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In general, we have

$$\sigma_{A_1} = \alpha \cdot \mathtt{T} + (1 - \alpha) \cdot \mathtt{B} \qquad \text{and} \qquad \sigma_{A_2} = \beta \cdot \mathtt{L} + (1 - \beta) \cdot \mathtt{R}$$

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$$g_{A_1}(\alpha, \beta) \;=\; \alpha(3\beta - 2) - 2\beta + 2 \;=\; -g_{A_2}(\alpha, \beta)$$

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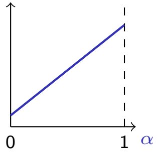
$$g_{A_1}(\alpha, \beta) = \alpha(3\beta - 2) - 2\beta + 2$$

$$\mathrm{BR}_{A_1}(\beta) = \begin{cases} \{1\} & \text{if } 3\beta - 2 > 0 \\ & \\ & \end{cases}$$



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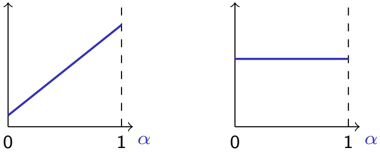

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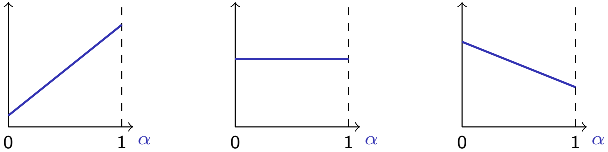

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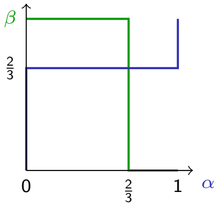

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Thus the following profile is an equilibrium in mixed strategies:

$$\sigma_{A_1} = \frac{2}{3} \cdot \mathtt{T} + \frac{1}{3} \cdot \mathtt{B} \qquad \text{and} \qquad \sigma_{A_2} = \frac{2}{3} \cdot \mathtt{L} + \frac{1}{3} \cdot \mathtt{R}$$

whose expected payoff is:

$$\left(\frac{2}{3}, -\frac{2}{3}\right)$$

Best response and Nash equilibrium

Proposition

Let $\mathbf{a}$ be a strategy profile.

$$\mathbf{a} \text{ is a Nash equilibrium if and only if } \mathbf{a} \in \mathrm{BR}(\mathbf{a})$$

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Nash Theorem [Nash50]

Any finite game admits mixed Nash equilibria.

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Key ingredient of the proof: Brouwer's fixpoint theorem
Or simply Kakutani's fixpoint theorem

Fixpoint theorems

Brouwer's fixpoint theorem

Let $X \subseteq \mathbb{R}^n$ be a convex, compact and nonempty set. Then every continuous function $f\colon X \to X$ has a fixpoint.

Kakutani's fixpoint theorem

Let X be a non-empty, compact and convex subset of $\mathbb{R}^n$. Let $f\colon X \to 2^X$ be a set-valued function on X with a closed graph and the property that $f(x)$ is non-empty and convex for all $x \in X$. Then f has a fixpoint.

Outline

1. Verification and game theory
2. What is a game?
3. A glimpse on strategic games
4. Games on graphs
   - The general model
   - Focus on a simple scenario
   - Adding probabilities to the setting?
   - Concurrent games
5. Conclusion

Which games do we need for verification?

Methodology

- Pick standard models used in model-checking
- Expand them with interaction capabilities

$\rightsquigarrow$ Games played on graphs

  - Several features in the graph: stochastic or deterministic
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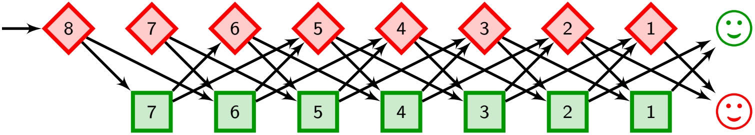
The Nim game modelled as a turn-based game



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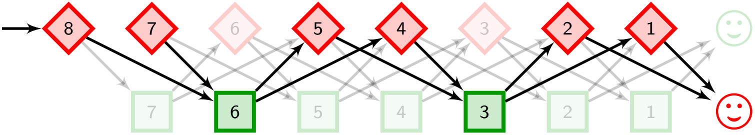

This is then just a matter of computing winning states (controller synthesis)

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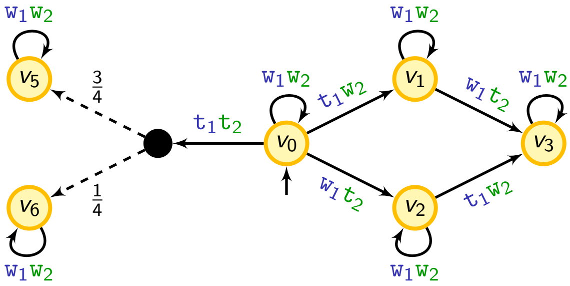
Multiplayer stochastic concurrent games

- Graph with stochastic nodes
- Multiple players: $\mathsf{Agt} = \{A_1, A_2, A_3, \dots\}$
- Concurrent moves: $\mathtt{a}_1\mathtt{a}_2\mathtt{a}_3 \dots \in \Sigma^{\mathsf{Agt}}$ means that player A_1 played $\mathtt{a}_1$, player A_2 played $\mathtt{a}_2$ and player A_3 played $\mathtt{a}_3$, ...
- Payoff functions $\mathsf{payoff}_A : V^\omega \to \mathbb{R}$ for every $A \in \mathsf{Agt}$

A simple model for the medium access control problem [KNPS19]



[KNPS19] Kwiatkowska, Norman, Parker, Santos. Equilibria-based probabilistic model checking for concurrent stochastic games *(FM'19)*.

How do we play those games?

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What kind of strategies?

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Mixed strategies
$\sigma_A : V^* \to \mathrm{Dist}(\Sigma)$

After history $h \in V^*$, player A will play each action $a \in \Sigma$ with probability $\sigma_A(h)$.

How do we play those games?

According to strategies!

What kind of strategies?

Mixed strategies
$\sigma_A : V^* \to \mathrm{Dist}(\Sigma)$

Deterministic strategies
$\sigma_A : V^* \to \Sigma$

For every $h \in V^*$, $\sigma_A(h)$ is a Dirac measure.

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$\sigma_A : V^* \to \mathrm{Dist}(\Sigma)$

Deterministic strategies
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Stationary strategies
$\sigma_A : V \to \mathrm{Dist}(\Sigma)$

If $h, h' \in V^*$ are s.t. $\mathit{last}(h) = \mathit{last}(h')$, then $\sigma_A(h) = \sigma_A(h')$.

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Memoryless strategies
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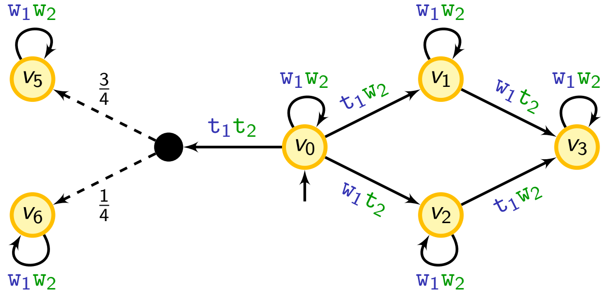
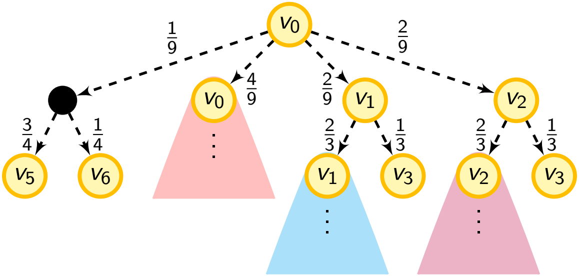
Strategy profile $\sigma = (\sigma_A)_{A \in \mathsf{Agt}}$

An example



Strategy for player A_i: $\sigma_{A_i}(h) = \frac{1}{3}\mathtt{t}_i + \frac{2}{3}\mathtt{w}_i$ if $\mathtt{t}_i$ available; $\sigma_i(h) = \mathtt{w}_i$ otherwise.



Payoffs

Given strategy profile $\sigma = (\sigma_A)_{A \in \mathsf{Agt}}$, the benefit $p(A)$ of player A from v_0 is given by:

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Examples

- $\phi_A \subseteq V^\omega$, and for $\rho \in V^\omega$,

$$\mathsf{payoff}_A(\rho) = \begin{cases} 1 & \text{if } \rho \models \phi_A \\ 0 & \text{otherwise} \end{cases}$$

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- payoff_A is a quantitative function on V^ω, for instance:
  - a mean-payoff function
  - a terminal-reward function

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- Turn-based games: V partitioned into all V_{A_i}'s

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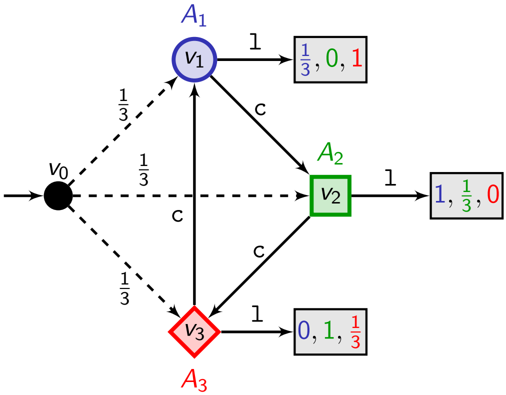

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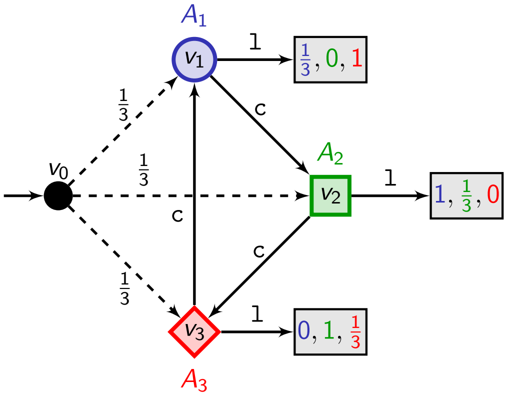

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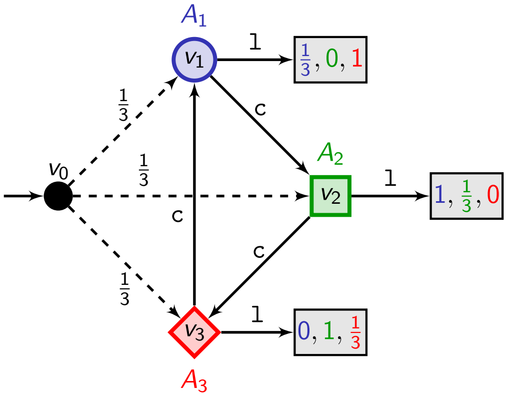

- Deterministic games
  If σ is pure and the game is deterministic, then profile σ has a single outcome $\mathrm{out}(\sigma)$, and

$$p_A(\sigma) = \mathrm{payoff}_A(\mathrm{out}(\sigma))$$

Nash equilibrium in this setting

Nash equilibrium

A mixed (resp. pure) strategy profile $\sigma = (\sigma_A)_{A \in \mathsf{Agt}}$ is a mixed (resp. pure) Nash equilibrium if no player can improve her payoff by unilaterally changing her strategy, that is, for every $A \in \mathsf{Agt}$, for every mixed (resp. pure) deviation σ'_A,

$$\mathbb{E}^{\sigma}_{v_0}(\mathsf{payoff}_A) \geq \mathbb{E}^{\sigma[A/\sigma'_A]}_{v_0}(\mathsf{payoff}_A)$$

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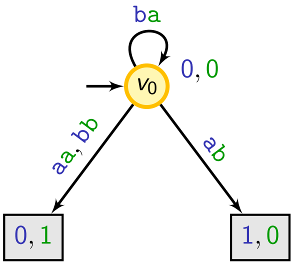

aa (that is, $\sigma_{A_i}(v_0) = \mathtt{a}$) is a (pure) Nash equilibrium

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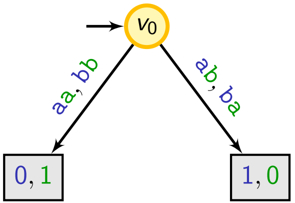
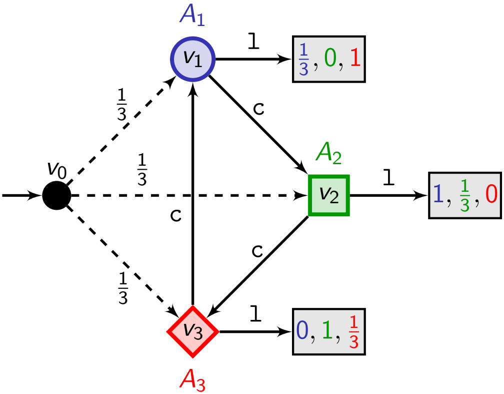
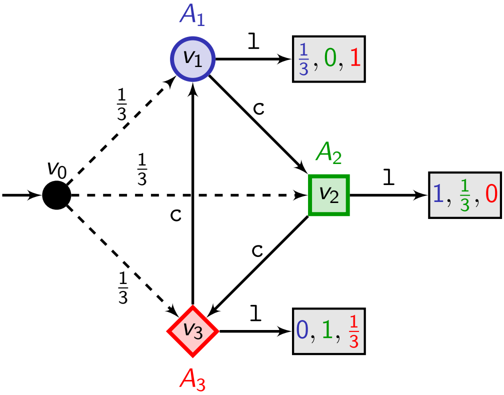
Example – Matching penny

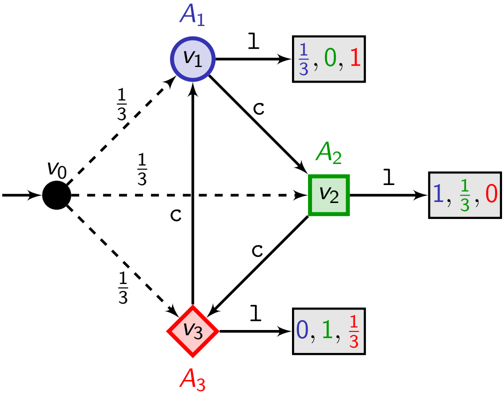


$\sigma_{A_i}(v_0) = \frac{1}{2} \cdot \mathsf{a} + \frac{1}{2} \cdot \mathsf{b}$ is the unique (mixed) Nash equilibrium


A_1
v_1
l
$\frac{1}{3}, 0, 1$
$\frac{1}{3}$
c
A_2
v_0
$\frac{1}{3}$
v_2
l
$1, \frac{1}{3}, 0$
c
c
$\frac{1}{3}$
v_3
l
$0, 1, \frac{1}{3}$
A_3



- There is no stationary Nash equilibrium



- There is no stationary Nash equilibrium
- There is a pure Nash equilibrium:
  - $v_0 v_i \mapsto \mathtt{c}$
  - $v_0 v_{i+1} \mapsto \mathtt{l}$
  - $v_0 v_i h \mapsto \mathtt{c}$

  It has payoff $(\frac{4}{9}, \frac{4}{9}, \frac{4}{9})$.

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  Does there always exist a Nash equilibrium?

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Kakutani's fixpoint theorem

Let X be a non-empty, compact and convex subset of $\mathbb{R}^n$. Let $f : X \to 2^X$ be a set-valued function on X with a closed graph and the property that $f(x)$ is non-empty and convex for all $x \in X$. Then f has a fixpoint.

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- Usually it applies to the best-response operator: if $\sigma \in \mathbb{S}$ ($\mathbb{S}$ is for stationary profiles), then

$$\mathsf{BR}(\sigma) = \left\{\sigma' \in \mathbb{S} \mid \forall A \in \mathsf{Agt},\ \sigma'_A \in \operatorname{argmax}_{\sigma''_A \in \mathbb{S}_A} \mathbb{E}_{v_0}^{\sigma[A/\sigma''_A]}(\mathsf{payoff}_A)\right\}$$

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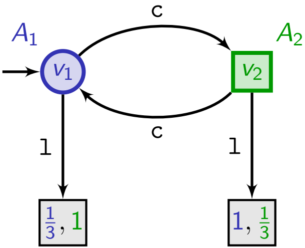

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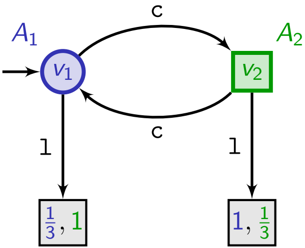

The first who leaves the loop loses!

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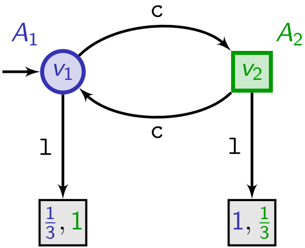

We note $(x_1, x_2) \in [0,1]^2$ for the profile σ s.t.

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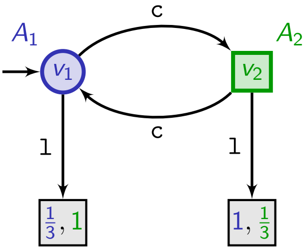

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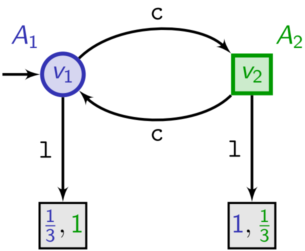

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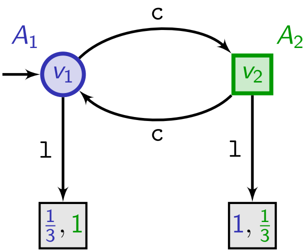
- $\mathrm{BR}((0, 0)) = \{(x_1, x_2) \mid x_1, x_2 > 0\}$

Does the standard theory apply?



We note $(x_1, x_2) \in [0,1]^2$ for the profile σ s.t.

$$\begin{cases} \sigma_{A_1}(v_1) & = x_1 \cdot \mathtt{l} + (1 - x_1) \cdot \mathtt{c} \\ \sigma_{A_2}(v_2) & = x_2 \cdot \mathtt{l} + (1 - x_2) \cdot \mathtt{c} \end{cases}$$

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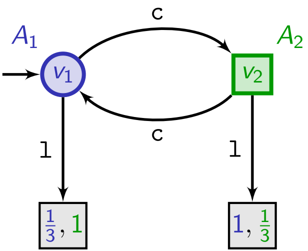

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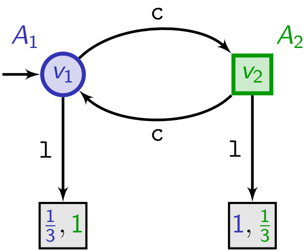

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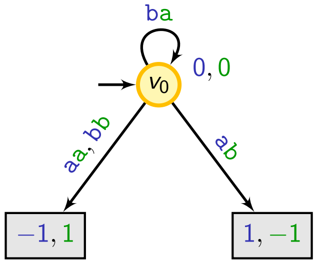
However there are infinitely many Nash equilibria:

all $(0, x_2)$ and $(x_1, 0)$ with $x_1, x_2 > 0$

No universal existence in general!



[UW11] Ummels, Wojtczak. The Complexity of Nash Equilibria in Limit-Average Games *(CONCUR'11)*.

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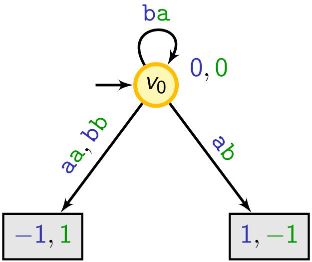

- By playing stationary strategy

$$\sigma_{A_2}(v_0) = (1-\epsilon)\cdot \mathtt{a} + \epsilon \cdot \mathtt{b},$$

A_2 ensures payoff $1 - 2\epsilon$

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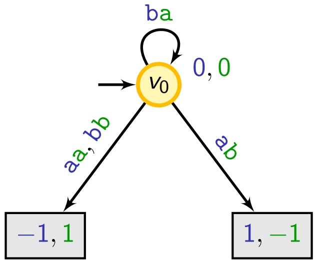

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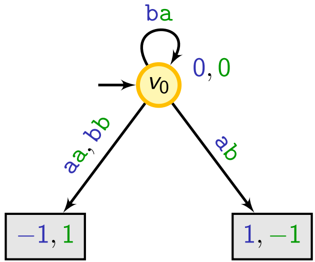

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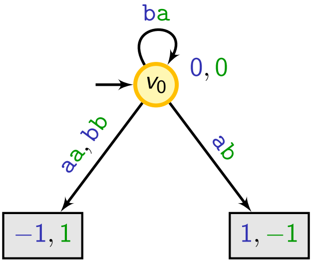

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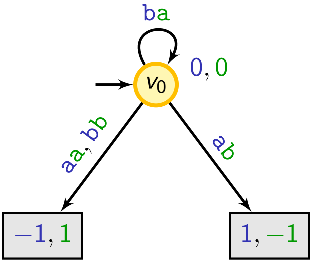

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$\rightsquigarrow$ There is no Nash equilibrium!

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2 What is a game?

3 A glimpse on strategic games

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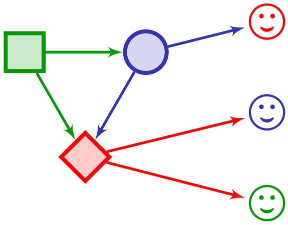
We focus on a simple scenario

Restrictions

- Turn-based games
- Payoffs given by ω-regular objectives: ϕ_A objective of player $A \in \mathsf{Agt}$
- Pure strategy profiles



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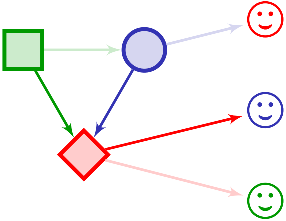

is a Nash equilibrium with payoff $(0, 1, 0)$

We focus on a simple scenario

Restrictions

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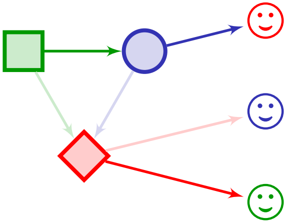


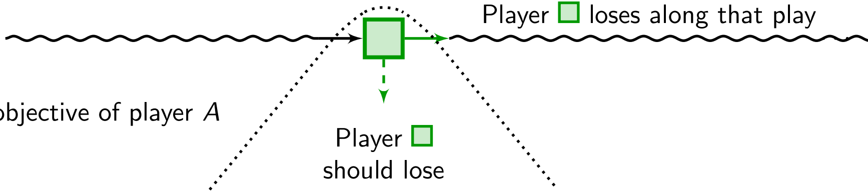
is not a Nash equilibrium

A simple characterization for ω-regular objectives

Player □ loses along that play

ϕ_A: objective of player A

A simple characterization for ω-regular objectives



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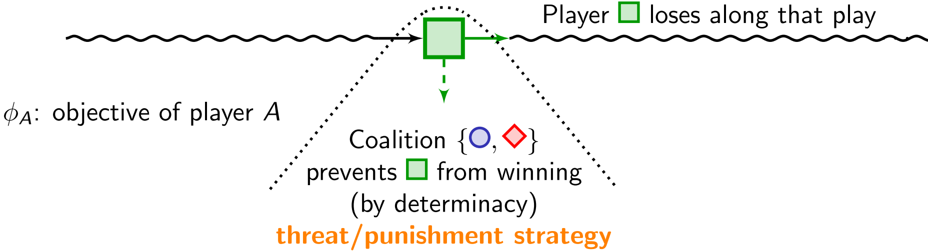
A simple characterization for ω-regular objectives



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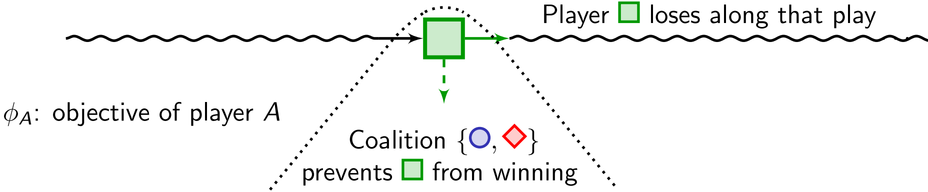
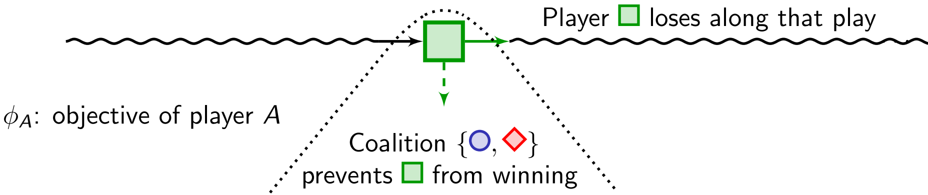


$$\neg\phi_{\square} \Rightarrow \mathbf{G}(p_{\square} \Rightarrow \mathbf{X}\, W_{\{\bigcirc,\diamond\}})$$

where $p_{\square}$ labels $\square$-states and $W_{\{\bigcirc,\diamond\}}$ is the set of winning states for the coalition $\{\bigcirc, \diamond\}$ for winning objective $\neg\phi_{\square}$.

A simple characterization for ω-regular objectives

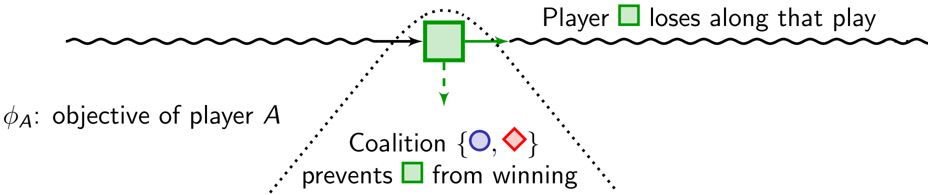


Main outcomes of Boolean Nash equilibria in turn-based games can be characterized by an LTL formula:

$$\Phi_{\mathsf{NE}} = \bigwedge_{A \in \mathsf{Agt}} \Big(\neg\phi_A \Rightarrow \mathbf{G}(p_A \Rightarrow \mathbf{X}\, W_{\{-A\}}) \Big)$$

where p_A labels A-states and $W_{\{-A\}}$ is the set of winning states for the coalition $\{-A\} \stackrel{\text{def}}{=} \mathsf{Agt} \setminus \{A\}$ against A for the objective $\neg\phi_A$. These sets should be precomputed.

A simple characterization for ω-regular objectives



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(valid for prefix-independent objectives)

Decidability of the constrained existence problem

Constrained existence problem

Given two thresholds $L, U \in \mathbb{Q}^+$, does there exists a Nash equilibrium σ such that for every $A \in \mathrm{Agt}$:

$$L_A \leq \mathbb{E}^{\sigma}_{v_0}(\mathrm{payoff}_A) \leq U_A?$$

[Umm08] Ummels. The Complexity of Nash Equilibria in Infinite Multiplayer Games *(FoSSaCS'08)*.

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Theorem [Umm08]

One can decide the *pure* constrained existence problem in finite turn-based multiplayer games for ω-regular objectives.

Examples of complexity results for single objectives:

Objectives	Reach.	Safety	Büchi	co-Büchi	Parity
Complexity	NP-c.		P-c.	NP-c.	

Note: it extends to "ω-regular" preference relations with a finite image.

[Umm08] Ummels. The Complexity of Nash Equilibria in Infinite Multiplayer Games *(FoSSaCS'08)*.

An example of NP-hardness result

By reduction from a SAT instance:

$$\varphi = \bigwedge_{1\leq i\leq n} C_i \qquad \text{with } C_i = \bigvee_{j=1}^{3} \ell_{i,j} \qquad \ell_{i,j} \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_k, \neg x_k\}$$

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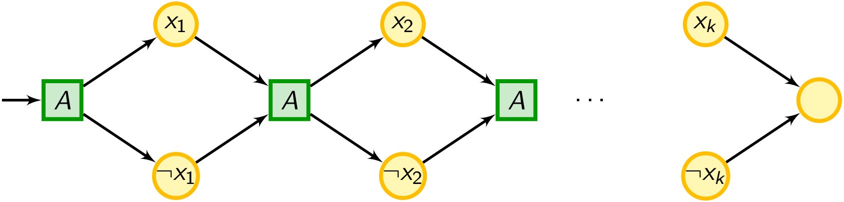

- Player A_i for clause C_i, with objective to reach $\{\ell_{i,j} \mid j = 1, 2, 3\}$
- Player A: reach the rightmost state

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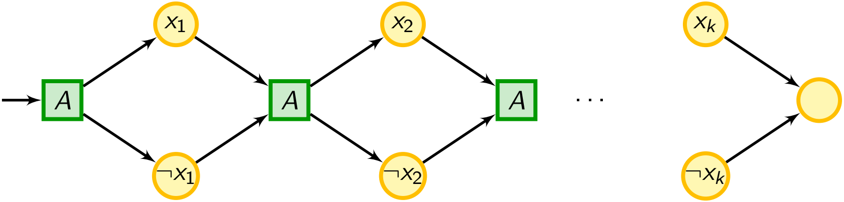

- Player A_i for clause C_i, with objective to reach $\{\ell_{i,j} \mid j = 1, 2, 3\}$
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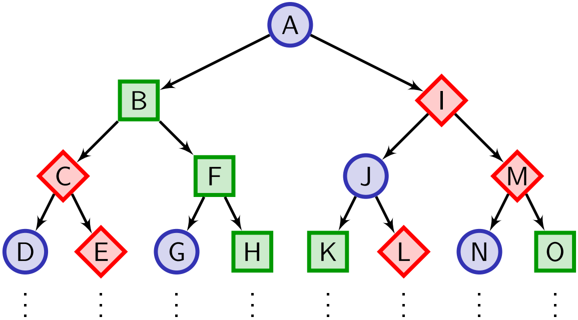
φ is satisfiable iff there is a Nash equilibrium with payoff 1 for everyone in the game

The universal existence problem: ω-regular objectives



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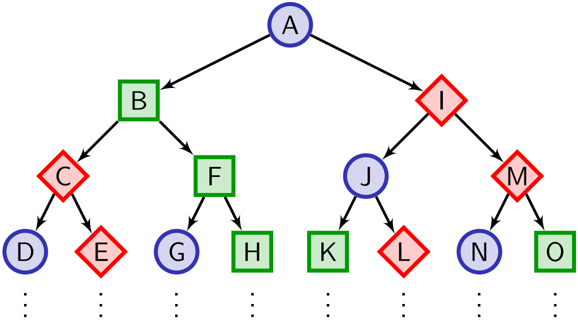
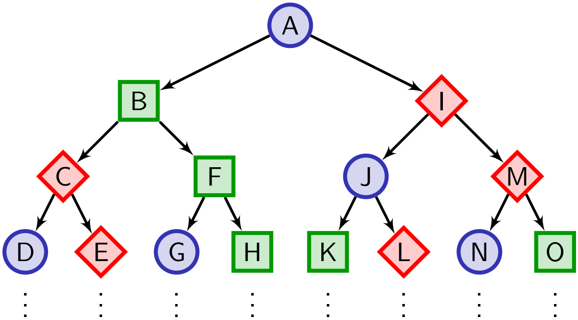


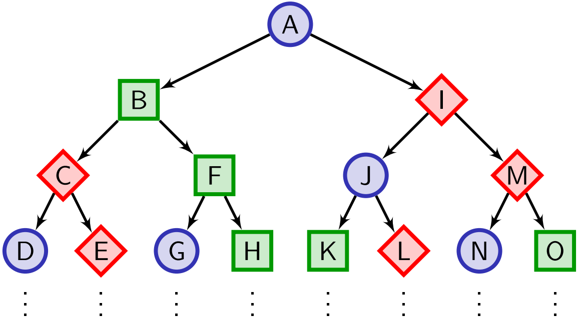
- If ◯ has a winning strategy from A, then ◯ should play it forever
- Otherwise ◯ plays any strategy, until (by chance) a new blue node, for instance J, is visited, from which ◯ has a winning strategy; ◯ then switches to such a winning strategy, forever

The universal existence problem: ω-regular objectives



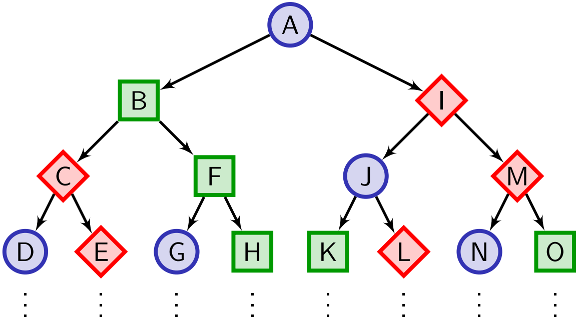
- If the game proceeds through B and has a winning strategy from B, then should play it forever
- If the game proceeds through B but has no winning strategy from B, then should play any strategy, until (by chance) a new green node, for instance H, is visited, from which has a winning strategy; then switches to such a winning strategy, forever

The universal existence problem: ω-regular objectives

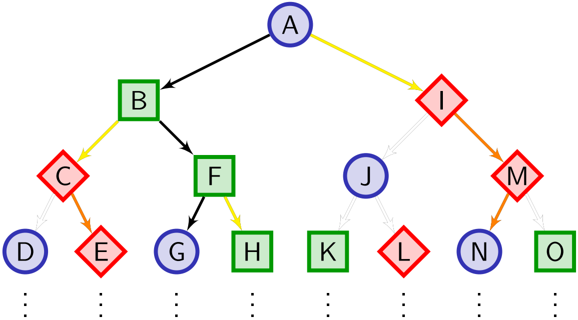


- If the game proceeds through C and ◇ has a winning strategy from C, then ◇ should play it forever
- If the game proceeds through C but ◇ has no winning strategy from C, then ◇ should play any strategy, until (by chance) a new red node, for instance E, is visited, from which ◇ has a winning strategy; ◇ then switches to such a winning strategy, forever

The universal existence problem: ω-regular objectives



- Outside the main outcome, all players play the adequate threat or punishment strategy: this is the coalition strategy that makes the deviator lose (NB: determinacy required!)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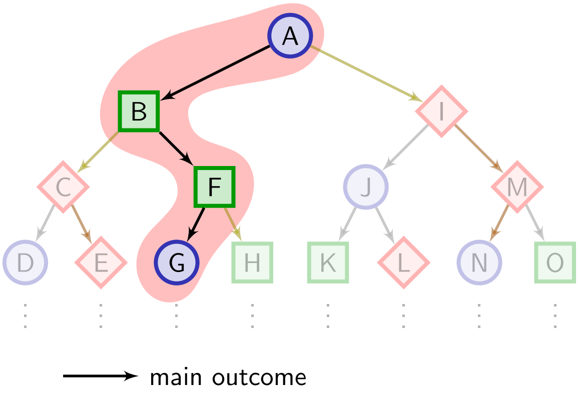
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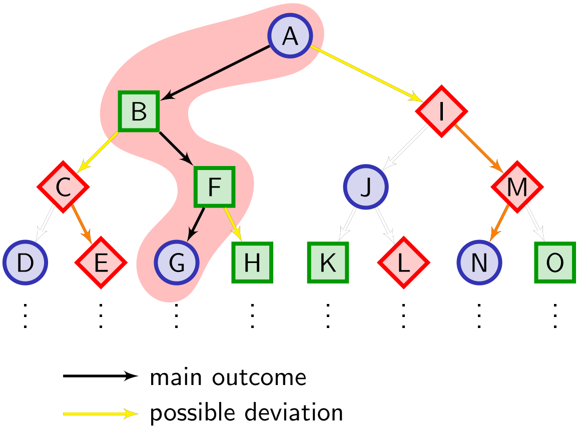
main outcome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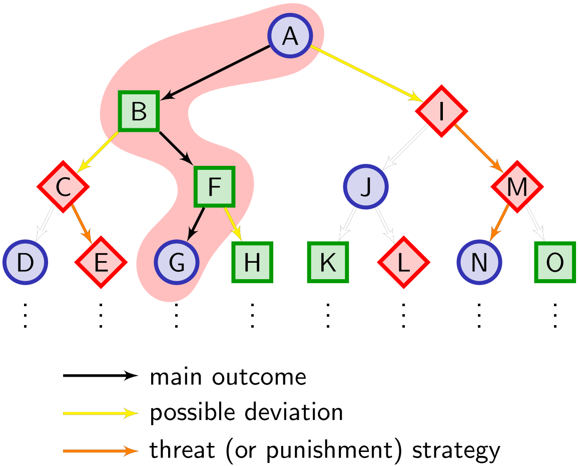
main outcome
possible deviation



A
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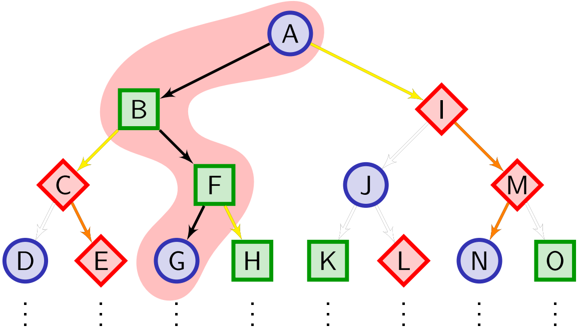
O
main outcome
possible deviation
threat (or punishment) strategy

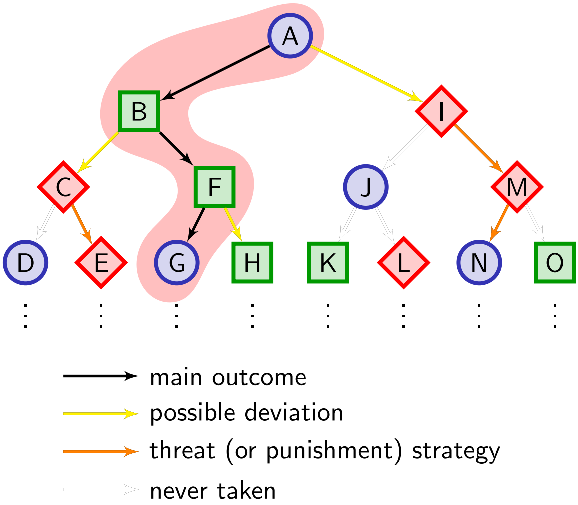


main outcome

possible deviation

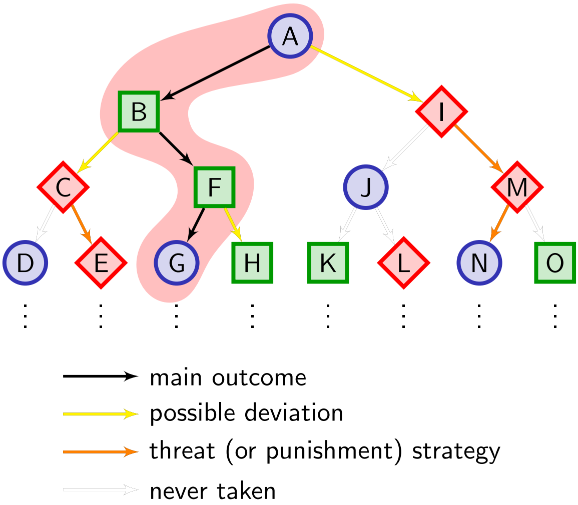
threat (or punishment) strategy

never taken



Questions:

- why is it correct?



Questions:

- why is it correct?
- what immediate extension can be handled?

The universal existence problem: ω-regular objectives

Universal existence [Umm11]

In infinite-duration turn-based deterministic games on finite graphs with ω-regular objectives, there is always a pure Nash equilibrium. Moreover, one can compute a witness.

[Umm11] Ummels. Stochastic multiplayer games: theory and algorithms *(PhD thesis)*.
[LeR13] Le Roux. Infinite sequential Nash equilibrium *(LMCS)*.

The universal existence problem

Universal existence [Umm11]

In infinite-duration turn-based deterministic games on finite graphs with ω-regular objectives, there is always a pure Nash equilibrium. Moreover, one can compute a witness.

Universal existence [LeR13]

In infinite turn-based deterministic games with Borel measurable countable preferences, with no ascending infinite chains, there is always a pure Nash equilibrium.

[Umm11] Ummels. Stochastic multiplayer games: theory and algorithms *(PhD thesis)*.
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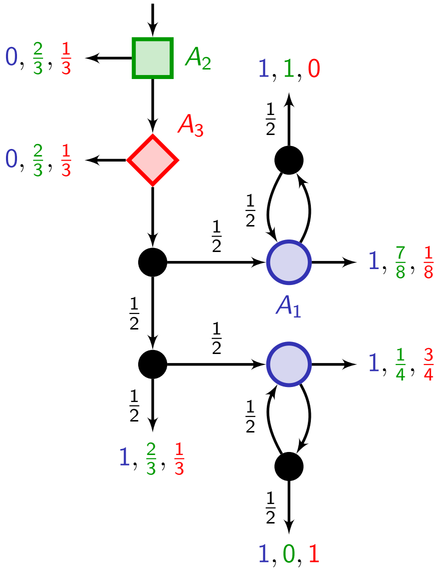
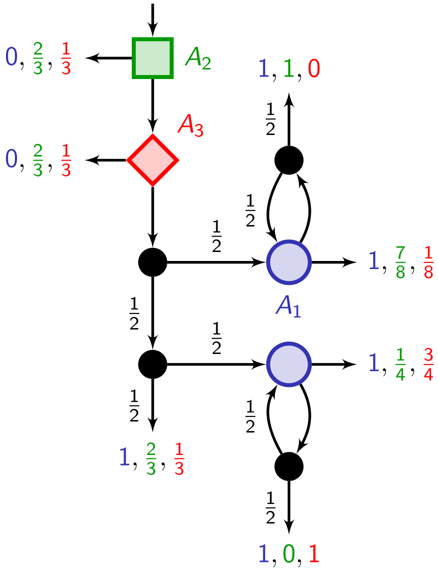
Stochastic turn-based games



[UW11] Ummels, Wojtczak. The Complexity of Nash Equilibria in Stochastic Multiplayer Games *(LMCS)*.

Stochastic turn-based games



Along a Nash equilibrium where $p_{A_1} \geq 1$:

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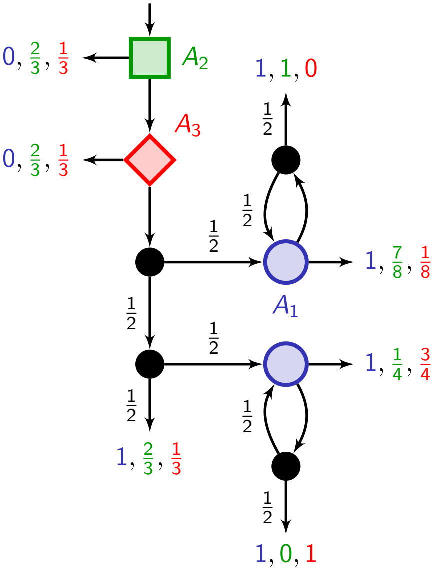
Stochastic turn-based games



Along a Nash equilibrium where $p_{A_1} \geq 1$:

- $p_{A_2} + p_{A_3} = 1$

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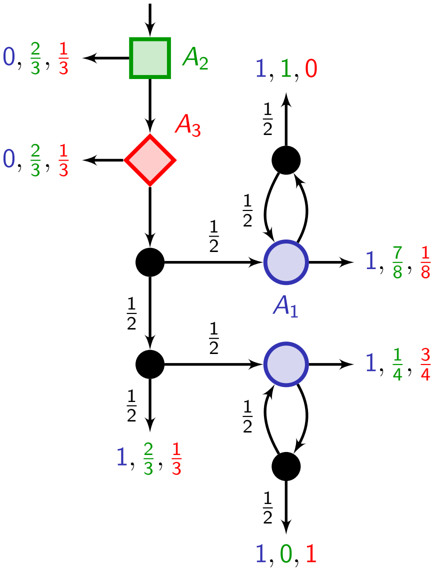
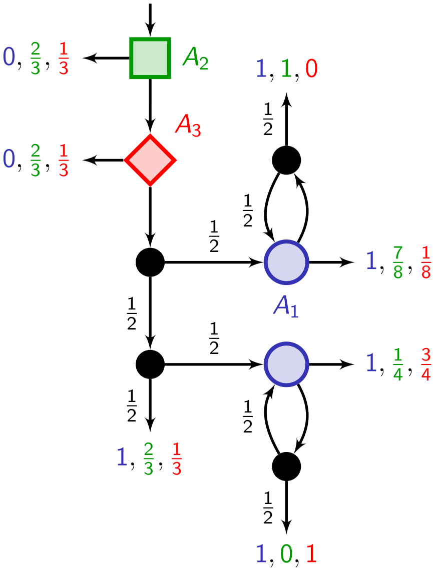
Stochastic turn-based games



Along a Nash equilibrium where $p_{A_1} \geq 1$:

- $p_{A_2} + p_{A_3} = 1$
- $p_{A_2} \geq \frac{2}{3}$ and $p_{A_3} \geq \frac{1}{3}$

Stochastic turn-based games



Along a Nash equilibrium where $p_{A_1} \geq 1$:

- $p_{A_2} + p_{A_3} = 1$
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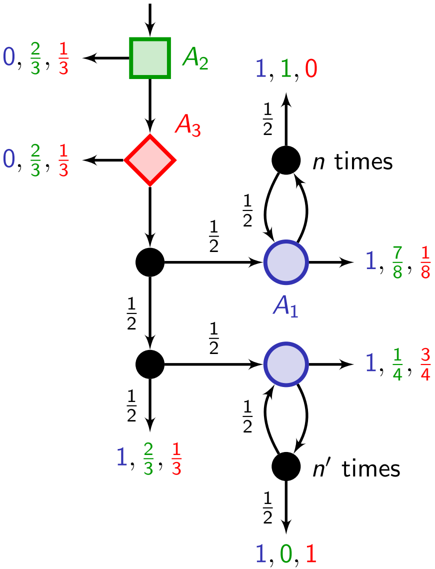
Stochastic turn-based games



Along a Nash equilibrium where $p_{A_1} \geq 1$:

- $p_{A_2} + p_{A_3} = 1$
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- $p_{A_2} = \frac{2}{3} + \frac{1}{16}\left(\frac{1}{2^{n'}} - \frac{1}{2^n}\right)$

[UW11] Ummels, Wojtczak. The Complexity of Nash Equilibria in Stochastic Multiplayer Games *(LMCS)*.

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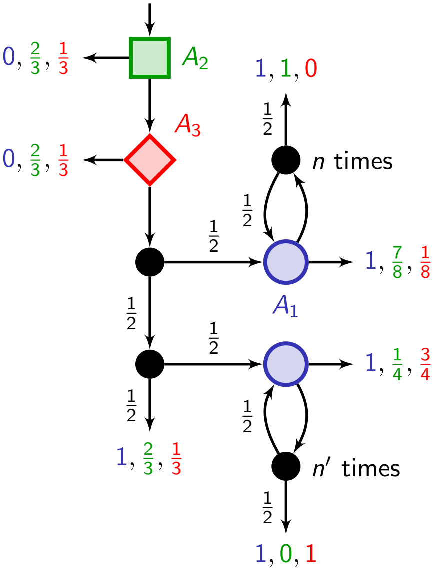

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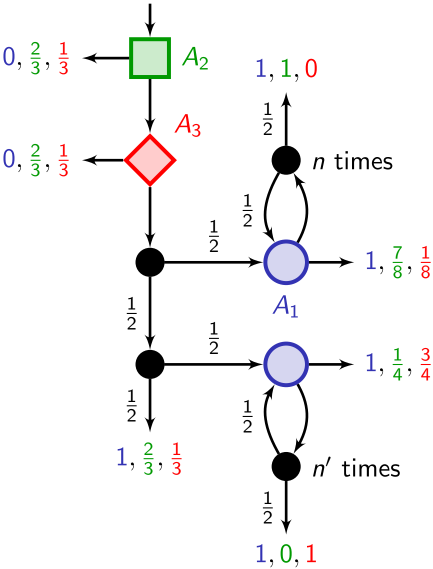

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$\rightsquigarrow$ $n = n'$

One can simulate a two-counter machine if we constrain $p_{A_1} \geq 1$!!

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Stochastic turn-based games

Undecidability results [UW11]

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Stochastic turn-based games

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- The constrained existence problem for pure strategies in stochastic turn-based games is undecidable.

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Stochastic turn-based games

Undecidability results [UW11]

- The constrained existence problem for pure strategies in stochastic turn-based games is undecidable.
- The constrained existence problem for mixed strategies in deterministic turn-based games is undecidable.

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Short summary for turn-based ω-regular games

[UW11,Umm11,LeR13]

- There always exists a Nash equilibrium for Boolean ω-regular objectives
- One can decide the constrained existence of a Nash equilibrium (and compute one!)
- One cannot decide the existence of a mixed (i.e. stochastic) Nash equilibrium

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- There always exists a Nash equilibrium for Boolean ω-regular objectives
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$\rightsquigarrow$ this is why we will restrict to pure equilibria in det. games

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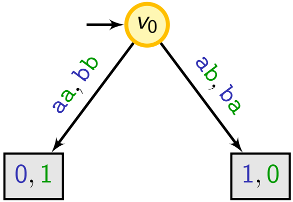
Outline

1. Verification and game theory
2. What is a game?
3. A glimpse on strategic games
4. Games on graphs
   - The general model
   - Focus on a simple scenario
   - Adding probabilities to the setting?
   - Concurrent games
5. Conclusion

Can this theory be extended to concurrent games?



There is no universal existence, even for simple Boolean objectives.

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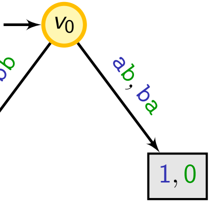

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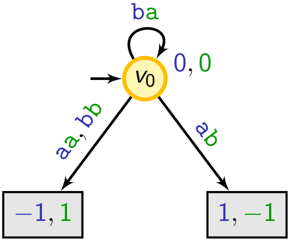

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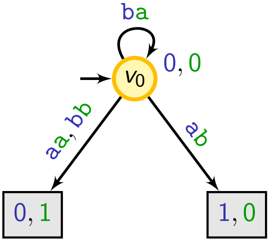

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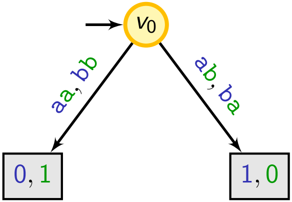

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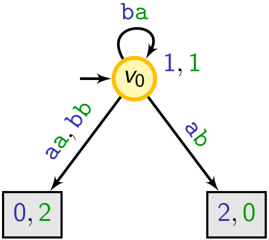

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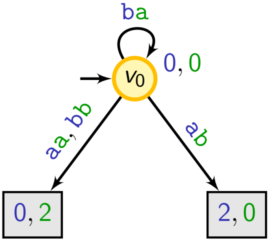


There is a pure Nash equilibrium

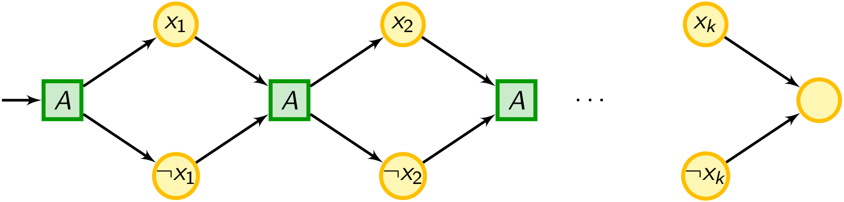
Existence becomes NP-hard

Hardness

The existence problem is NP-hard for reachability objectives.

By reduction from a SAT instance:

$$\varphi = \bigwedge_{1 \leq i \leq n} C_i \qquad \text{with } C_i = \bigvee_{j=1}^{3} \ell_{i,j} \qquad \ell_{i,j} \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_k, \neg x_k\}$$



φ is satisfiable iff there is a Nash equilibrium with payoff 1 for everyone in the game

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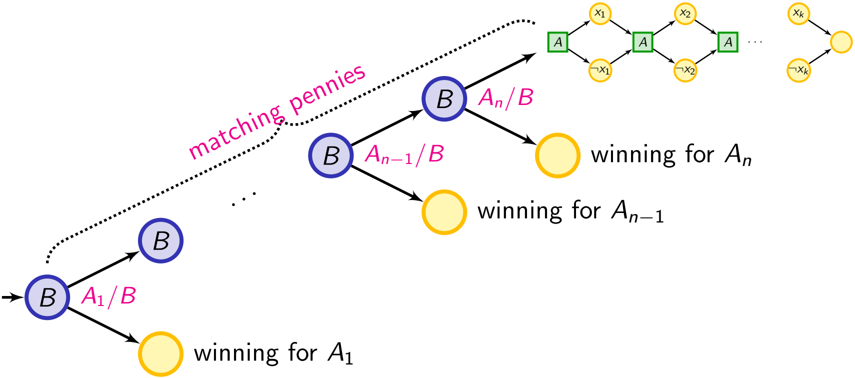

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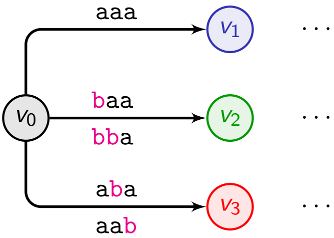

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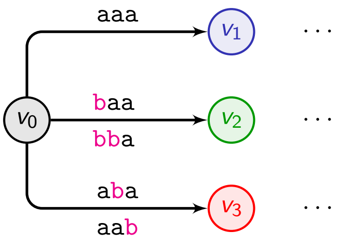

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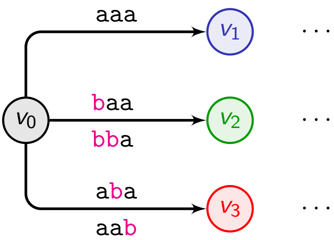

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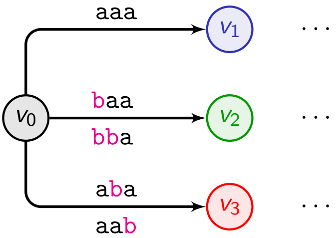

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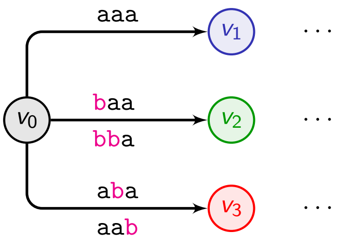

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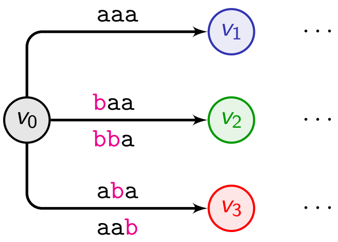

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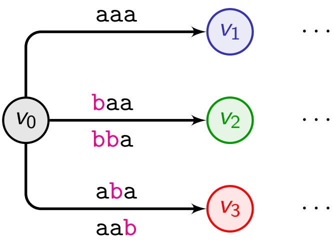

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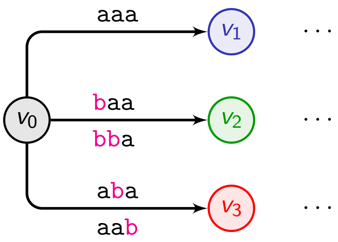

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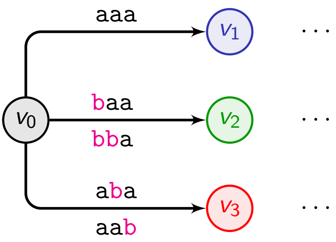

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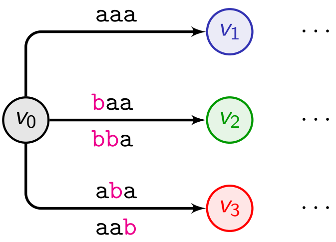

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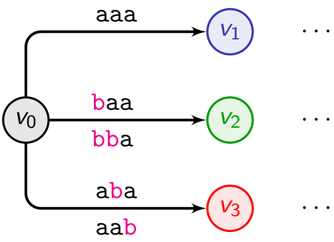

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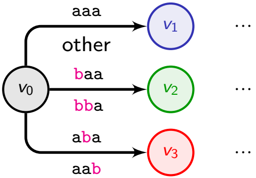
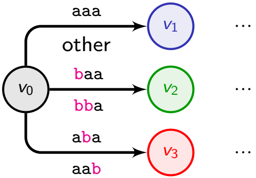
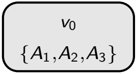
Construction of the suspect game abstraction



Two players: Eve (light)
Adam (dark)

Construction of the suspect game abstraction





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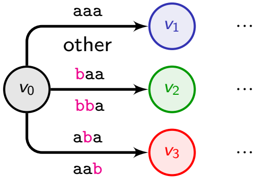

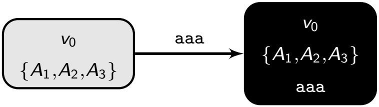

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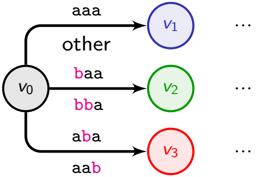

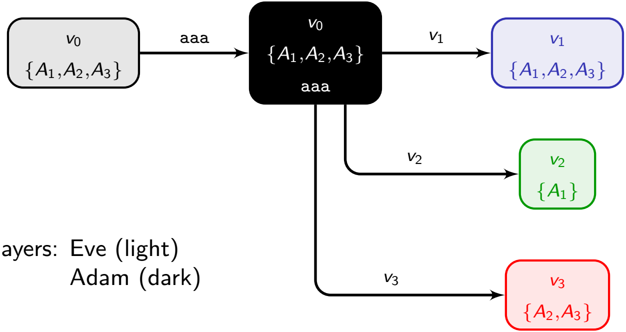

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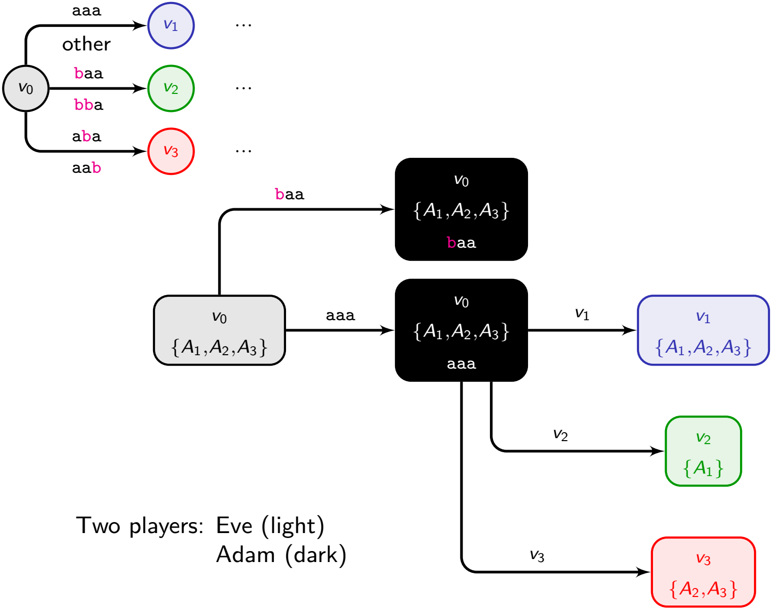

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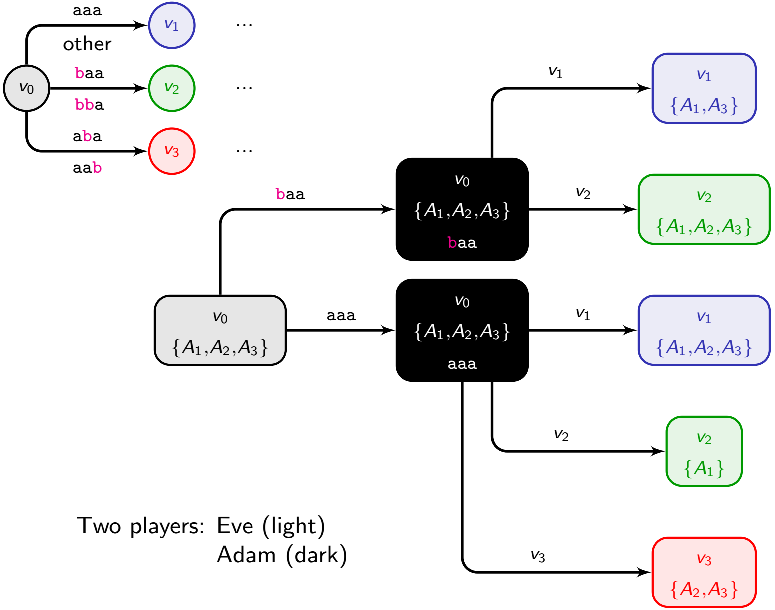

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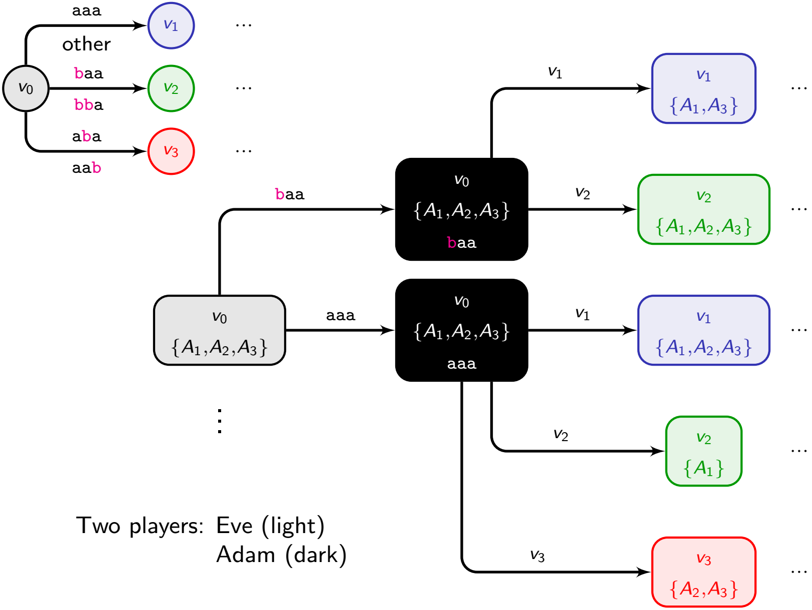

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Correctness of the suspect game construction

Winning condition

A strategy ζ for Eve in the suspect game is winning for some $\alpha \in \mathbb{R}^{\mathsf{Agt}}$ if the unique outcome of ζ where Adam complies to Eve has payoff α, and for every other outcome ρ of ζ, for every $A \in \mathsf{susp}(\rho)$, $\mathsf{payoff}_A(\rho) \leq \alpha_A$.

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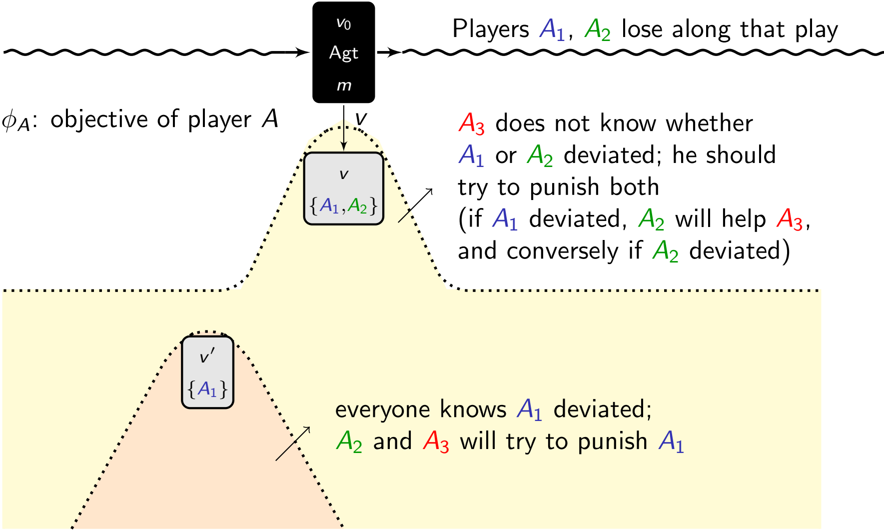
Correctness

Let $\alpha \in \mathbb{R}^{\mathsf{Agt}}$. There is a Nash equilibrium in the original game with payoff α if and only if Eve has a winning strategy for α in the suspect game.


v_0
Agt
m
Players A_1, A_2 lose along that play
ϕ_A: objective of player A
v
v
$\{A_1, A_2\}$
A_3 does not know whether A_1 or A_2 deviated; he should try to punish both (if A_1 deviated, A_2 will help A_3, and conversely if A_2 deviated)
v'
$\{A_1\}$
everyone knows A_1 deviated; A_2 and A_3 will try to punish A_1

From an algorithmic point-of-view

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- In the orange part: compute the winner (Eve or Adam) of the zero-sum game, where Eve's objective is $\neg\phi_{A_1}$ (Eve wants to show that there is no profitable deviation for A_1)

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The approach can be extended to various settings!

Some results

Examples of complexity results

- For single objectives:

Objectives	Reach.	Safety	Büchi	co-Büchi	Parity
Complexity	NP-c.		P-c.	NP-c.	$P^{NP}_{\parallel}$-c.

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- For combinations of Büchi objectives:

Combinations	Subset	Lexico.	Count.	Bool. circuit
Complexity	P-c.	NP-.c		PSPACE-c.

[BBMU15] Bouyer, Brenguier, Markey, Ummels. Pure Nash equilibria in concurrent deterministic games *(LMCS)*

Some results

Examples of complexity results

- For single objectives:

Objectives	Reach.	Safety	Büchi	co-Büchi	Parity
Complexity	NP-c.		P-c.	NP-c.	$P_{\parallel}^{NP}$-c.

- For combinations of Büchi objectives:

Combinations	Subset	Lexico.	Count.	Bool. circuit
Complexity	P-c.	NP-.c		PSPACE-c.

- For combinations of reachability objectives:

Combinations	Subset	Lexico.	Count.	Bool. circuit
Complexity	NP-c.	PSPACE-.c		

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Extensions of this approach

Partial information monitoring

- Public signal [Bou18]
- Communication graphs [BT19]

[Bou18] Bouyer. Games on graphs with a public signal monitoring *(FoSSaCS'18)*.
[BT19] Bouyer, Tomasset. Nash equilibria in games over graphs equipped with a communication mechanism *(MFCS'19)*.
[Bre16] Brenguier. Robust equilibria in mean-payoff games *(FoSSaCS'16)*.
[COT18] Condurache, Oualhadj, Troquard. The complexity of rational synthesis for concurrent games *(CONCUR'18)*.

Extensions of this approach

Partial information monitoring

- Public signal [Bou18]
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Other solution concepts

- Robust equilibria [Bre16]
- Rational synthesis [COT18]

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Outline

1. Verification and game theory
2. What is a game?
3. A glimpse on strategic games
4. Games on graphs
   - The general model
   - Focus on a simple scenario
   - Adding probabilities to the setting?
   - Concurrent games
5. Conclusion

Wrap-up

General objective

- Import game theory solutions to the verification field, where interactivity plays also a role

  Ex: Distributed systems interacting in some environment

Applications?

- Smart grids: decentralized control of EV charging [GBLM19]
  - stochastic setting
  - ad-hoc approximated solutions

- Cassting project: smart houses that produce energy with solar panels [BDGHM16]
  - deterministic setting
  - setting with universal existence
  - exact computation

- PRISM-games: medium access control, Aloha protocol, robot coordination, power control [KNPS19]
  - stochastic setting
  - approximated value iteration for computing ϵ-SPE

[BDGHM16] Brihaye, Dhar, Geeraerts, Haddad, Monmege. Efficient energy distribution in a smart grid using multi-player games *(Cassting'16)*
[KNPS19] Kwiatkowska, Norman, Parker, Santos. Equilibria-based probabilistic model checking for concurrent stochastic games *(FM'19)*.
[GBLM19] González, Bouyer, Lasaulce, Markey. Optimisation en présence de contraintes en probabilités et processus markoviens contrôlés *(GRETSI'19)*

Wrap-up

General objective

- Import game theory solutions to the verification field, where interactivity plays also a role

  Ex: Distributed systems interacting in some environment
- Relevant questions:
  - assumptions made in the game theory field relevant?
  - solution concepts adapted to the context?

Wrap-up

General objective

- Import game theory solutions to the verification field, where interactivity plays also a role

  Ex: Distributed systems interacting in some environment
- Relevant questions:
  - assumptions made in the game theory field relevant?
  - solution concepts adapted to the context?

Nash equilibria in games on graphs

- The setting of pure Nash equilibria in turn-based det. games rather well-understood
- Probabilistic setting much more complicated
- Concurrent games: a rather generic approach based on the suspect game construction

Going further?

More relevant solution concepts?

- Temporal aspects weakens the concept of Nash equilibrium:
  Will a rational agent/process focus on punishing a deviator, instead of pursuing her own objective?
- Another solution concept: subgame-perfect equilibrium