Verification and Game Theory

Patricia Bouyer

LSV, CNRS & ENS Paris-Saclay
Université Paris-Saclay, Cachan, France

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Outline

1. Verification and game theory

2. Games on graphs
   - The general model
   - Focus on a simple scenario
   - Adding probabilities to the setting?
   - Concurrent games

3. Conclusion
# Computer programming

## Computer programming is a difficult task
- understand deeply the initial problem;
- find a solution;
- write the program correctly.

## Software bugs
- It is an error, a failure in a computer program or system that induces an incorrect result.
- It may have catastrophic consequences.
Software bugs

Bug example
In August 2005, a Malaysian Airlines MH124 (Boeing 777) that was on autopilot suddenly ascended 2,000 feet.

Bug consequences
- loss of confidence from users’ point of view,
- loss of credibility from institutions’ point of view,
- large financial loss,
- human loss,

⇒ Real need to verify the correctness of a program!
The model-checking approach to verification

Real system
plane,…

\[ \equiv \]

Specification
arrive safely,…
The model-checking approach to verification

- **Real system**
  - plane, ...

- **Specification**
  - arrive safely, ...

- **Abstract model**
  - automaton, ...

- **Logic formula**
  - FO, LTL, ...

The model-checking approach involves comparing the real system with the abstract model, and then checking if the specification holds in both the real system and the abstract model.
The model-checking approach to verification

Real system
plane, ...

| = |

Specification
arrive safely, ...

Algorithm

Abstract model
automaton, ...

| = |

Logic formula
FO, LTL, ...

YES/NO
The autopilot case

**Requirement:** to arrive safely in every weather condition,
The autopilot case

**Requirement:** to arrive safely *in every weather condition*, while minimising the fuel consumption.
Controlling computer systems

- **Real system**
  - plane, ... 

- **Environment**
  - weather, ... 

- **(Quant.) Spec.**
  - arrive safely, 
  - energy cons., ...
Controlling computer systems

Real system
plane,...

Environment weather,...

Safe or optimal solution?

(Quant.) Spec.
arrive safely,
energy cons.,...
Controlling computer systems

Real system
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Safe or optimal solution?

(Quant.) Spec.
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Abstract model
game

Controller strategy

Logic. formula
Quant. constraint
Controlling computer systems

Real system
plane,...

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Abstract model
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Controller
strategy

Logic. formula
Quant. constraint

Algorithm

NO/YES + A controller
The autopilot case

**Requirement:** to arrive safely *in every weather condition,*
The autopilot case

Requirement: to arrive safely *in every weather condition*,
taking into account the other planes,
The autopilot case

Requirement: to arrive safely in every weather condition, taking into account the other planes, while minimising the fuel consumption.
Controlling complex interactive computer systems

Real systems
planes,...

Quant. Spec.
energy cons.,…

Environment weather,…
Controlling complex interactive computer systems

Real systems
planes,...

Environment weather,...

Optimal or stable solution?

Quant. Spec.
energy cons.,...
Controlling complex interactive computer systems

- Real systems: planes, ...
- Environment: weather, ...
- Quant. Spec.: energy cons., ...
- Optimal or stable solution?

- Abstract model: game
- Equilibrium?
- Payoff function

NO/YES + An equilibrium
Controlling complex interactive computer systems

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NO/YES + An equilibrium
Controlling complex interactive computer systems

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Algorithm

- Abstract model: game
- Equilibrium?
- Payoff function: Multi-agent logic

NO/YES + An equilibrium
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Which games do we need for verification?

**Methodology**

- Pick standard models used in model-checking
- Expand them with interaction capabilities
- Games played on graphs
  - Several features in the graph: stochastic or deterministic
  - Several options for interaction: turn-based vs concurrent, pure vs mixed strategies
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The Nim game modelled as a turn-based game
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The Nim game modelled as a turn-based game

This is then just a matter of computing winning states (controller synthesis)
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Multiplayer stochastic concurrent games

- Graph with stochastic nodes
- Multiple players: \( \text{Agt} = \{A_1, A_2, A_3, \ldots \} \)
- Concurrent moves: \( a_1a_2a_3 \cdots \in \Sigma^{\text{Agt}} \) means that player \( A_1 \) played \( a_1 \), player \( A_2 \) played \( a_2 \) and player \( A_3 \) played \( a_3 \), ...
- Payoff functions \( \text{payoff}_A : V^\omega \rightarrow \mathbb{R} \) for every \( A \in \text{Agt} \)

A simple model for the medium access control problem [KNPS19]
How do we play those games?

According to strategies!
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What kind of strategies?
How do we play those games?

According to strategies!

What kind of strategies?

Mixed strategies

\( \sigma_A : V^* \rightarrow \text{Dist}(\Sigma) \)

After history \( h \in V^* \), player \( A \) will play each action \( b \in \Sigma \) with probability \( \sigma_A(h)(b) \).
How do we play those games?

According to strategies!

What kind of strategies?

Mixed strategies
\[ \sigma_A : V^* \to \text{Dist}(\Sigma) \]

Deterministic strategies
\[ \sigma_A : V^* \to \Sigma \]

For every \( h \in V^* \), \( \sigma_A(h) \) is a Dirac measure.
How do we play those games?

According to strategies!

What kind of strategies?

Mixed strategies
\[ \sigma_A : V^* \rightarrow \text{Dist}(\Sigma) \]

Deterministic strategies
\[ \sigma_A : V^* \rightarrow \Sigma \]

Stationary strategies
\[ \sigma_A : V \rightarrow \text{Dist}(\Sigma) \]

If \( h, h' \in V^* \) are s.t. \( \text{last}(h) = \text{last}(h') \), then \( \sigma_A(h) = \sigma_A(h') \).
How do we play those games?

According to strategies!

What kind of strategies?

- **Mixed strategies**
  \[ \sigma_A : V^* \rightarrow \text{Dist}(\Sigma) \]

- **Deterministic strategies**
  \[ \sigma_A : V^* \rightarrow \Sigma \]

- **Stationary strategies**
  \[ \sigma_A : V \rightarrow \text{Dist}(\Sigma) \]

- **Memoryless strategies**
  \[ \sigma_A : V \rightarrow \Sigma \]
How do we play those games?

According to strategies!

What kind of strategies?

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Deterministic strategies
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Stationary strategies
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Memoryless strategies
\( \sigma_A : V \rightarrow \Sigma \)

Strategy profile \( \sigma = (\sigma_A)_{A \in \text{Agt}} \)
An example

Strategy for player $A_i$: $\sigma_A(i) = \begin{cases} v_0 & \text{if } t_i \text{ available} \\ w_i & \text{otherwise} \end{cases}$
An example

Strategy for player $A_i$: $\sigma_{A_i}(h) = \begin{cases} \frac{1}{3} \cdot t_i + \frac{2}{3} \cdot w_i & \text{if } t_i \text{ available} \\ w_i & \text{otherwise} \end{cases}$
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Strategy for player $A_i$: $\sigma_{A_i}(h) = \begin{cases} 
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Payoffs

Given strategy profile $\sigma = (\sigma_A)_{A \in \text{Agt}}$, the benefit $p_A(\sigma)$ of player $A$ from $v_0$ is given by:

$$p_A(\sigma) = \mathbb{E}^\sigma_{v_0} (\text{payoff}_A)$$
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**Examples**

- $\phi_A \subseteq V^\omega$, and for $\rho \in V^\omega$,

$$\text{payoff}_A(\rho) = \begin{cases} 1 & \text{if } \rho \models \phi_A \\ 0 & \text{otherwise} \end{cases}$$

Then, $p_A(\sigma) = \mathbb{P}_{\nu_0}^\sigma (\phi_A)$. 
Given strategy profile $\sigma = (\sigma_A)_{A \in \text{Agt}}$, the benefit $p_A(\sigma)$ of player $A$ from $v_0$ is given by:

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  0 & \text{otherwise}
  \end{cases}$$

  Then, $p_A(\sigma) = \mathbb{P}^\sigma_{v_0}(\phi_A)$.

- $\text{payoff}_A$ is a quantitative function on $V^\omega$, for instance:
  - a mean-payoff function
  - a terminal-reward function
Subclasses of interest

- **Turn-based** games: $V$ partitioned into all $V_{A_i}$'s

Deterministic games

If $\sigma$ is pure and the game is deterministic, then profile $\sigma$ has a single outcome $\text{out}(\sigma)$, and $p_{A_i}(\sigma) = \text{payoff}_{A_i}(\text{out}(\sigma))$
Subclasses of interest

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![Game Diagram]

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![Diagram of turn-based games]

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- **Turn-based games**: $V$ partitioned into all $V_{A_i}$’s

![Diagram showing a game graph with nodes $V_0, V_1, V_3, V_2$, and actions $A_1, A_2, A_3$.]

- **Deterministic games**
  If $\sigma$ is pure and the game is deterministic, then profile $\sigma$ has a single outcome $\text{out}(\sigma)$, and

$$p_A(\sigma) = \text{payoff}_A(\text{out}(\sigma))$$
Nash equilibrium

A mixed (resp. pure) strategy profile \( \sigma = (\sigma_A)_{A \in \text{Agt}} \) is a mixed (resp. pure) Nash equilibrium if no player can improve her payoff by unilaterally changing her strategy, that is, for every \( A \in \text{Agt} \), for every mixed (resp. pure) deviation \( \sigma'_A \),

\[
E_{\nu_0}(\text{payoff}_A) \geq E_{\nu_0}[A/\sigma'_A](\text{payoff}_A)
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Nash equilibrium in this setting

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Example

\[\begin{array}{c}
0, 1 \\
0, 0 \\
a_a, bb \\
ab
\end{array}\]

\[\begin{array}{c}
ba \\
0, 0 \\
\text{a}a, bb \\
\text{a}b
\end{array}\]

aa (that is, $\sigma_{A_i}(v_0) = a$) is a (pure) Nash equilibrium
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\[
\mathbb{E}_{\nu_0}^\sigma(\text{payoff}_A) \geq \mathbb{E}_{\nu_0}^{\sigma[A/\sigma'_A]}(\text{payoff}_A)
\]

Example – Matching penny

\[
\sigma_{A_i}(\nu_0) = \frac{1}{2} \cdot a + \frac{1}{2} \cdot b
\]
is the unique (mixed) Nash equilibrium

\[
\sigma_A = (\sigma_A, \sigma_B) = \left( \frac{1}{2}, \frac{1}{2} \right)
\]
is the unique (mixed) Nash equilibrium
There is no stationary Nash equilibrium.

There is a pure Nash equilibrium:

\[ v_0 \rightarrow v_1 \mapsto \frac{1}{3}, 0, 1 \]
\[ v_0 \rightarrow v_3 \mapsto 0, 1, \frac{1}{3} \]

\[ v_1 \rightarrow A_1 \mapsto \frac{1}{3}, 0, 1 \]
\[ v_1 \rightarrow A_2 \mapsto \frac{1}{3}, 0, 1 \]
\[ v_1 \rightarrow A_3 \mapsto \frac{1}{3}, 0, 1 \]

\[ v_2 \rightarrow A_2 \mapsto 1, \frac{1}{3}, 0 \]
\[ v_2 \rightarrow A_3 \mapsto 1, \frac{1}{3}, 0 \]

It has payoff \((\frac{4}{9}, \frac{4}{9}, \frac{4}{9})\).
There is no stationary Nash equilibrium.

There is a pure Nash equilibrium:

- **v0** $\rightarrow \frac{1}{3}, 0, 1$
- **v1** $\rightarrow \frac{1}{3}, 0, 1$
- **v2** $\rightarrow 1, \frac{1}{3}, 0$
- **v3** $\rightarrow 0, 1, \frac{1}{3}$

It has payoff (\(\frac{19}{48}\)).
There is no stationary Nash equilibrium.

There is a pure Nash equilibrium:
- $v_0, v_i \mapsto c$
- $v_0, v_{i+1} \mapsto 1$
- $v_0, v_h \mapsto c$

It has payoff $(\frac{4}{9}, \frac{4}{9}, \frac{4}{9})$. 
Problems of interest

- Universal existence:

  Does there exist a Nash equilibrium in all games?
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  *Does there exist a Nash equilibrium in all games?*

- **Existence problem:**

  *Given a game, does there exist a Nash equilibrium?*
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  \textit{Given a game, does there exist a Nash equilibrium which satisfies some constraint?}
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- **Synthesis of witness (simple?) profiles?**
  
  *Do strategy profiles require randomness? Memory?*
Problems of interest

- Universal existence:
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- Synthesis of witness (simple?) profiles?
  \textit{Do strategy profiles require randomness? Memory?}
Does the standard theory apply?

- Nash theorem does not apply (requires a finite number of pure strategies)

Kakutani's fixpoint theorem

Let $X$ be a non-empty, compact and convex subset of $\mathbb{R}^n$. Let $f: X \to 2^X$ be a set-valued function on $X$ with a closed graph and the property that $f(x)$ is non-empty and convex for all $x \in X$. Then $f$ has a fixpoint.

Usually it applies to the best-response operator: if $\sigma \in S$ (where $S$ is for stationary profiles), then $\text{BR}(\sigma) = \{ \sigma' \in S \mid \forall A \in \text{Agt}, \sigma'_A \in \arg\max_{\sigma''_A \in S_A} \sigma''_A \in \text{E} \sigma[A/\sigma''] \}$.
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- But do the related fixed point theorems apply?

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Does the standard theory apply?

We note \((x_1, x_2) \in [0, 1]^2\) for the profile \(\sigma\) s.t.

\[
\begin{align*}
\sigma A_1(v_1) &= x_1 \cdot l + (1 - x_1) \cdot c \\
\sigma A_2(v_2) &= x_2 \cdot l + (1 - x_2) \cdot c
\end{align*}
\]

The first who leaves the loop loses!

For every \(x_1, x_2 > 0\), \(\text{BR}((x_1, x_2)) = (0, 0)\)

\(\text{BR}((0, 0)) = \{(x_1, x_2) | x_1, x_2 > 0\}\)

The graph of \(\text{BR}\) is not closed.

Kakutani's theorem does not apply.

However there are infinitely many Nash equilibria: all \((0, x_2)\) and \((x_1, 0)\) with \(x_1, x_2 > 0\).
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- By playing stationary strategy

\[ \sigma_{A_2}(v_0) = (1 - \epsilon) \cdot a + \epsilon \cdot b, \]

\[ A_2 \text{ ensures payoff } 1 - 2\epsilon \]
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  \( \leadsto \) There is no Nash equilibrium!

---

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2 Games on graphs
   - The general model
   - Focus on a simple scenario
   - Adding probabilities to the setting?
   - Concurrent games

3 Conclusion
We focus on a simple scenario

Restrictions

- Turn-based games
- Payoffs given by \( \omega \)-regular objectives: \( \phi_A \) objective of player \( A \in \text{Agt} \)
- Pure strategy profiles
We focus on a simple scenario

Restrictions
- Turn-based games
- Payoffs given by $\omega$-regular objectives: $\phi_A$ objective of player $A \in \text{Agt}$
- Pure strategy profiles

is a Nash equilibrium with payoff $(0, 1, 0)$
We focus on a simple scenario

**Restrictions**

- Turn-based games
- Payoffs given by $\omega$-regular objectives: $\phi_A$ objective of player $A \in \text{Agt}$
- Pure strategy profiles

is not a Nash equilibrium
A simple characterization for $\omega$-regular objectives

Player □ loses along that play

$\phi_A$: objective of player $A$
A simple characterization for $\omega$-regular objectives

$\phi_A$: objective of player $A$

player $A$ loses along that play

Player $A$ should lose
A simple characterization for $\omega$-regular objectives

$\phi_A$: objective of player $A$

Coalition $\{\bigcirc, \Diamond\}$ prevents $\square$ from winning (by determinacy)

threat/punishment strategy

Player $\square$ loses along that play
A simple characterization for $\omega$-regular objectives

$\phi_A$: objective of player $A$

Coalition $\{\bigcirc, \bigdiamond\}$ prevents $\square$ from winning

$\neg\phi_{\square} \Rightarrow G(p_{\square} \Rightarrow X W_{\{\bigcirc, \bigdiamond\}})$

where $p_{\square}$ labels $\square$-states and $W_{\{\bigcirc, \bigdiamond\}}$ is the set of winning states for the coalition $\{\bigcirc, \bigdiamond\}$ for winning objective $\neg\phi_{\square}$. 
A simple characterization for $\omega$-regular objectives

$\phi_A$: objective of player $A$

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Main outcomes of Boolean Nash equilibria in turn-based games can be characterized by an LTL formula:

$$
\Phi_{NE} = \bigwedge_{A \in \text{Agt}} \left( \neg \phi_A \Rightarrow G(p_A \Rightarrow X W_{\{\neg A\}}) \right)
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where $p_A$ labels $A$-states and $W_{\{\neg A\}}$ is the set of winning states for the coalition $\{\neg A\} \overset{\text{def}}{=} \text{Agt} \setminus \{A\}$ against $A$ for the objective $\neg \phi_A$. These sets should be precomputed.
A simple characterization for \( \omega \)-regular objectives

\( \phi_A \): objective of player \( A \)

Coalition \( \{\bigcirc, \blacklozenge\} \) prevents \( \square \) from winning

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Main outcomes of Boolean Nash equilibria in turn-based games can be characterized by an LTL formula:

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where \( \rho_A \) labels \( A \)-states and \( W_{\{-A\}} \) is the set of winning states for the coalition \( \{-A\} \overset{\text{def}}{=} \text{Agt} \setminus \{A\} \) against \( A \) for the objective \( \neg \phi_A \). These sets should be precomputed.

(valid for prefix-independent objectives)
Decidability of the constrained existence problem

Constrained existence problem

Given two thresholds \( L, U \in \mathbb{Q}^+ \), does there exists a Nash equilibrium \( \sigma \) such that for every \( A \in \text{Agt} \):

\[
L_A \leq E_{\nu_0}(\text{payoff}_A) \leq U_A?
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Decidability of the constrained existence problem

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$$L_A \leq \mathbb{E}^\sigma_{v_0}(\text{payoff}_A) \leq U_A?$$

Theorem [Umm08]

One can decide the pure constrained existence problem in finite turn-based multiplayer games for $\omega$-regular objectives.

Examples of complexity results for single objectives:

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Note: it extends to “$\omega$-regular” preference relations with a finite image.
An example of NP-hardness result

By reduction from a SAT instance:

$$\varphi = \bigwedge_{1 \leq i \leq n} C_i \quad \text{with} \quad C_i = \bigvee_{j=1}^{3} \ell_{i,j} \quad \ell_{i,j} \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_k, \neg x_k\}$$
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- Player \( A_i \) for clause \( C_i \), with objective to reach \( \{ \ell_{i,j} \mid j = 1, 2, 3 \} \)
- Player \( A \): reach the rightmost state
An example of NP-hardness result

By reduction from a SAT instance:

$$\varphi = \bigwedge_{1 \leq i \leq n} C_i$$  with $$C_i = \bigvee_{j=1}^{3} \ell_{i,j}$$  $$\ell_{i,j} \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_k, \neg x_k\}$$

- Player $$A_i$$ for clause $$C_i$$, with objective to reach $$\{\ell_{i,j} \mid j = 1, 2, 3\}$$
- Player $$A$$: reach the rightmost state

$$\varphi$$ is satisfiable iff there is a Nash equilibrium with payoff 1 for everyone in the game
The universal existence problem: \( \omega \)-regular objectives
The universal existence problem: $\omega$-regular objectives

- If $\bigcirc$ has a winning strategy from A, then $\bigcirc$ should play it forever.
- Otherwise $\bigcirc$ plays any strategy, until (by chance) a new blue node, for instance J, is visited, from which $\bigcirc$ has a winning strategy; $\bigcirc$ then switches to such a winning strategy, forever.
The universal existence problem: $\omega$-regular objectives

If the game proceeds through B and □ has a winning strategy from B, then □ should play it forever.

If the game proceeds through B but □ has no winning strategy from B, then □ should play any strategy, until (by chance) a new green node, for instance H, is visited, from which □ has a winning strategy; □ then switches to such a winning strategy, forever.
The universal existence problem: $\omega$-regular objectives

If the game proceeds through C and $\Diamond$ has a winning strategy from C, then $\Diamond$ should play it forever.

If the game proceeds through C but $\Diamond$ has no winning strategy from C, then $\Diamond$ should play any strategy, until (by chance) a new red node, for instance E, is visited, from which $\Diamond$ has a winning strategy; $\Diamond$ then switches to such a winning strategy, forever.
The universal existence problem: $\omega$-regular objectives

- Outside the main outcome, all players play the adequate threat or punishment strategy: this is the coalition strategy that makes the deviator lose (NB: determinacy required!)
Questions:
why is it correct?
what immediate extension can be handled?

main outcome
Questions:
- Why is it correct?
- What immediate extension can be handled?

Diagram:
- Black arrow: main outcome
- Yellow arrow: possible deviation
- Orange arrow: threat (or punishment) strategy
Questions:
- Why is it correct?
- What immediate extension can be handled?
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The universal existence problem: $\omega$-regular objectives

Universal existence [Umm11]

In infinite-duration turn-based deterministic games on finite graphs with $\omega$-regular objectives, there is always a pure Nash equilibrium. Moreover, one can compute a witness.

The universal existence problem

**Universal existence [Umm11]**

In infinite-duration turn-based deterministic games on finite graphs with $\omega$-regular objectives, there is always a pure Nash equilibrium. Moreover, one can compute a witness.

**Universal existence [LeR13]**

In infinite turn-based deterministic games with Borel measurable countable preferences, with no ascending infinite chains, there is always a pure Nash equilibrium.

Outline

1 Verification and game theory

2 Games on graphs
   - The general model
   - Focus on a simple scenario
   - Adding probabilities to the setting?
   - Concurrent games

3 Conclusion
Along a Nash equilibrium where $p_{A_1} \geq 1$:

\[
p_{A_2} + p_{A_3} = 1
\]

\[
p_{A_2} = 2n + 1
\]

One can simulate a two-counter machine if we constrain $p_{A_1} \geq 1$.

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Stochastic turn-based games

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$\leadsto n = n'$

One can simulate a two-counter machine if we constrain $p_{A_1} \geq 1$!!

Stochastic turn-based games

Undecidability results [UW11]

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Undecidability results [UW11]

- The constrained existence problem for pure strategies in stochastic turn-based games is undecidable.
Stochastic turn-based games

**Undecidability results [UW11]**

- The constrained existence problem for pure strategies in stochastic turn-based games is undecidable.
- The constrained existence problem for mixed strategies in deterministic turn-based games is undecidable.

There always exists a Nash equilibrium for Boolean $\omega$-regular objectives

One can decide the constrained existence of a Nash equilibrium (and compute one!)

One cannot decide the existence of a mixed (i.e. stochastic) Nash equilibrium
Short summary for turn-based $\omega$-regular games

**[UW11,Umm11,LeR13]**

- There always exists a Nash equilibrium for Boolean $\omega$-regular objectives
- One can decide the constrained existence of a Nash equilibrium (and compute one!)
- One cannot decide the existence of a mixed (i.e. stochastic) Nash equilibrium

\[ \sim \text{ this is why we will restrict to pure equilibria in det. games} \]
Outline

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3. Conclusion
Can this theory be extended to concurrent games?

There is no universal existence, even for simple Boolean objectives.
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Existence becomes NP-hard

**Hardness**

The existence problem is NP-hard for reachability objectives.

By reduction from a SAT instance:

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ϕ is satisfiable iff there is a Nash equilibrium in the game.
Who is a suspect? Who knows what?

$\text{susp}(v_0, v_2) = \{A_1\}$

Everyone knows that $A_1$ is the deviator.

$\text{susp}(v_0, v_3) = \{A_1, A_2, A_3\}$

$A_1$ knows that the deviator is either $A_2$ or $A_3$; $A_2$ knows the identity of the deviator; and so does $A_3$.

Assume that the normal move is $v_0 \xrightarrow{aaa} v_1$.

What does that mean if the game proceeds to $v_2$?

Either player $A_1$ deviated alone (playing $b$ instead of $a$);

or $A_3$ deviated alone (playing $b$ instead of $a$).

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Two players: Eve (light)
Adam (dark)
Construction of the suspect game abstraction

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Correctness of the suspect game construction

Winning condition

A strategy $\zeta$ for Eve in the suspect game is winning for some $\alpha \in \mathbb{R}^{\text{Agt}}$ if the unique outcome of $\zeta$ where Adam complies to Eve has payoff $\alpha$, and for every other outcome $\rho$ of $\zeta$, for every $A \in \text{susp}(\rho)$, payoff $A(\rho) \leq \alpha_A$. 

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**Correctness**

Let $\alpha \in \mathbb{R}^\text{Agt}$. There is a Nash equilibrium in the original game with payoff $\alpha$ if and only if Eve has a winning strategy for $\alpha$ in the suspect game.
Players $A_1, A_2$ lose along that play

$A_3$ does not know whether $A_1$ or $A_2$ deviated; he should try to punish both (if $A_1$ deviated, $A_2$ will help $A_3$, and conversely if $A_2$ deviated)

everyone knows $A_1$ deviated; $A_2$ and $A_3$ will try to punish $A_1$
From an algorithmic point-of-view

In the orange part: compute the winner (Eve or Adam) of the zero-sum game, where Eve's objective is $\neg \varphi_A^1$ (Eve wants to show that there is no profitable deviation for $A^1$).

We remove the orange part, and replace the root vertex by a winning state for the previously computed winner.

In the yellow part: compute the winner (Eve or Adam) of the zero-sum game, where Eve's objective is $(\neg \varphi_A^1 \land \neg \varphi_A^2) \lor \text{Reach}(\text{win}_Eve)$, where $\text{win}_Eve$ is an already computed winning state for Eve.

It is then just a matter to find an infinite play satisfying the appropriate property.

The approach can be extended to various settings!
From an algorithmic point-of-view

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Some results

Examples of complexity results

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Extensions of this approach

Partial information monitoring

- Public signal [Bou18]
- Communication graphs [BT19]

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[Bou18] Bouyer. Games on graphs with a public signal monitoring (FoSSaCS’18).
[Bre16] Brenguier. Robust equilibria in mean-payoff games (FoSSaCS’16).
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Partial information monitoring
- Public signal [Bou18]
- Communication graphs [BT19]

Other solution concepts
- Robust equilibria [Bre16]
- Rational synthesis [COT18]

[Bou18] Bouyer. Games on graphs with a public signal monitoring (FoSSaCS’18).
[Bre16] Brenguier. Robust equilibria in mean-payoff games (FoSSaCS’16).
Outline

1 Verification and game theory

2 Games on graphs
   • The general model
   • Focus on a simple scenario
   • Adding probabilities to the setting?
   • Concurrent games

3 Conclusion
Wrap-up

General objective

- Import game theory solutions to the verification field, where interactivity plays also a role
  Ex: Distributed systems interacting in some environment
Applications?

- **Smart grids**: decentralized control of EV charging [GBLM19]
  - stochastic setting
  - ad-hoc approximated solutions

- **Cassting project**: smart houses that produce energy with solar panels [BDGHM16]
  - deterministic setting
  - setting with universal existence
  - exact computation

- **PRISM-games**: medium access control, Aloha protocol, robot coordination, power control [KNPS19]
  - stochastic setting
  - approximated value iteration for computing $\epsilon$-SPE

[BDGHM16] Brihaye, Dhar, Geeraerts, Haddad, Monmege. Efficient energy distribution in a smart grid using multi-player games (Cassting'16)
Wrap-up

General objective

- Import game theory solutions to the verification field, where interactivity plays also a role
  Ex: Distributed systems interacting in some environment

- Relevant questions:
  - assumptions made in the game theory field relevant?
  - solution concepts adapted to the context?
**Wrap-up**

**General objective**
- Import game theory solutions to the verification field, where interactivity plays also a role
  - Ex: Distributed systems interacting in some environment
- **Relevant questions:**
  - assumptions made in the game theory field relevant?
  - solution concepts adapted to the context?

**Nash equilibria in games on graphs**
- The setting of pure Nash equilibria in turn-based det. games rather well-understood
- Probabilistic setting much more complicated
- Concurrent games: a rather generic approach based on the suspect game construction
More relevant solution concepts?

- Temporal aspects weakens the concept of Nash equilibrium: *Will a rational agent/process focus on punishing a deviator, instead of pursuing her own objective?*
- Another solution concept: **subgame-perfect equilibrium** (SPE)
Going further?

More relevant solution concepts?
- Temporal aspects weakens the concept of Nash equilibrium: *Will a rational agent/process focus on punishing a deviator, instead of pursuing her own objective?*
- Another solution concept: subgame-perfect equilibrium (SPE)

Universal existence [Umm06]
For Boolean parity (even Borel) objectives, there is always a pure SPE.

A simple example with no SPE [LP14]

\[
\begin{align*}
v_1 & \quad x \quad v_2 \\
\text{A}_1 & \colon x >_{A_1} y >_{A_1} z \\
\text{A}_2 & \colon y >_{A_2} z >_{A_2} x
\end{align*}
\]
Going further?

A simple example with no SPE [LP14]

\[ A_1: x >_A_1 y >_A_1 z \]
\[ A_2: y >_A_2 z >_A_2 x \]
Going further?

More relevant solution concepts?

- Temporal aspects weakens the concept of Nash equilibrium: *Will a rational agent/process focus on punishing a deviator, instead of pursuing her own objective?*
- Another solution concept: *subgame-perfect equilibrium* (SPE)