On finite-memory determinacy of games on graphs

Patricia Bouyer
LSV, CNRS, Univ. Paris-Saclay, ENS Paris-Saclay
France

Based on joint work with Stéphane Le Roux, Youssouf Oualhadj,
Mickael Randour, Pierre Vandenhove
(Published at CONCUR’20)
The talk in one slide

Strategy synthesis for two-player games

• Find good and simple controllers for systems interacting with an antagonistic environment
The talk in one slide

**Strategy synthesis for two-player games**

- Find good and simple controllers for systems interacting with an antagonistic environment

**« Good »?**

- Performance w.r.t. objectives / payoffs / preference relations
The talk in one slide

Strategy synthesis for two-player games

- Find good and simple controllers for systems interacting with an antagonistic environment

« Good »?
- Performance w.r.t. objectives / payoffs / preference relations

« Simple »?
- Memoryless strategies
- Finite-memory strategies
Strategy synthesis for two-player games

- Find good and simple controllers for systems interacting with an antagonistic environment

« Good »?
- Performance w.r.t. objectives / payoffs / preference relations

« Simple »?
- Memoryless strategies
- Finite-memory strategies

When are simple strategies sufficient to play optimally?
The setting - Example of a game

Reachability winning condition for $P_1$

\[ P_1 \rightarrow P_2 \]
The setting - Example of a game

Reachability winning condition for $P_1$

Use of colors to define winning condition/preference relation

$$\ast \in (\circ + \bullet)^\omega$$
The setting - Example of a game

Reachability winning condition for $P_1$
The setting - Example of a game

Reachability winning condition for $P_1$

The game is played using strategies:

$$\sigma_i : S^*S_i \rightarrow E$$
Families of strategies

$$\sigma_i : S^*S_i \rightarrow E$$
Families of strategies

\[ \sigma_i : S^*S_i \to E \]

Subclasses of interest
Families of strategies

\[ \sigma_i : S^i S_i \rightarrow E \]

Subclasses of interest

- Memoryless strategy: \( \sigma_i : S_i \rightarrow E \)

« Reach the target »
Families of strategies

\[ \sigma_i : S^*S_i \rightarrow E \]

**Subclasses of interest**

- Memoryless strategy: \( \sigma_i : S_i \rightarrow E \)
- Finite-memory strategy: \( \sigma_i \) defined by a finite-state Mealy machine

« Reach the target »

« Visit both \( s_1 \) and \( s_2 \) »

Every odd visit to \( s_0 \), go to \( s_1 \)
Every even visit to \( s_0 \), go to \( s_2 \)
Families of strategies

$$\sigma_i : S^*S_i \to E$$

- Memoryless strategy: $$\sigma_i : S_i \to E$$
- Finite-memory strategy: $$\sigma_i$$ defined by a finite-state Mealy machine

Subclasses of interest

- « Reach the target with energy 0 »
  - Loop 5 times in the initial state

- « Reach the target »
  - « Visit both $$s_1$$ and $$s_2$$ »
  - Every odd visit to $$s_0$$, go to $$s_1$$
  - Every even visit to $$s_0$$, go to $$s_2$$
The setting - Preference relation

A preference relation $\sqsubseteq$ is a total preorder on $C^\omega$.

$\pi \sqsubseteq \pi'$ and $\pi' \sqsubseteq \pi$ means that $\pi$ and $\pi'$ are equally appreciated.

$\pi \sqsubseteq \pi'$ and $\pi' \not\sqsubseteq \pi$ means that $\pi'$ is preferred over $\pi$. 
The setting - Preference relation

A preference relation $\succeq$ is a total preorder on $C^\omega$.

$\pi \succeq \pi'$ and $\pi' \succeq \pi$ means that $\pi$ and $\pi'$ are equally appreciated.

$\pi \succeq \pi'$ and $\pi' \not\succeq \pi$ means that $\pi'$ is preferred over $\pi$.

Examples

- $W \subseteq C^\omega$ winning condition:
  $\pi \succeq \pi'$ if either $\pi' \in W$ or $\pi \not\in W$.
- Quantitative real payoff $f$
  $\pi \succeq \pi'$ if $f(\pi) \leq f(\pi')$.
  Ex: MP, AE, TP.
The setting - Preference relation

A preference relation $\sqsubseteq$ is a total preorder on $C^\omega$.

$\pi \sqsubseteq \pi'$ and $\pi' \sqsubseteq \pi$ means that $\pi$ and $\pi'$ are equally appreciated.

$\pi \sqsubseteq \pi'$ and $\pi' \not\sqsubseteq \pi$ means that $\pi'$ is preferred over $\pi$.

Examples

- $W \subseteq C^\omega$ winning condition:
  $\pi \sqsubseteq \pi'$ if either $\pi' \in W$ or $\pi \notin W$

- Quantitative real payoff $f$
  $\pi \sqsubseteq \pi'$ if $f(\pi) \leq f(\pi')$
  Ex: MP, AE, TP

Zero-sum assumption:
- Preference of $P_1$ is $\sqsubseteq$
- Preference of $P_2$ is $\sqsubseteq^{-1}$
Payoffs based on energy

Focus on two memoryless strategies
Payoffs based on energy

Focus on two memoryless strategies

• Constraint on the energy level (EL)
Payoffs based on energy

Focus on two memoryless strategies

- Constraint on the energy level (EL)
- Mean-payoff (MP): long-run average payoff per transition
Payoffs based on energy

Focus on two memoryless strategies

- Constraint on the energy level (EL)
- Mean-payoff (MP): long-run average payoff per transition
Payoffs based on energy

Focus on two memoryless strategies

• Constraint on the energy level (EL)
• Mean-payoff (MP): long-run average payoff per transition
• Total-payoff (TP)
Payoffs based on energy

Focus on two memoryless strategies

- Constraint on the energy level (EL)
- Mean-payoff (MP): long-run average payoff per transition
- Total-payoff (TP)
Payoffs based on energy

Focus on two memoryless strategies

• Constraint on the energy level (EL)
• Mean-payoff (MP): long-run average payoff per transition
• Total-payoff (TP)
• Average-energy (AE)
Optimality of strategies
Optimality of strategies

\[ \text{Out}(\sigma_n) \uparrow \subseteq \text{Out}(\sigma'_n) \uparrow \]
Optimality of strategies

\[ \text{Out}(\sigma_\alpha) \subseteq \text{Out}(\sigma'_\alpha) \]

\[ \Rightarrow \sigma_\alpha \text{ is better than } \sigma'_\alpha \]
Optimality of strategies

\[ \text{Out}(\omega_n) \subseteq \text{Out}(\sigma_n') \]

\[ \Rightarrow \sigma_n \text{ is better than } \sigma_n' \]

\[ \sigma_n \text{ is optimal whenever it is better than any other } \sigma_n' \]
Optimality of strategies

To be distinguished from:
- \( \epsilon \)-optimal
- Subgame-perfect optimal (in our case: Nash equilibria)

\[
\text{Out}(\sigma_n) \uparrow \subseteq \text{Out}(\sigma_n') \uparrow
\]

\( \Rightarrow \sigma_n \) is better than \( \sigma_n' \)

\( \sigma_n \) optimal whenever it is better than any other \( \sigma_n' \)
A focus on memoryless strategies
When are memoryless strategies sufficient to play optimally?

Quite often!
When are memoryless strategies sufficient to play optimally?

Quite often!

Examples

• Reachability, safety, Büchi, parity, MP, EL $\geq 0$, TP, AE, etc…
When are memoryless strategies sufficient to play optimally?

Quite often!

Examples

- Reachability, safety, Büchi, parity, MP, EL \( \geq 0 \), TP, AE, etc…

Can we characterize when they are?
When are memoryless strategies sufficient to play optimally?

Quite often!

Examples

- Reachability, safety, Büchi, parity, MP, EL ≥ 0, TP, AE, etc…

Can we characterize when they are?

YES!
When are memoryless strategies sufficient to play optimally?

Quite often!

Examples

- Reachability, safety, Büchi, parity, MP, EL $\geq 0$, TP, AE, etc…

Can we characterize when they are?

YES!

And this is a beautiful result by Gimbert and Zielonka, CONCUR’05
The memoryless story

Sufficient conditions
The memoryless story

**Sufficient conditions**

- Sufficient conditions to guarantee memoryless optimal strategies for both players [GZ04, AR17]
The memoryless story

Sufficient conditions

• Sufficient conditions to guarantee memoryless optimal strategies for both player [GZ04,AR17]
• Sufficient conditions to guarantee memoryless optimal strategies for one player (« half-positional ») [Kop06,Gim07,GK14]
The memoryless story

Sufficient conditions

- Sufficient conditions to guarantee memoryless optimal strategies for both player [GZ04,AR17]
- Sufficient conditions to guarantee memoryless optimal strategies for one player (« half-positional ») [Kop06,Gim07,GK14]

Characterization of the preference relations admitting optimal memoryless strategies for both players in all finite games [GZ05]
The Gimbert-Zielonka characterization for memory less determinacy (1)
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let $\sqsubseteq$ be a preference relation.
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let $\succeq$ be a preference relation.

It is said:

- monotone whenever
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let $\preceq$ be a preference relation.

It is said:

- **monotone** whenever
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let $\sqsubseteq$ be a preference relation.

It is said:

- monotone whenever

\[
\begin{array}{c}
\vdash \\
\Rightarrow
\end{array}
\]
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let $\sqsubseteq$ be a preference relation.

It is said:

- **monotone** whenever

\[ \ldots \quad \Rightarrow \quad \ldots \]
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let \(\succeq\) be a preference relation. It is said:

- **monotone whenever**

\[ \begin{array}{cccc}
\succ & \succ & \cdots & \Rightarrow \\
\succ & \succ & \cdots & \\
\end{array} \]

- **selective whenever**

\[ \begin{array}{cccc}
\succ & \succ & \cdots & \\
\succ & \succ & \cdots & \\
\end{array} \]
Let \( \sqsubseteq \) be a preference relation.

It is said:

- **monotone whenever**

  \[
  \vdash \quad \Pi \\
  \implies \quad \Pi
  \]

- **selective whenever**

  \[
  \vdash \quad \Pi \\
  \implies \quad \Pi
  \]
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let $\succeq$ be a preference relation.

It is said:

- **monotone** whenever

  \[ \cdots \ circ \cdots = \cdots \]

- **selective** whenever

  \[ \cdots \ circ \cdots \Rightarrow \cdots \]

\[ \text{[GZ05]} \]
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let $\succeq$ be a preference relation. It is said:

- **monotone** whenever
  
  $\begin{align*}
  \vdash & \Rightarrow \\
  & \vdash
  \end{align*}$

- **selective** whenever
  
  $\begin{align*}
  & \vdash \\
  \vdash & \Rightarrow
  \end{align*}$
The Gimbert-Zielonka characterization for memory less determinacy (1)

Let $\succ$ be a preference relation.

It is said:

- **monotone** whenever

- **selective** whenever

\[ [GZ05] \]
The Gimbert-Zielonka characterization for memory less determinacy (2)
The two following assertions are equivalent:

1. All finite games have memoryless optimal strategies for both players.
2. Both $\varepsilon$ and $\varepsilon^{-1}$ are monotone and selective.
The Gimbert-Zielonka characterization for memory less determinacy (2)

Characterization - Two-player games

The two following assertions are equivalent:
1. All finite games have memoryless optimal strategies for both players
2. Both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are monotone and selective

Characterization - One-player games

The two following assertions are equivalent:
1. All finite $P_1$-games have (uniform) memoryless optimal strategies
2. $\mathcal{E}$ is monotone and selective
Why? Proof hint (1)
Why? Proof hint (1)

Assume all $P_1$-games have optimal memoryless strategies.
Assume all $P_1$-games have optimal memoryless strategies.
Assume all $P_1$-games have optimal memoryless strategies.
Why? Proof hint (1)

Assume all $P_1$-games have optimal memoryless strategies.
Why? Proof hint (1)

Assume all $P_1$-games have optimal memoryless strategies.
Why? Proof hint (2)

Assume $\Xi$ is monotone and selective.

The case of one-player games

One best choice between $\circ$ and $\triangleright$ (monotony)

No reason to swap at $t$ (selectivity)

No memory required at $t$!
Applications

Lifting theorem

- If in all finite one-player game for player $P_i$, $P_i$ has uniform memoryless optimal strategies, then both players have memoryless optimal strategies in all finite two-player games.
Applications

Lifting theorem

- If in all finite one-player game for player $P_i$, $P_i$ has uniform memoryless optimal strategies, then both players have memoryless optimal strategies in all finite two-player games.

Very powerful and extremely useful in practice!
Applications

Lifting theorem

• If in all finite one-player game for player $P_i$, $P_i$ has uniform memoryless optimal strategies, then both players have memoryless optimal strategies in all finite two-player games.

Very powerful and extremely useful in practice!

Discussion

• Easy to analyse the one-player case (graph analysis)
  - Mean-payoff, average-energy [BMRLl15]
• Allows to deduce properties in the two-player case
Discussion of examples
Discussion of examples

- Reachability, safety:
Discussion of examples

Examples

• Reachability, safety:
  - Monotone (though not prefix-independent)
Discussion of examples

Examples

- Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective
Discussion of examples

Examples

- Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective
- Parity, mean-payoff:
Discussion of examples

Examples

- Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective
- Parity, mean-payoff:
  - Prefix-independent hence monotone
Discussion of examples

Examples

• Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective

• Parity, mean-payoff:
  - Prefix-independent hence monotone
  - Selective
Discussion of examples

Examples

• Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective
• Parity, mean-payoff:
  - Prefix-independent hence monotone
  - Selective
• Priority mean payoff [GZ05]
Discussion of examples

- Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective
- Parity, mean-payoff:
  - Prefix-independent hence monotone
  - Selective
- Priority mean payoff [GZ05]
- Average-energy games [BMRLL15]
Discussion of examples

Examples

- Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective
- Parity, mean-payoff:
  - Prefix-independent hence monotone
  - Selective
- Priority mean payoff \([GZ05]\)
- Average-energy games \([BMRLL15]\)
  - Lifting theorem!!
Discussion
Discussion

Winning condition for $P_1$:

$$((MP \in Q) \land \text{Büchi}(A)) \lor \text{coBüchi}(B)$$
Discussion

Winning condition for $P_1$:

$$(\text{MP } \in \mathbb{Q}) \land \text{Büchi}(A) \lor \text{coBüchi}(B)$$

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} c_i \in \mathbb{Q}$$

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} c_i \in \mathbb{Q}$$
Discussion

Winning condition for $P_1$:

$$(((MP \in Q) \land \text{Büchi}(A)) \lor \text{coBüchi}(B))$$
Discussion

Winning condition for $P_1$:

\[((\text{MP} \in \mathbb{Q}) \land \text{Büchi}(A)) \lor \text{coBüchi}(B)\]

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
Discussion

Winning condition for $P_1$:

$((MP \in Q) \land \text{Büchi}(A)) \lor \text{coBüchi}(B)$

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
- Hence: the winning condition is monotone and selective
Discussion

Winning condition for $P_1$:

$$((MP \in \mathbb{Q}) \land \text{B"{u}chi}(A)) \lor \text{coB"{u}chi}(B)$$

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
- Hence: the winning condition is monotone and selective
Discussion

Winning condition for $P_1$:

$$((MP \in \mathbb{Q}) \land \text{Büchi}(A)) \lor \text{coBüchi}(B)$$

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
- Hence: the winning condition is monotone and selective

![Game diagram]

How should $P_1$ play this game?
Discussion

Winning condition for $P_1$:

$$((MP \in \mathbb{Q}) \land \text{Büchi}(A)) \lor \text{coBüchi}(B)$$

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
- Hence: the winning condition is monotone and selective

- $P_1$ wins this game:

![Game diagram]

How should $P_1$ play this game?

- $P_1$ wins this game:
Discussion

Winning condition for $P_1$:

$$((\text{MP} \in \mathbb{Q}) \land \text{Büchi}(A)) \lor \text{coBüchi}(B)$$

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
- Hence: the winning condition is monotone and selective

How should $P_1$ play this game?

- $P_1$ wins this game:
  - Infinitely often, give the hand back to $P_2$
Discussion

Winning condition for $P_1$:

\[ ((M \in \mathbb{Q}) \land \text{Büchi}(A)) \lor \text{coBüchi}(B) \]

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
- Hence: the winning condition is monotone and selective

How should $P_1$ play this game?

- $P_1$ wins this game:
  - Infinitely often, give the hand back to $P_2$
  - Play for a long time the edge labelled $(0,B)$ to approach 0
Discussion

Winning condition for $P_1$:

$((MP \in \mathbb{Q}) \land \text{Büchi}(A)) \lor \text{coBüchi}(B)$

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
- Hence: the winning condition is monotone and selective

How should $P_1$ play this game?

- $P_1$ wins this game:
  - Infinitely often, give the hand back to $P_2$
  - Play for a long time the edge labelled $(0,B)$ to approach 0
  - Play for a long time the edge labelled $(1,B)$ to approach 1
Discussion

Winning condition for $P_1$:

$$((MP \in \mathbb{Q}) \land \text{Büchi}(A)) \lor \text{coBüchi}(B)$$

- In all one-player games, $P_1$ has a memoryless uniform optimal strategy
- Hence: the winning condition is monotone and selective

$P_1$ wins this game:
- Infinitely often, give the hand back to $P_2$
- Play for a long time the edge labelled $(0,B)$ to approach $0$
- Play for a long time the edge labelled $(1,B)$ to approach $1$
- It requires infinite memory!
Discussion

Winning condition for $P_1$:

$$((MP \in Q) \land \text{B"uchi}(A)) \lor \text{coB"uchi}(B)$$

If only $\subseteq$ is monotone and selective, $P_1$ might not have a memoryless optimal strategy.
Finite-memory strategies
We need memory!

Objectives/preference relations become more and more complex
We need memory!

Objectives/preference relations become more and more complex

- Büchi($A$) $\land$ Büchi($B$) requires finite memory
We need memory!

Objectives/preference relations become more and more complex

- Büchi(A) ∧ Büchi(B) requires finite memory
- \( MP_1 \geq 0 \land MP_2 \geq 0 \) requires infinite memory
Can we lift [GZ05] to finite memory?
Can we lift \cite{GZ05} to finite memory?

A priori no...
Can we lift [GZ05] to finite memory?

A priori no...

Consider the following winning condition for $P_1$:

$$\lim \inf \sum_{i=1}^{n} c_i = +\infty \text{ or } \exists \infty n \text{ s.t. } \sum_{i=1}^{n} c_i = 0$$
Can we lift [GZ05] to finite memory?

A priori no...

Consider the following winning condition for $P_1$:

$$\liminf_{n} \sum_{i=1}^{n} c_i = +\infty \lor \exists \infty n \text{ s.t. } \sum_{i=1}^{n} c_i = 0$$

- Optimal finite-memory strategies in one-player games
Can we lift [GZ05] to finite memory?

A priori no...

Consider the following winning condition for $P_1$:

$$\liminf_n \sum_{i=1}^{n} c_i = +\infty \text{ or } \exists \infty n \text{ s.t. } \sum_{i=1}^{n} c_i = 0$$

- Optimal finite-memory strategies in one-player games
- But not in two-player games!!
Can we lift [GZ05] to finite memory?

A priori no...

Consider the following winning condition for $P_1$:

$$\lim \inf \sum_{i=1}^{n} c_i = + \infty \quad \text{or} \quad \exists \infty \text{ s.t. } \sum_{i=1}^{n} c_i = 0$$

- Optimal finite-memory strategies in one-player games
- But not in two-player games!!

$P_1$ wins but uses infinite memory!
How do we formalize finite memory?
How do we formalize finite memory?

Standardly
How do we formalize finite memory?

Standardly

- A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{\text{init}}, \alpha_{\text{upd}}, \alpha_{\text{next}})$ where $M$ is finite, $m_{\text{init}} \in M$, $\alpha_{\text{upd}} : M \times S \to M$ and $\alpha_{\text{next}} : M \times S_i \to E$
How do we formalize finite memory?

Standardly

- A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{init}, \alpha_{upd}, \alpha_{next})$ where $M$ is finite, $m_{init} \in M$,

  $\alpha_{upd}: M \times S \rightarrow M$ and $\alpha_{next}: M \times S_i \rightarrow E$

- $(M, m_{init}, \alpha_{upd})$ is a memory mechanism
How do we formalize finite memory?

Standardly

- A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{\text{init}}, \alpha_{\text{upd}}, \alpha_{\text{next}})$ where $M$ is finite, $m_{\text{init}} \in M$,

  $\alpha_{\text{upd}} : M \times S \rightarrow M$ and $\alpha_{\text{next}} : M \times S_i \rightarrow E$

  - $(M, m_{\text{init}}, \alpha_{\text{upd}})$ is a memory mechanism
  - $\alpha_{\text{next}}$ gives the next move
A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{\text{init}}, \alpha_{\text{upd}}, \alpha_{\text{next}})$ where $M$ is finite, $m_{\text{init}} \in M$, $\alpha_{\text{upd}} : M \times S \rightarrow M$ and $\alpha_{\text{next}} : M \times S_i \rightarrow E$

- $(M, m_{\text{init}}, \alpha_{\text{upd}})$ is a memory mechanism
- $\alpha_{\text{next}}$ gives the next move

How do we formalize finite memory?

Standardly
A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{\text{init}}, \alpha_{\text{upd}}, \alpha_{\text{next}})$ where $M$ is finite, $m_{\text{init}} \in M$, $\alpha_{\text{upd}} : M \times S \to M$ and $\alpha_{\text{next}} : M \times S_i \to E$.

- $(M, m_{\text{init}}, \alpha_{\text{upd}})$ is a memory mechanism
- $\alpha_{\text{next}}$ gives the next move
How do we formalize finite memory?

Standardly

- A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{\text{init}}, \alpha_{\text{upd}}, \alpha_{\text{next}})$ where $M$ is finite, $m_{\text{init}} \in M$,
  $\alpha_{\text{upd}} : M \times S \to M$ and $\alpha_{\text{next}} : M \times S_i \to E$
  - $(M, m_{\text{init}}, \alpha_{\text{upd}})$ is a memory mechanism
  - $\alpha_{\text{next}}$ gives the next move

To have an abstract theorem...
How do we formalize finite memory?

Standardly

- A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{\text{init}}, \alpha_{\text{upd}}, \alpha_{\text{next}})$ where $M$ is finite, $m_{\text{init}} \in M$,
  $\alpha_{\text{upd}} : M \times S \rightarrow M$ and $\alpha_{\text{next}} : M \times S_i \rightarrow E$
  - $(M, m_{\text{init}}, \alpha_{\text{upd}})$ is a memory mechanism
  - $\alpha_{\text{next}}$ gives the next move

To have an abstract theorem...

- The memory mechanism should not speak about information specific to particular games, hence:
How do we formalize finite memory?

Standardly

- A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{\text{init}}, \alpha_{\text{upd}}, \alpha_{\text{next}})$ where $M$ is finite, $m_{\text{init}} \in M$,
  
  $\alpha_{\text{upd}} : M \times S \to M$ and $\alpha_{\text{next}} : M \times S_i \to E$

  - $(M, m_{\text{init}}, \alpha_{\text{upd}})$ is a memory mechanism
  - $\alpha_{\text{next}}$ gives the next move

To have an abstract theorem...

- The memory mechanism should not speak about information specific to particular games, hence:
  - $\alpha_{\text{upd}}$ should not speak of states
How do we formalize finite memory?

**Standardly**

- A strategy $\sigma_i$ of player $P_i$ has finite memory if it can be encoded as a Mealy machine $(M, m_{init}, \alpha_{upd}, \alpha_{next})$ where $M$ is finite, $m_{init} \in M$,
  $\alpha_{upd} : M \times S \to M$ and $\alpha_{next} : M \times S_i \to E$
- $(M, m_{init}, \alpha_{upd})$ is a memory mechanism
- $\alpha_{next}$ gives the next move

**To have an abstract theorem...**

- The memory mechanism should not speak about information specific to particular games, hence:
  - $\alpha_{upd}$ should not speak of states
  - $\alpha_{upd}$ can speak of colors
  (notion of « chromatic strategy » by Kopczynski)
Arena-independent memory management
Arena-independent memory management

Memory skeleton

\[ \mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}}) \text{ with } m_{\text{init}} \in M \text{ and } \alpha_{\text{upd}} : M \times C \rightarrow M \]
Arena-independent memory management

\[ \mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}}) \text{ with } m_{\text{init}} \in M \text{ and } \alpha_{\text{upd}} : M \times C \to M \]

Memory skeleton

\[ \begin{array}{c}
A \quad A \\
 m_1 \quad m_2 \\
 B \\
B
\end{array} \]
Arena-independent memory management

Memory skeleton

\[ \mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}}) \text{ with } m_{\text{init}} \in M \text{ and } \alpha_{\text{upd}} : M \times C \to M \]

Not yet a strategy!
Arena-independent memory management

Memory skeleton

- $\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})$ with $m_{\text{init}} \in M$ and $\alpha_{\text{upd}} : M \times C \rightarrow M$

![Memory skeleton diagram]

Strategy with memory $\mathcal{M}$

- Additional next-move function: $\alpha_{\text{next}} : M \times S_i \rightarrow E$

Not yet a strategy!
Arena-independent memory management

**Memory skeleton**

- $\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})$ with $m_{\text{init}} \in M$ and $\alpha_{\text{upd}} : M \times C \to M$

**Strategy with memory $\mathcal{M}$**

- Additional next-move function: $\alpha_{\text{next}} : M \times S_i \to E$

The above skeleton is sufficient for the winning condition

$\text{Büchi}(A) \land \text{Büchi}(B)$
Example
Example

Game arena $\mathcal{A}$:
Example

Game arena $\mathcal{A}$:

- $(s_1, m_1) \mapsto (s_1, s_2)$
- $(s_1, m_2) \mapsto (s_1, s_1)$
- $(s_2, m_1) \mapsto (s_2, s_2)$
- $(s_2, m_2) \mapsto (s_2, s_1)$
Example

Game arena $\mathcal{A}$:

Product game $\mathcal{A} \times \mathcal{M}$:

$\mathcal{M}$:

- $(s_1, m_1) \mapsto (s_1, s_2)$
- $(s_1, m_2) \mapsto (s_1, s_1)$
- $(s_2, m_1) \mapsto (s_2, s_2)$
- $(s_2, m_2) \mapsto (s_2, s_1)$
Example

Game arena $\mathcal{A}$:

Product game $\mathcal{A} \times \mathcal{M}$:

- One can however not apply the [GZ05] result to product games!
Memory-dependent monotony and selectivity
Memory-dependent monotony and selectivity

Let $\preceq$ be a preference relation and $\mathcal{M}$ a memory skeleton.
Memory-dependent monotony and selectivity

Let $\preceq$ be a preference relation and $\mathcal{M}$ a memory skeleton.

It is said:

- $\mathcal{M}$-monotone whenever
Memory-dependent monotony and selectivity

Let $\succeq$ be a preference relation and $\mathcal{M}$ a memory skeleton.

It is said:

- $\mathcal{M}$-monotone whenever

\[
\begin{array}{c}
\text{m} \\
\text{m}
\end{array}
\qquad \Rightarrow
\begin{array}{c}
\text{m} \\
\text{m}
\end{array}
\]
Memory-dependent monotony and selectivity

Let $\mathcal{E}$ be a preference relation and $\mathcal{M}$ a memory skeleton.

It is said:

- $\mathcal{M}$-monotone whenever

\[ m \]
Memory-dependent monotony and selectivity

Let \( \preceq \) be a preference relation and \( M \) a memory skeleton.

It is said:

- \( M \)-monotone whenever

- \( M \)-selective whenever
Memory-dependent monotony and selectivity

Let $\preceq$ be a preference relation and $M$ a memory skeleton.

It is said:

- $M$-monotone whenever

- $M$-selective whenever
Memory-dependent monotony and selectivity

Let $\preceq$ be a preference relation and $\mathcal{M}$ a memory skeleton.

It is said:

- $\mathcal{M}$-monotone whenever

- $\mathcal{M}$-selective whenever

\[ \preceq \text{ Max} \]
Memory-dependent monotony and selectivity

Let $\mathcal{E}$ be a preference relation and $\mathcal{M}$ a memory skeleton.

It is said:

- **$\mathcal{M}$-monotone** whenever

  $\Downarrow$

  $\{m, \ldots\}$

  $\Rightarrow$

  $\{\ldots\}$

- **$\mathcal{M}$-selective** whenever

  $\Downarrow$

  $\Downarrow$

  $\Downarrow$

  $\Downarrow$

  $\Rightarrow$

  $\Rightarrow$

  $\Rightarrow$

  $\Rightarrow$

  $\Rightarrow$

  $\Rightarrow$

  $\Rightarrow$
Memory-dependent monotony and selectivity

Let $\leq$ be a preference relation and $\mathcal{M}$ a memory skeleton.

It is said:

- $\mathcal{M}$-monotone whenever

- $\mathcal{M}$-selective whenever

We look at how $\mathcal{M}$ classifies prefixes and cycles.
Formal definitions of $\mathcal{M}$-monotony and $\mathcal{M}$-selectivity

**Definition ($\mathcal{M}$-monotony)**

Let $\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})$ be a memory skeleton. A preference relation $\sqsubseteq$ is $\mathcal{M}$-monotone if for all $m \in M$, for all $K_1, K_2 \in \mathcal{R}(C)$,

$$\exists w \in L_{m_{\text{init}}, m}, [wK_1] \sqsubseteq [wK_2] \implies \forall w' \in L_{m_{\text{init}}, m}, [w'K_1] \sqsubseteq [w'K_2].$$

**Definition ($\mathcal{M}$-selectivity)**

Let $\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})$ be a memory skeleton. A preference relation $\sqsubseteq$ is $\mathcal{M}$-selective if for all $w \in C^*$, $m = \hat{\alpha}_{\text{upd}}(m_{\text{init}}, w)$, for all $K_1, K_2 \in \mathcal{R}(C)$ such that $K_1, K_2 \subseteq L_{m, m}$, for all $K_3 \in \mathcal{R}(C)$,

$$[w(K_1 \cup K_2)^* K_3] \sqsubseteq [wK_1^*] \cup [wK_2^*] \cup [wK_3].$$
Our characterization for $M$-determinacy
Our characterization for $\mathcal{M}$-determinacy

Characterization - Two-player games
Our characterization for $\mathcal{M}$-determinacy

The two following assertions are equivalent:
1. All finite games have optimal $\mathcal{M}$-strategies for both players
2. Both $\exists$ and $\exists^{-1}$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective
Our characterization for $\mathcal{M}$-determinacy

The two following assertions are equivalent:

1. All finite games have optimal $\mathcal{M}$-strategies for both players
2. Both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective
Our characterization for $\mathcal{M}$-determinacy

Characterization - Two-player games

The two following assertions are equivalent:
1. All finite games have optimal $\mathcal{M}$-strategies for both players
2. Both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective

Characterization - One-player games

The two following assertions are equivalent:
1. All finite $P_1$-games have (uniform) optimal $\mathcal{M}$-strategies
2. $\mathcal{E}$ is $\mathcal{M}$-monotone and $\mathcal{M}$-selective
Our characterization for $\mathcal{M}$-determinacy

**Characterization - Two-player games**

The two following assertions are equivalent:

1. All finite games have optimal $\mathcal{M}$-strategies for both players
2. Both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective

**Characterization - One-player games**

The two following assertions are equivalent:

1. All finite $P_1$-games have (uniform) optimal $\mathcal{M}$-strategies
2. $\mathcal{E}$ is $\mathcal{M}$-monotone and $\mathcal{M}$-selective

$\Rightarrow$ We recover [GZ05] with $\mathcal{M} = \mathcal{M}_{\text{triv}}$
Applications
Applications

Transfer/Lifting theorem

• If in all finite one-player game for player $P_i$, $P_i$ has optimal $\mathcal{M}_i$-strategies, then both players have optimal $\mathcal{M}_1 \times \mathcal{M}_2$-strategies in all finite two-player games.
Applications

Transfer/Lifting theorem

• If in all finite one-player game for player $P_i$, $P_i$ has optimal $M_i$-strategies, then both players have optimal $M_1 \times M_2$-strategies in all finite two-player games.

Very powerful and extremely useful in practice!
Applications

Transfer/Lifting theorem

• If in all finite one-player game for player $P_i$, $P_i$ has optimal $\mathcal{M}_i$-strategies, then both players have optimal $\mathcal{M}_1 \times \mathcal{M}_2$-strategies in all finite two-player games.

Very powerful and extremely useful in practice!

Subclasses of games

• If both $\Xi$ and $\Xi^{-1}$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective, then both players have optimal memoryless strategies in all $\mathcal{M}$-covered games.
Memory-covered arenas
Memory-covered arenas

If the game has enough information from $M$, then memoryless strategies will be sufficient.
Memory-covered arenas

If the game has enough information from $M$, then memoryless strategies will be sufficient.

Covered arenas = same properties as product arenas
Memory-covered arenas

If the game has enough information from $M$, then memoryless strategies will be sufficient.

Covered arenas = same properties as product arenas

![Diagram of memory-covered arenas]
Memory-covered arenas

If the game has enough information from $\mathcal{M}$, then memoryless strategies will be sufficient.

Covered arenas $\approx$ same properties as product arenas.
Example of application

\[ \in \text{defined by a conjunction of reachability } \text{Reach}(\red{\bigcirc}) \land \text{Reach}(\green{\bigcirc}) \]
Example of application

∈ defined by a conjunction of reachability $\text{Reach}(\bullet) \land \text{Reach}(\circ)$

$M_1$

$C \setminus \{\bullet\}$

Diagram: $m_1 \rightarrow m_2 \rightarrow C$
Example of application

\( \sqsubseteq \) defined by a conjunction of reachability \( \text{Reach}(\bullet) \land \text{Reach}(\circ) \)

\( M_1 \)

\( C \setminus \{ \bullet \} \quad \xrightarrow{m_1} \quad m_1 \quad \xrightarrow{m_2} \quad m_2 \quad \xrightarrow{C} \quad C \)

\( \sqsubseteq \) is \( M_1 \)-monotone, but not \( M_1 \)-selective
Example of application

$\Xi$ defined by a conjunction of reachability $\text{Reach}(\bigcirc) \land \text{Reach}(\bigcirc)$

$\mathcal{M}_1$

$C \setminus \{\} \xrightarrow{m_1} m_2 \xrightarrow{C}$

$\Xi$ is $\mathcal{M}_1$-monotone, but not $\mathcal{M}_1$-selective

$\mathcal{M}_2$

$C \setminus \{\}, C \setminus \{\} \xrightarrow{m_1} m_2 \xrightarrow{C}$

$C \setminus \{\} \xrightarrow{m_3} \{\}$
Example of application

\[ \mathcal{E} \text{ defined by a conjunction of reachability } \text{Reach}(\bullet) \wedge \text{Reach}(\circ) \]

\[ M_1 \]

\[ C \setminus \{\bullet\} \rightarrow m_1 \rightarrow m_2 \rightarrow C \]

\[ M_2 \]

\[ C \setminus \{\bullet, \bullet\} \rightarrow m_1 \rightarrow m_2 \rightarrow C \]

\[ C \setminus \{\bullet\} \rightarrow m_3 \rightarrow C \setminus \{\bullet\} \]

\[ \mathcal{E} \text{ is } M_1\text{-monotone, but not } M_1\text{-selective} \]

\[ \mathcal{E} \text{ is } M_2\text{-selective} \]
Example of application

\( \Xi \) defined by a conjunction of reachability \( \text{Reach}(\bullet) \land \text{Reach}(\circ) \)

\( M_1 \)
\[ C \setminus \{ \bullet \} \xrightarrow{m_1} m_2 \xrightarrow{} C \]

\( \Xi \) is \( M_1 \)-monotone, but not \( M_1 \)-selective

\( M_2 \)
\[ C \setminus \{ \bullet, \circ \} \xrightarrow{m_1} m_2 \xrightarrow{} C \]
\[ C \setminus \{ \bullet, \circ \} \xrightarrow{m_3} \]
\[ C \setminus \{ \bullet \} \]

\( \Xi \) is \( M_2 \)-selective

\( \Xi \) is \( M_1 \)-monotone and \( M_2 \)-selective
Example of application

$\Xi$ defined by a conjunction of reachability $\text{Reach}(\bullet) \land \text{Reach}(\bullet)$

$M_1$

$C \setminus \{\bullet\} \rightarrow m_1 \rightarrow m_2 \rightarrow C$

$\Xi$ is $M_1$-monotone, but not $M_1$-selective

$M_2$

$C \setminus \{\bullet, \bullet\} \rightarrow m_1 \rightarrow m_2 \rightarrow C$

$\Xi$ is $M_2$-selective

$C \setminus \{\bullet\}$

$\Xi$ is $M_1$-monotone and $M_2$-selective

$\Xi^{-1}$ is $M_1$-monotone and $M_{\text{triv}}$-selective
Example of application

\[ \mathcal{E} \text{ defined by a conjunction of reachability } \text{Reach}(\bullet) \land \text{Reach}(\bigcirc) \]

\( M_1 \)

\( C \setminus \{ \bullet \} \rightarrow m_1 \rightarrow m_2 \rightarrow C \)

\( M_1 \) is \( M_1 \)-monotone, but not \( M_1 \)-selective

\( M_2 \)

\( C \setminus \{ \bullet, \bigcirc \} \rightarrow m_1 \rightarrow m_2 \rightarrow C \)

\( C \setminus \{ \bigcirc \} \rightarrow m_3 \rightarrow m_2 \rightarrow C \)

\( M_2 \) is \( M_2 \)-selective

\( M_1 \) is \( M_1 \)-monotone and \( M_2 \)-selective

\( M_1^{-1} \) is \( M_1 \)-monotone and \( M_{triv} \)-selective

\( \Rightarrow \) Memory \( M_2 \) is sufficient for both players!!
Conclusion

A generalization of [GZ05]

• To arena-independent finite memory
• Applies to generalized reachability or parity, lower- and upper-bounded (multi-dimension) energy games
Conclusion

A generalization of [GZ05]

- To arena-independent finite memory
- Applies to generalized reachability or parity, lower- and upper-bounded (multi-dimension) energy games

Limitations

- Does only capture arena-independent finite memory
- Hard to generalize (remember counter-example)
- Does not apply to multi-dim. MP, MP+parity, energy+MP (infinite memory)
Conclusion
Conclusion

Other approaches

- Sufficient conditions giving half-memory management results
- Compositionality w.r.t. objectives [LPR18]
Conclusion

Other approaches

- Sufficient conditions giving half-memory management results
- Compositionality w.r.t. objectives [LPR18]

Further work

- Understand the arena-dependent framework
- Infinite arenas
- Probabilistic setting
- Other concepts (Nash equilibria)