On the optimal reachability problem in weighted timed automata and games

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Based on former works with Thomas Brihaye, Kim G. Larsen, Nicolas Markey, etc...
And on recent work with Samy Jaziri and Nicolas Markey

Time-dependent systems

- We are interested in timed systems

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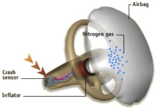

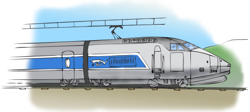

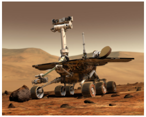

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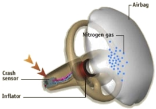
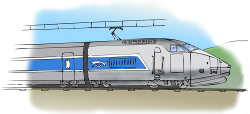
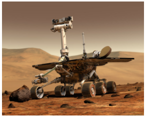






- ... and in their analysis and control

An example: The task graph scheduling problem

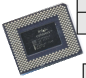
Compute $D \times (C \times (A + B)) + (A + B) + (C \times D)$ using two processors:

P_1 (fast):

time	
+	2 picoseconds
×	3 picoseconds

energy	
idle	10 Watt
in use	90 Watts

P_2 (slow):



time	
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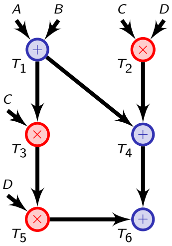

[BFLM10] Bouyer, Fahrenberg, Larsen, Markey. Quantitative Analysis of Real-Time Systems using Priced Timed Automata.

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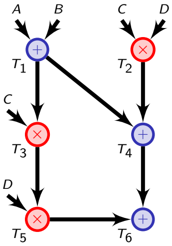

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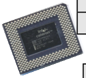

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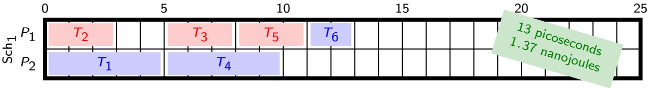

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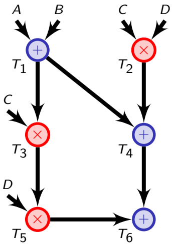

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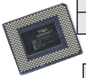

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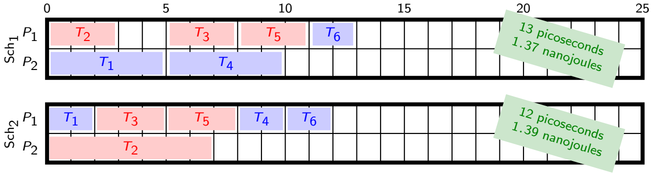

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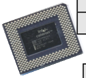

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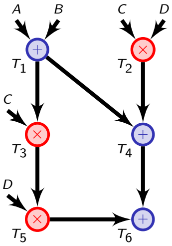

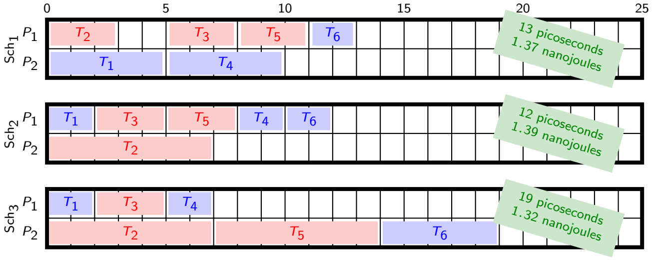

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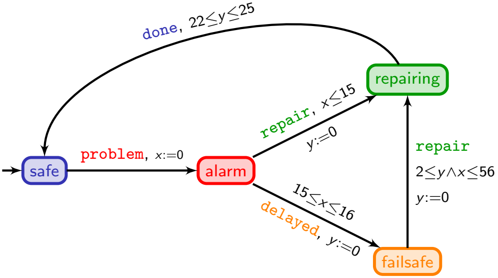
Outline

1. Timed automata
2. Weighted timed automata
3. Timed games
4. Weighted timed games
5. Conclusion

The model of timed automata



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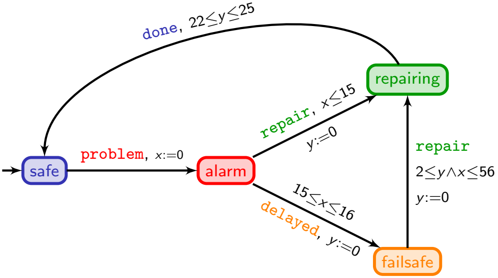


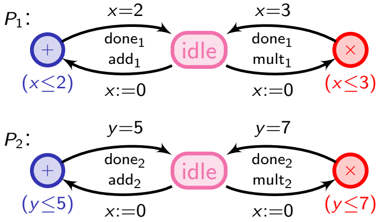
	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe	
x	0		23		0		15.6		15.6	⋯
y	0		23		23		38.6		0	

	failsafe	$\xrightarrow{2.3}$	failsafe	$\xrightarrow{\texttt{repair}}$	repairing	$\xrightarrow{22.1}$	repairing	$\xrightarrow{\texttt{done}}$	safe
⋯	15.6		17.9		17.9		40		40
	0		2.3		0		22.1		22.1

Modelling the task graph scheduling problem

Modelling the task graph scheduling problem

- Processors



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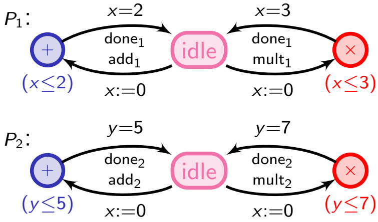
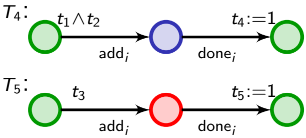
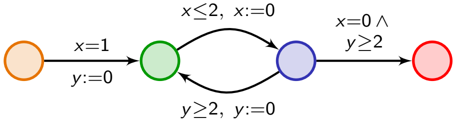


- Tasks



A schedule is a path in the product automaton

Analyzing timed automata



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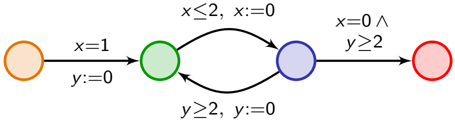

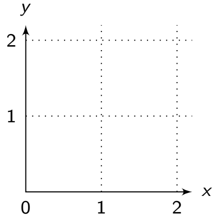

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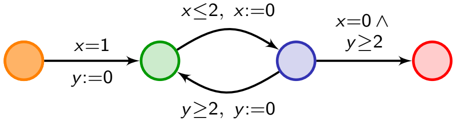

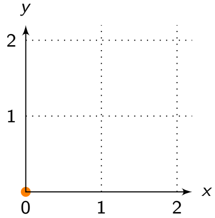

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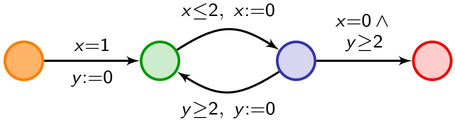

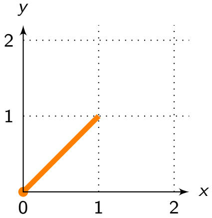

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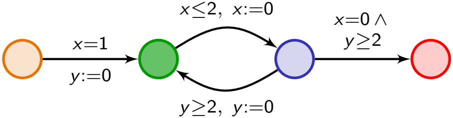

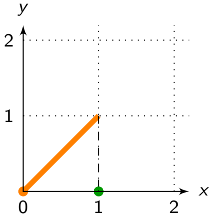

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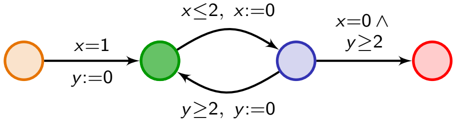

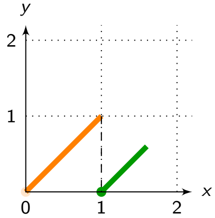

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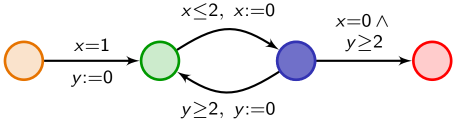

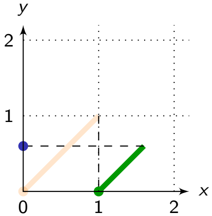

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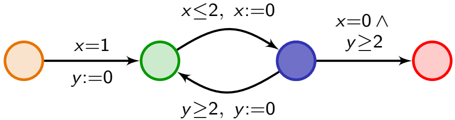

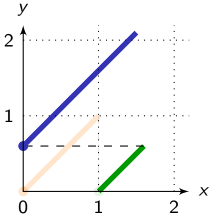

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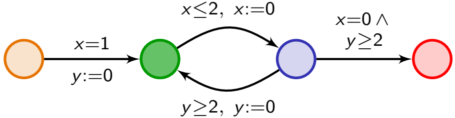

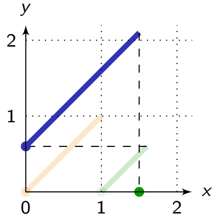

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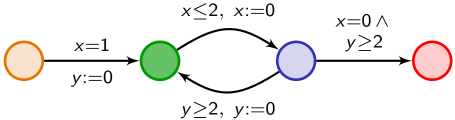

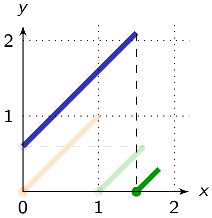

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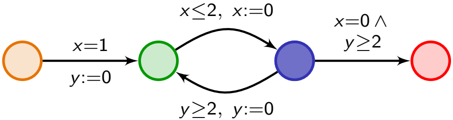

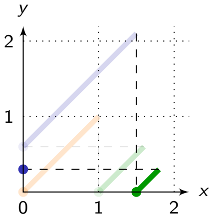

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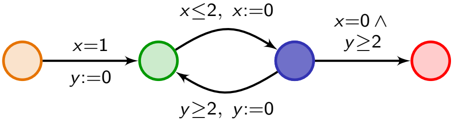

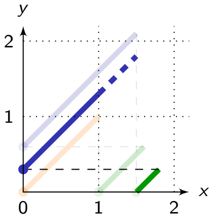

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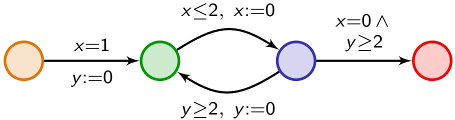

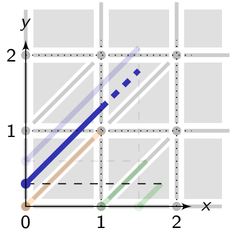

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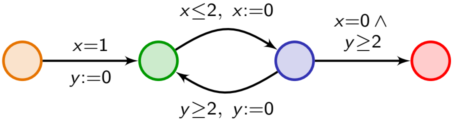
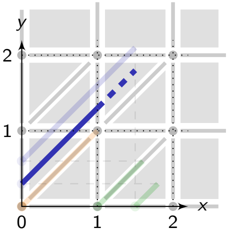


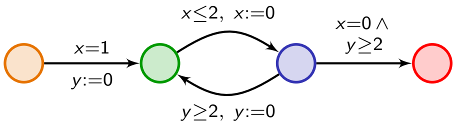


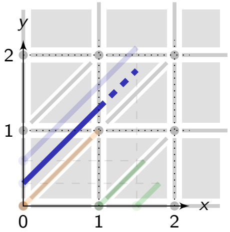
Theorem [AD94]

Reachability in timed automata is decidable (as well as many other important properties). It is PSPACE-complete.

- Technical tool: region abstraction

Analyzing timed automata





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- Efficient symbolic technics based on zones, implemented in tools

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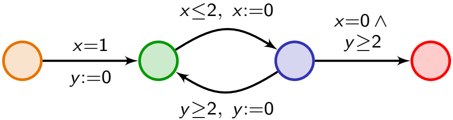

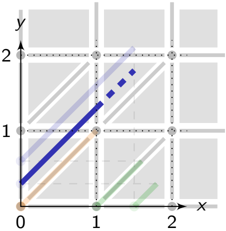

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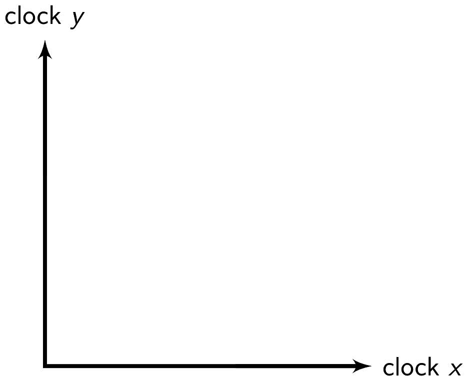
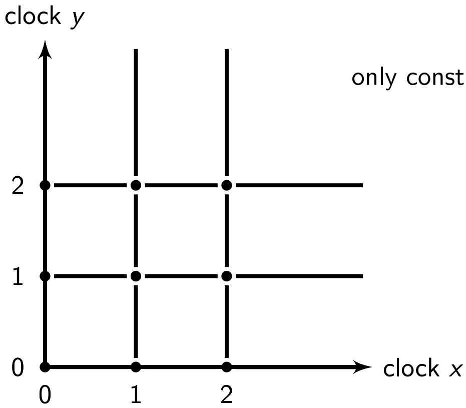
Skip regions

Technical tool: Region abstraction



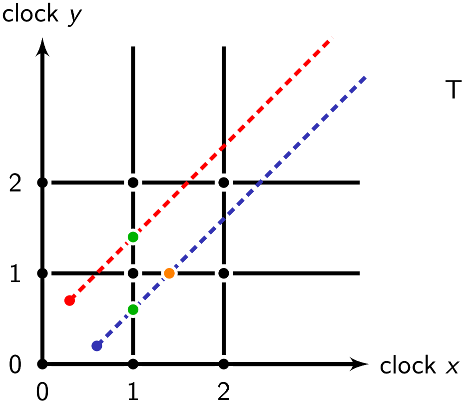
Technical tool: Region abstraction



only constraints: $x \sim c$ with $c \in \{0, 1, 2\}$
$y \sim c$ with $c \in \{0, 1, 2\}$

- "compatibility" between regions and constraints

Technical tool: Region abstraction

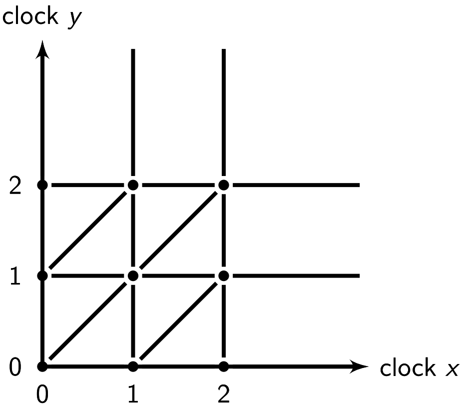


The path 

- can be fired from ●
- cannot be fired from ●

- “compatibility” between regions and constraints
- “compatibility” between regions and time elapsing

Technical tool: Region abstraction



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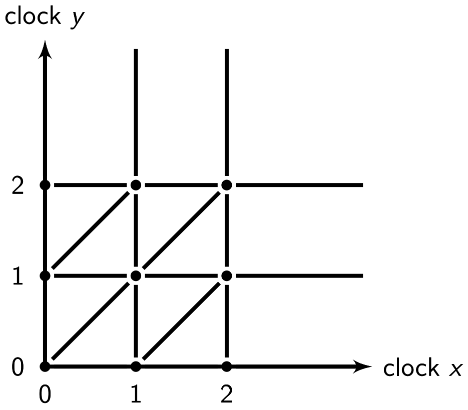

- "compatibility" between regions and constraints
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$\rightsquigarrow$ This is a finite time-abstract bisimulation!

Time-abstract bisimulation

This is a relation between • and • such that:

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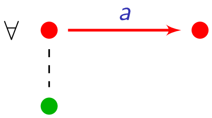

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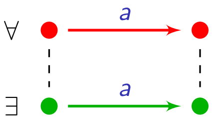

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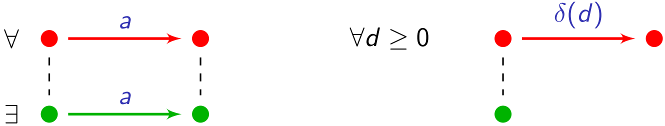

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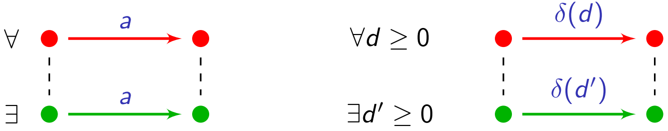


Time-abstract bisimulation

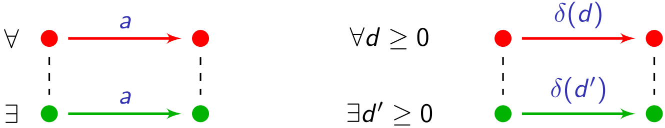
This is a relation between $\bullet$ and $\bullet$ such that:

$$\begin{tikzcd} \forall & \bullet \arrow[r, "a"] \arrow[d, dash, dashed] & \bullet \arrow[d, dash, dashed] \\ \exists & \bullet \arrow[r, "a"] & \bullet \end{tikzcd}
\qquad
\begin{tikzcd} \forall d \geq 0 & \bullet \arrow[r, "\delta(d)"] \arrow[d, dash, dashed] & \bullet \arrow[d, dash, dashed] \\ \exists d' \geq 0 & \bullet \arrow[r, "\delta(d')"] & \bullet \end{tikzcd}$$

... and vice-versa (swap $\bullet$ and $\bullet$).

Time-abstract bisimulation

This is a relation between • and • such that:



... and vice-versa (swap • and •).

Consequence

$$\forall \quad (\ell_1, v_1) \xrightarrow{d_1, a_1} (\ell_2, v_2) \xrightarrow{d_2, a_2} (\ell_3, v_3) \xrightarrow{d_3, a_3} \quad \cdots$$

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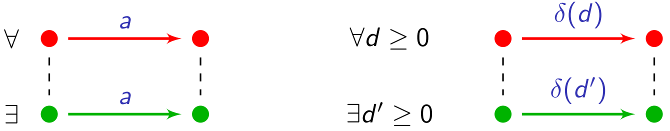


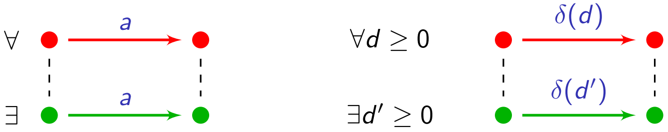
... and vice-versa (swap • and •).

Consequence

$$
\begin{array}{ccccccccc}
\forall & (\ell_1, v_1) & \xrightarrow{d_1, a_1} & (\ell_2, v_2) & \xrightarrow{d_2, a_2} & (\ell_3, v_3) & \xrightarrow{d_3, a_3} & \cdots & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
& (\ell_1, R_1) & \xrightarrow{a_1} & (\ell_2, R_2) & \xrightarrow{a_2} & (\ell_3, R_3) & \xrightarrow{a_3} & \cdots & \text{with } v_i \in R_i
\end{array}
$$

Time-abstract bisimulation

This is a relation between • and • such that:



... and vice-versa (swap • and •).

Consequence

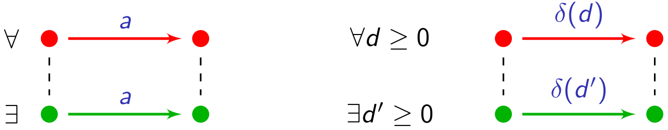
$$\begin{tikzcd}[column sep=small] \forall & (\ell_1, v_1) \arrow[r, "d_1, a_1"] \arrow[d] & (\ell_2, v_2) \arrow[r, "d_2, a_2"] \arrow[d] & (\ell_3, v_3) \arrow[r, "d_3, a_3"] \arrow[d] & \cdots \\ & (\ell_1, R_1) \arrow[r, "a_1"] & (\ell_2, R_2) \arrow[r, "a_2"] & (\ell_3, R_3) \arrow[r, "a_3"] & \cdots \end{tikzcd}$$

with $v_i \in R_i$

$\forall v'_1 \in R_1$

Time-abstract bisimulation

This is a relation between • and • such that:



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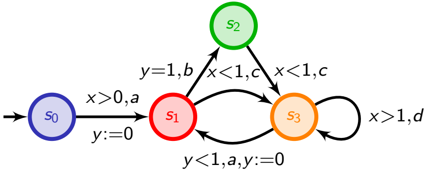
\begin{tikzcd}
\forall & (\ell_1, v_1) \arrow[r, "d_1, a_1"] \arrow[d] & (\ell_2, v_2) \arrow[r, "d_2, a_2"] \arrow[d] & (\ell_3, v_3) \arrow[r, "d_3, a_3"] \arrow[d] & \cdots & \\
 & (\ell_1, R_1) \arrow[r, "a_1"] \arrow[d] & (\ell_2, R_2) \arrow[r, "a_2"] \arrow[d] & (\ell_3, R_3) \arrow[r, "a_3"] \arrow[d] & \cdots & \text{with } v_i \in R_i \\
\forall v_1' \in R_1 \ \exists & (\ell_1, v_1') \arrow[r, "d_1', a_1"] & (\ell_2, v_2') \arrow[r, "d_2', a_2"] & (\ell_3, v_3') \arrow[r, "d_3', a_3"] & \cdots & \text{with } v_i' \in R_i
\end{tikzcd}
$$

Technical tool: Region abstraction – An example [AD94]



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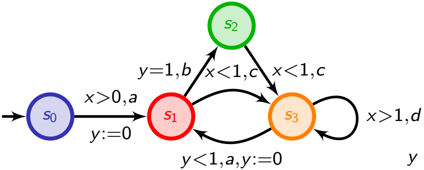

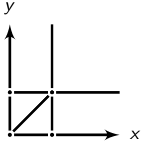

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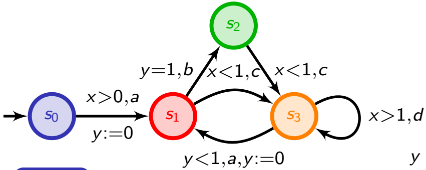

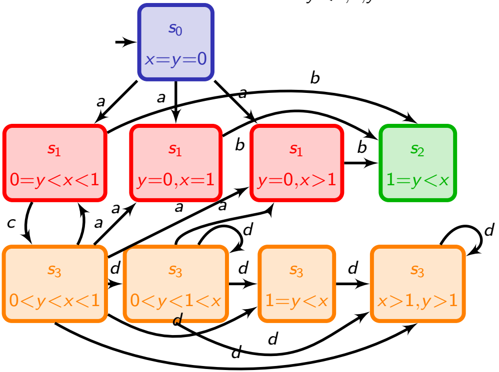

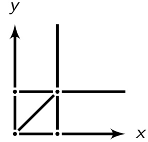

Outline

Modelling resources in timed systems

- System resources might be relevant and even crucial information

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  - energy consumption,
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  - price to pay,
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$\leadsto$ timed automata are not powerful enough!

- A possible solution: use hybrid automata

a discrete control (the mode of the system)
$+$ continuous evolution of the variables within a mode

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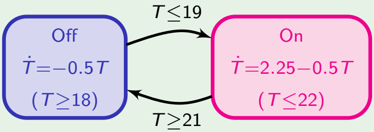
The thermostat example



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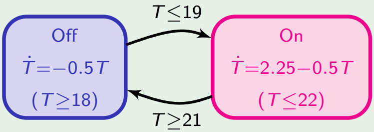
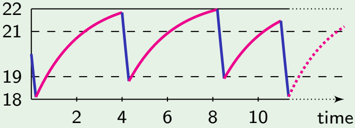
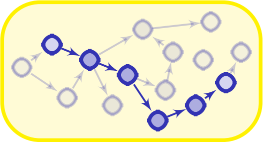




Ok...



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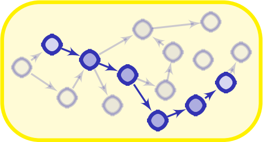
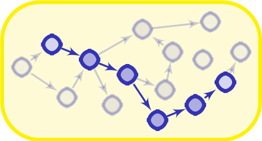


Easy...

Ok...



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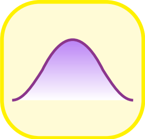

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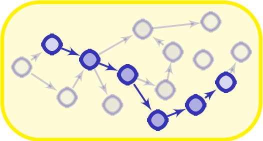

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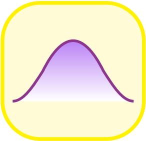

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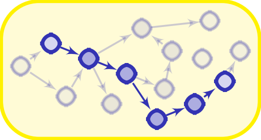
Ok... but?



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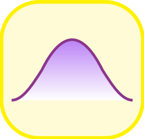

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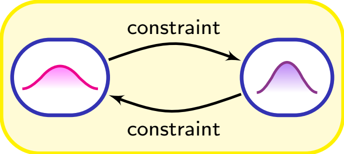

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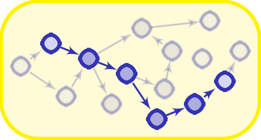

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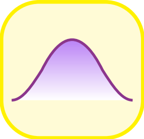

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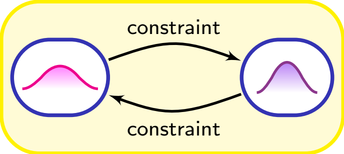

Hard!

Modelling resources in timed systems

- System resources might be relevant and even crucial information
  - energy consumption,
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  $\rightsquigarrow$ timed automata are not powerful enough!

- A possible solution: use hybrid automata

Theorem [HKPV95]

The reachability problem is undecidable in hybrid automata. Even for the simplest, the so-called stopwatch automata (clocks can be stopped).

[HKPV95] Henzinger, Kopke, Puri, Varaiya. What's decidable wbout hybrid automata? *(SToC'95)*.

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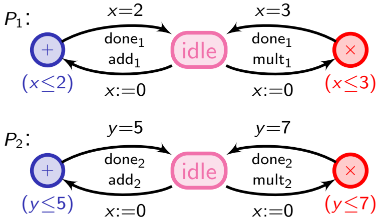
- An alternative: weighted/priced timed automata [ALP01,BFH+01]
  $\rightsquigarrow$ hybrid variables do not constrain the system
  hybrid variables are observer variables

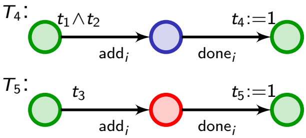
[HKPV95] Henzinger, Kopke, Puri, Varaiya. What's decidable wbout hybrid automata? *(SToC'95)*.
[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.

Modelling the task graph scheduling problem

- Processors



- Tasks



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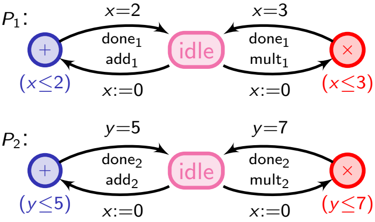

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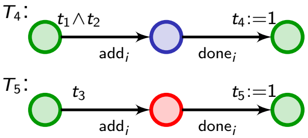
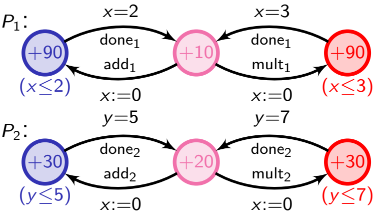
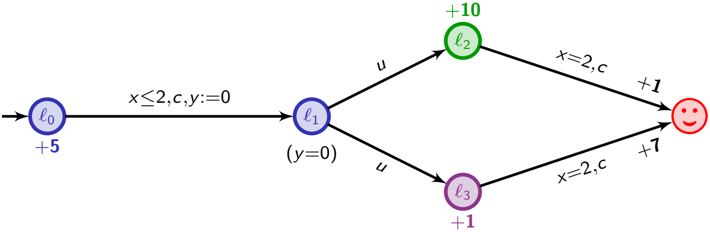


- Modelling energy



A good schedule is a path in the product automaton with a low cost

Weighted/priced timed automata [ALP01,BFH+01]



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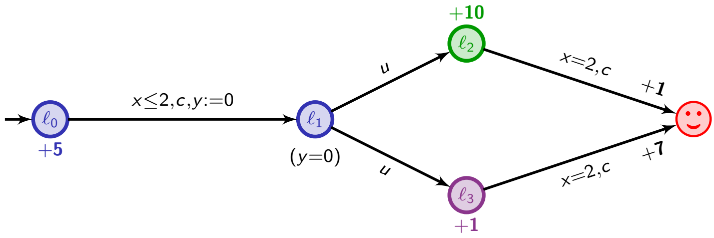

$$\begin{array}{ccccccccccccc}
 & \ell_0 & \xrightarrow{1.3} & \ell_0 & \xrightarrow{c} & \ell_1 & \xrightarrow{u} & \ell_3 & \xrightarrow{0.7} & \ell_3 & \xrightarrow{c} & ☺ \\
x & 0 & & 1.3 & & 1.3 & & 1.3 & & 2 & & \\
y & 0 & & 1.3 & & 0 & & 0 & & 0.7 & &
\end{array}$$

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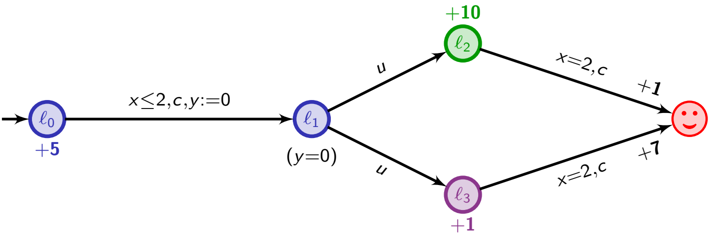

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost :

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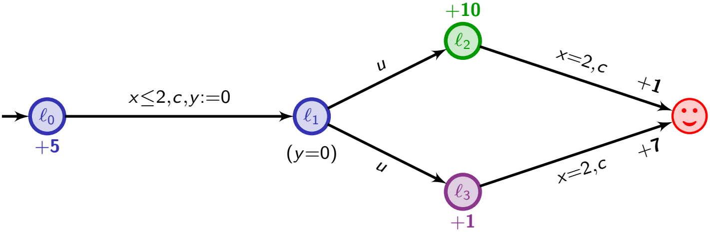

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5

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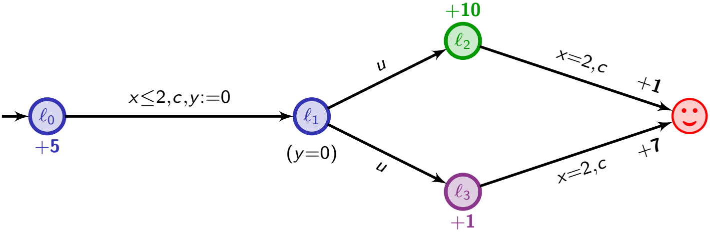

$$
\begin{array}{lccccccccccc}
 & \ell_0 & \xrightarrow{1.3} & \ell_0 & \xrightarrow{c} & \ell_1 & \xrightarrow{u} & \ell_3 & \xrightarrow{0.7} & \ell_3 & \xrightarrow{c} & ☺ \\
x & 0 & & 1.3 & & 1.3 & & 1.3 & & 2 & & \\
y & 0 & & 1.3 & & 0 & & 0 & & 0.7 & &
\end{array}
$$

cost : 6.5 + 0

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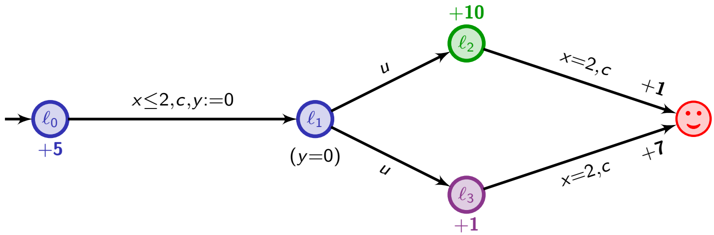

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x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0

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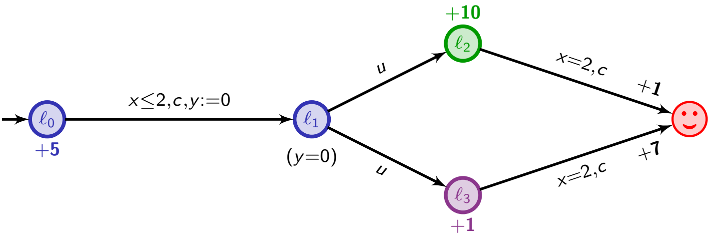

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
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cost : 6.5 + 0 + 0 + 0.7

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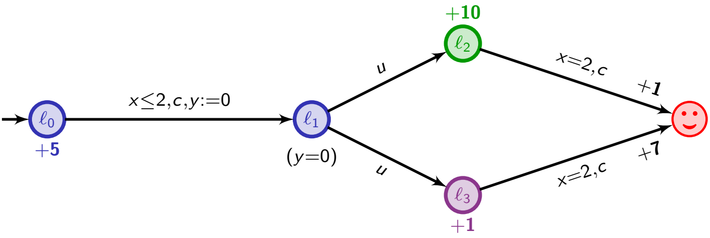

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0 + 0.7 + 7

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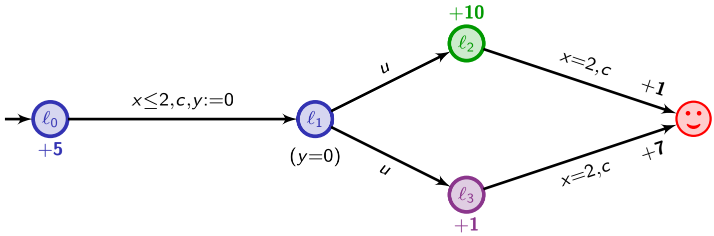

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺	
x	0		1.3		1.3		1.3		2			
y	0		1.3		0		0		0.7			
cost :		6.5	+	0	+	0	+	0.7	+	7	=	14.2

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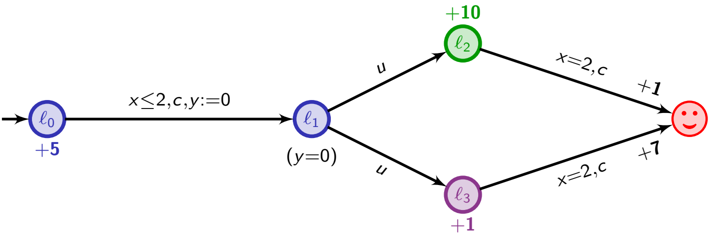
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.

Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching 🙂?

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
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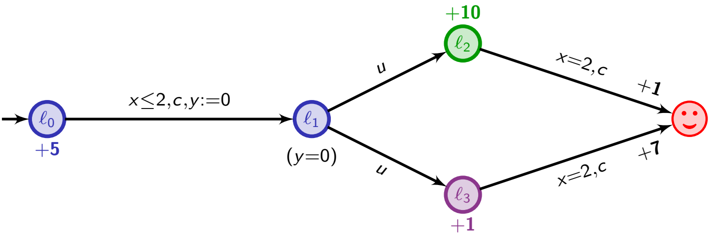

Question: what is the optimal cost for reaching 🙂 ?

$$5t + 10(2 - t) + 1$$

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
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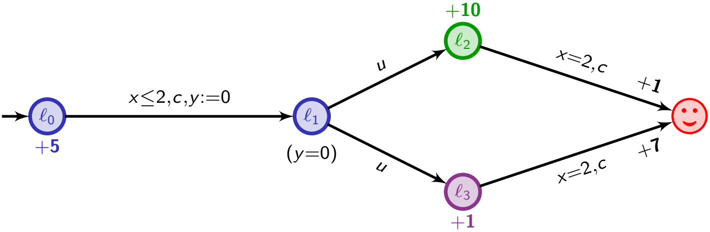
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching ☺ ?

$$5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7$$

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
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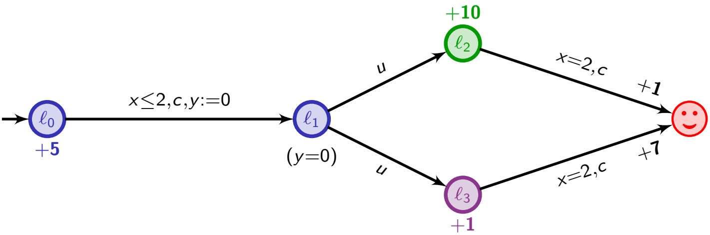
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching ☺?

$$\min\left(5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7 \right)$$

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
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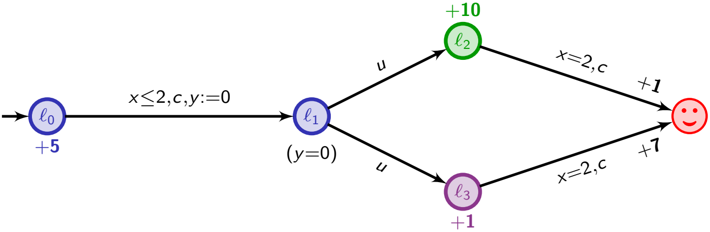
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching 🙂?

$$\inf_{0 \leq t \leq 2} \min \left(5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7 \right) = 9$$

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
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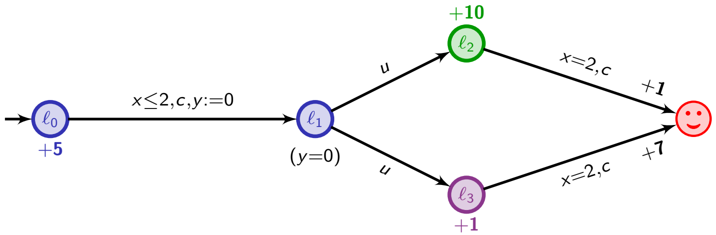
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching ☺ ?

$$\inf_{0\leq t\leq 2} \min\left(\, 5t+10(2-t)+1 \,,\, 5t+(2-t)+7 \,\right) = 9$$

$\rightsquigarrow$ *strategy:* leave immediately ℓ_0, go to ℓ_3, and wait there 2 t.u.

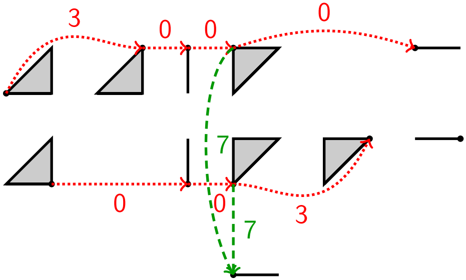
[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.

Optimal-cost reachability

Theorem [ALP01,BFH+01,BBBR07]

In weighted timed automata, the optimal cost is an integer and can be computed in PSPACE.

- Technical tool: a refinement of the regions, the corner-point abstraction



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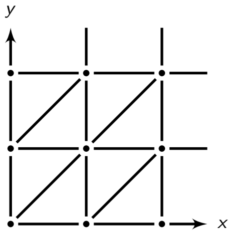
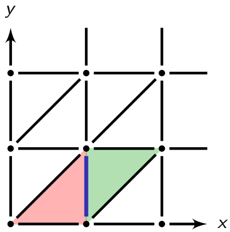
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.
[BBBR07] Bouyer, Brihaye, Bruyère, Raskin. On the optimal reachability problem *(Formal Methods in System Design)*.

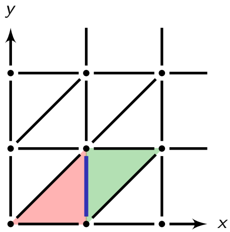
Technical tool: the corner-point abstraction



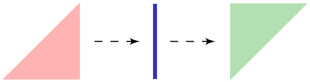
Technical tool: the corner-point abstraction



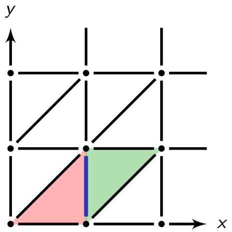
Technical tool: the corner-point abstraction



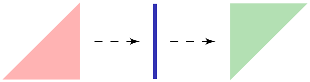
Abstract time successors:



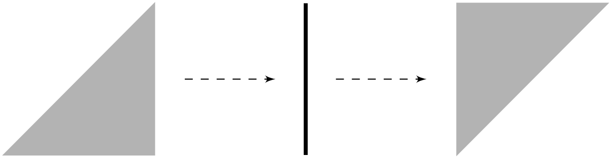
Technical tool: the corner-point abstraction



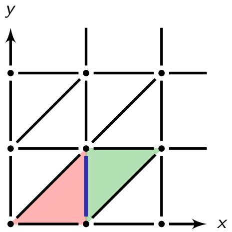
Abstract time successors:



Concrete time successors:



Technical tool: the corner-point abstraction



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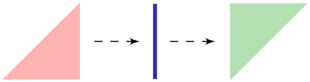

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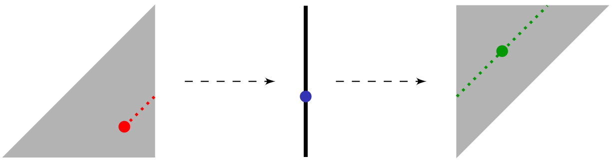
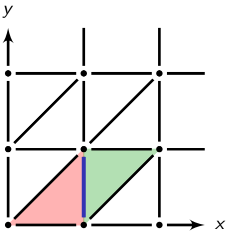


Technical tool: the corner-point abstraction



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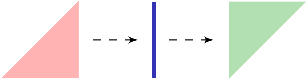

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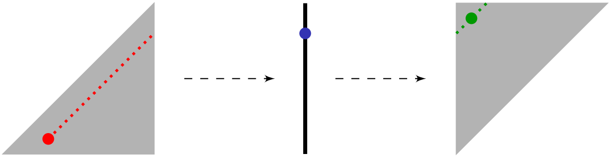

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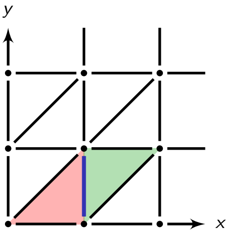

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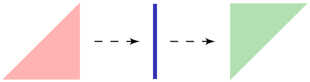

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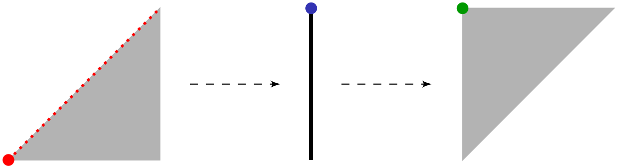
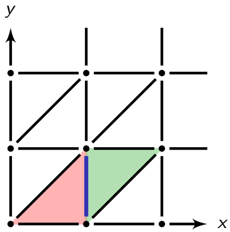


Technical tool: the corner-point abstraction



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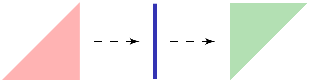
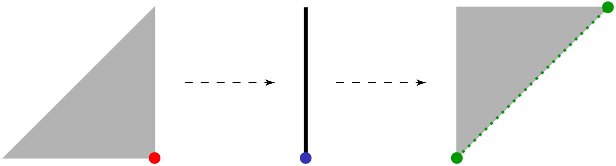


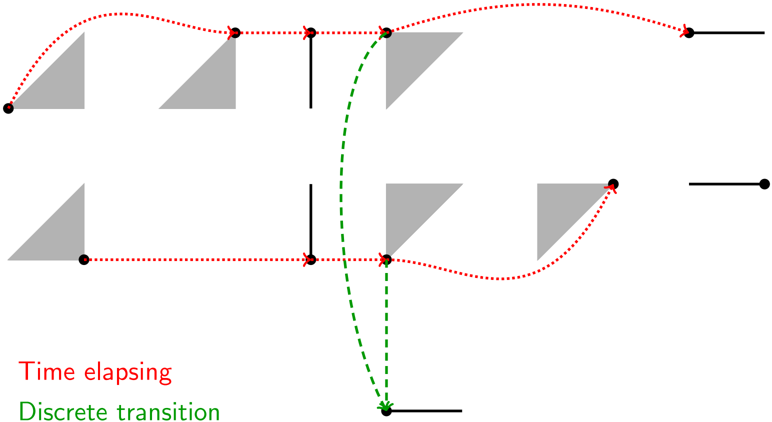
Concrete time successors:



Technical tool: the corner-point abstraction

Idea

only record information about the extremal points



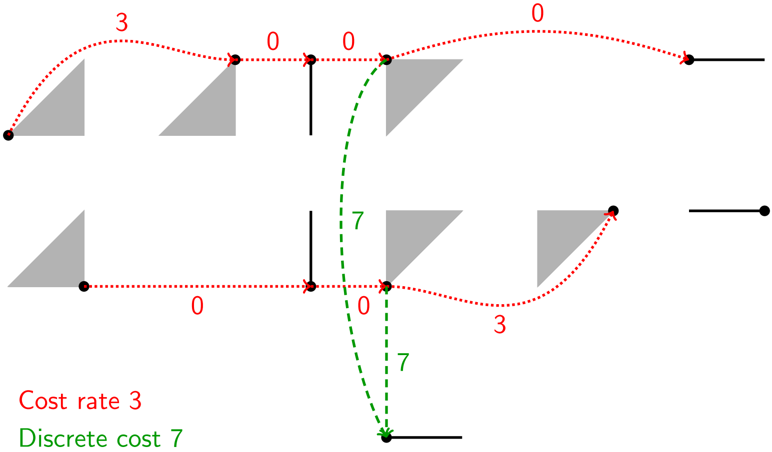
Time elapsing

Discrete transition

Technical tool: the corner-point abstraction

Idea

only record information about the extremal points



From timed to discrete behaviours

Optimal reachability as a linear programming problem

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow{t_1} \circ \xrightarrow{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \left\{ \begin{array}{l} t_1+t_2\leq 2 \\ \\ \end{array} \right.$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \begin{cases} t_1+t_2\leq 2 \\ t_2+t_3+t_4\geq 5 \end{cases}$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\begin{array}{cccccc} T_1 & T_2 & T_3 & T_4 & T_5 & \\ \circ \xrightarrow[y:=0]{t_1} & \circ \xrightarrow[x\le 2]{t_2} & \circ \xrightarrow{t_3} & \circ \xrightarrow[y\ge 5]{t_4} & \circ \xrightarrow{t_5} & \circ \end{array} \quad \cdots \quad \begin{cases} t_1+t_2\le 2 & T_2\le 2 \\ t_2+t_3+t_4\ge 5 & T_4-T_1\ge 5 \end{cases}$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\begin{array}{cccccc} & T_1 & T_2 & T_3 & T_4 & T_5 \end{array}$$

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \begin{cases} t_1+t_2\leq 2 & T_2\leq 2 \\ t_2+t_3+t_4\geq 5 & T_4-T_1\geq 5 \end{cases}$$

Lemma

Let Z be a bounded zone and f be a function

$$f : (T_1, ..., T_n) \mapsto \sum_{i=1}^{n} c_i T_i + c$$

well-defined on $\overline{Z}$. Then $inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\begin{array}{cccccc} & T_1 & T_2 & T_3 & T_4 & T_5 \end{array}$$

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \begin{cases} t_1+t_2\leq 2 & T_2\leq 2 \\ t_2+t_3+t_4\geq 5 & T_4-T_1\geq 5 \end{cases}$$

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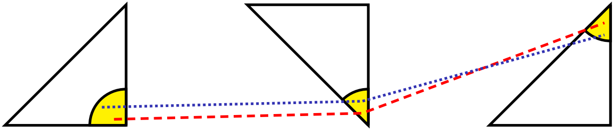
$\rightsquigarrow$ for every finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{\text{cp}}$ such that

$$\text{cost}(\Pi) \leq \text{cost}(\pi)$$

[Π is a "corner-point projection" of π]

From discrete to timed behaviours

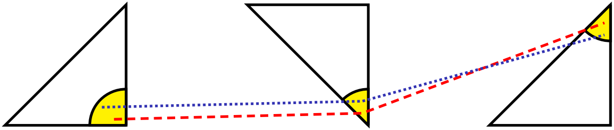
Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$,

From discrete to timed behaviours

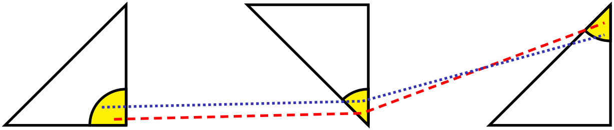
Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$,

From discrete to timed behaviours

Approximation of abstract paths:

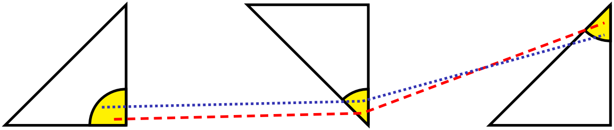


For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

From discrete to timed behaviours

Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

For every $\eta > 0$, there exists $\varepsilon > 0$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon \Rightarrow |\mathrm{cost}(\Pi) - \mathrm{cost}(\pi_\varepsilon)| < \eta$$

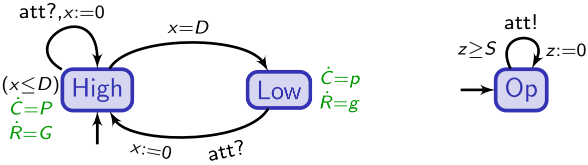
Use of the corner-point abstraction

It is a very interesting abstraction, that can be used in several other contexts:

- for mean-cost optimization [BBL04,BBL08]
- for discounted-cost optimization [FL08]
- for all concavely-priced timed automata [JT08]
- for deciding frequency objectives [BBBS11,Sta12]
- ...

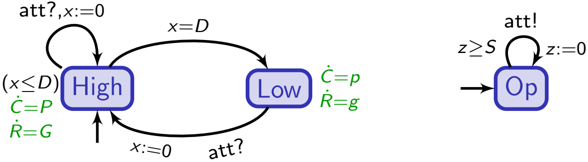
[BBL04] Bouyer, Brinksma, Larsen. Staying Alive As Cheaply As Possible *(HSCC'04)*.
[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.
[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.
[JT08] Judziński, Trivedi. Concavely-priced timed automata *(FORMATS'08)*.
[BBBS11] Bertrand, Bouyer, Brihaye, Stainer. Emptiness and universality problems in timed automata with positive frequency *(ICALP'11)*.
[Sta12] Stainer. Frequencies in forgetful timed automata *(FORMATS'12)*.

Going further 1: mean-cost optimization



[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.

Going further 1: mean-cost optimization

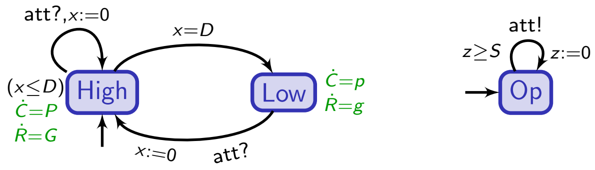


$\rightsquigarrow$ compute optimal infinite schedules that minimize

$$\text{mean-cost}(\pi) = \limsup_{n \to +\infty} \frac{\text{cost}(\pi_n)}{\text{reward}(\pi_n)}$$

[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.

Going further 1: mean-cost optimization



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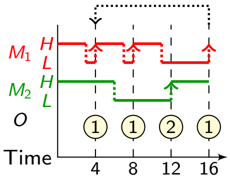
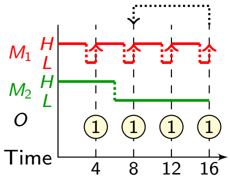


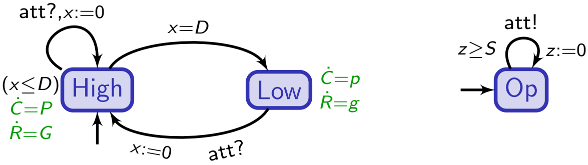
Schedule with ratio ≈1.455



Schedule with ratio ≈1.478

[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.

Going further 1: mean-cost optimization



$\rightsquigarrow$ compute optimal infinite schedules that minimize

$$\text{mean-cost}(\pi) = \limsup_{n \to +\infty} \frac{\text{cost}(\pi_n)}{\text{reward}(\pi_n)}$$

Theorem [BBL08]

In weighted timed automata, the optimal mean-cost can be compute in PSPACE.

$\rightsquigarrow$ the corner-point abstraction can be used

[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.

From timed to discrete behaviours

- **Finite behaviours:** based on the following property

Lemma

Let Z be a bounded zone and f be a function

$$f : (t_1, \ldots, t_n) \mapsto \frac{\sum_{i=1}^{n} c_i t_i + c}{\sum_{i=1}^{n} r_i t_i + r}$$

well-defined on $\overline{Z}$. Then $inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

From timed to discrete behaviours

- **Finite behaviours:** based on the following property

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$\rightsquigarrow$ for every finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{\mathsf{cp}}$ s.t.
mean-cost$(\Pi) \leq$ mean-cost(π)

From timed to discrete behaviours

- **Finite behaviours:** based on the following property

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Let Z be a bounded zone and f be a function

$$f : (t_1, \ldots, t_n) \mapsto \frac{\sum_{i=1}^{n} c_i t_i + c}{\sum_{i=1}^{n} r_i t_i + r}$$

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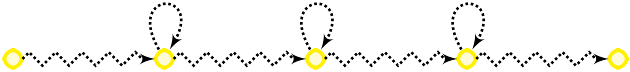
$\rightsquigarrow$ for every finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{\text{cp}}$ s.t.

$$\text{mean-cost}(\Pi) \leq \text{mean-cost}(\pi)$$

- **Infinite behaviours:** decompose each sufficiently long projection into cycles:



The (acyclic) linear part will be negligible!

From timed to discrete behaviours

- **Finite behaviours:** based on the following property

Lemma

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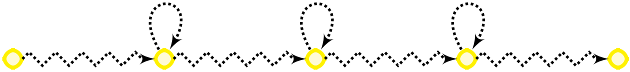
well-defined on $\overline{Z}$. Then $inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

$\rightsquigarrow$ for every finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{\text{cp}}$ s.t.

$$\text{mean-cost}(\Pi) \leq \text{mean-cost}(\pi)$$

- **Infinite behaviours:** decompose each sufficiently long projection into cycles:

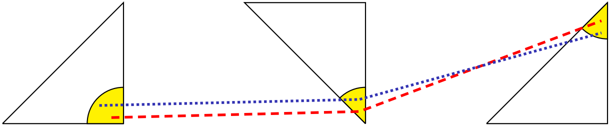


The (acyclic) linear part will be negligible!

$\rightsquigarrow$ the optimal cycle of $\mathcal{A}_{\text{cp}}$ is better than any infinite path of $\mathcal{A}$!

From discrete to timed behaviours

Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$,

From discrete to timed behaviours

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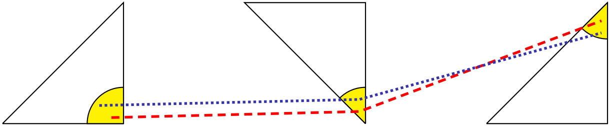

For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$,

From discrete to timed behaviours

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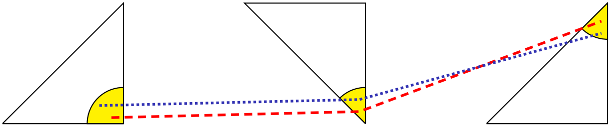
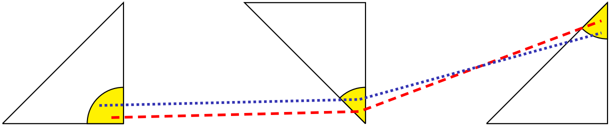


For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

From discrete to timed behaviours

Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

For every $\eta > 0$, there exists $\varepsilon > 0$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon \Rightarrow |\text{mean-cost}(\Pi) - \text{mean-cost}(\pi_\varepsilon)| < \eta$$

Going further 2: concavely-priced cost functions

$\rightsquigarrow$ A general abstract framework for quantitative timed systems

Theorem [JT08]

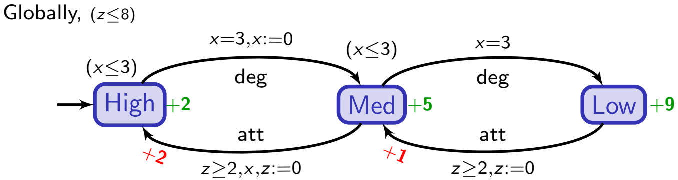
In concavely-priced timed automata, optimal cost is computable, if we restrict to quasi-concave cost functions. For the following cost functions, the (decision) problem is even PSPACE-complete:

- optimal-time and optimal-cost reachability;
- optimal discrete discounted cost;
- optimal mean-cost.

$\rightsquigarrow$ the corner-point abstraction can be used

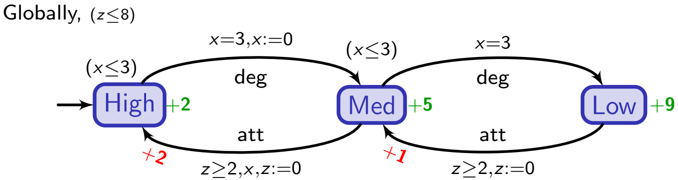
[JT08] Judziński, Trivedi. Concavely-priced timed automata *(FORMATS'08)*.

Going further 3: discounted-time cost optimization



[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.

Going further 3: discounted-time cost optimization



$\rightsquigarrow$ compute optimal infinite schedules that minimize
discounted cost over time

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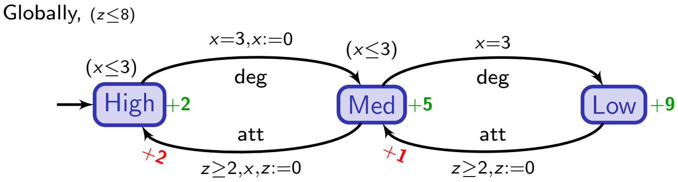
Going further 3: discounted-time cost optimization



$\rightsquigarrow$ compute optimal infinite schedules that minimize

$$\text{discounted-cost}_{\lambda}(\pi) = \sum_{n\geq 0} \lambda^{T_n} \int_{t=0}^{\tau_{n+1}} \lambda^t \text{cost}(\ell_n)\, \mathrm{d}t + \lambda^{T_{n+1}} \text{cost}(\ell_n \xrightarrow{a_{n+1}} \ell_{n+1})$$

if $\pi = (\ell_0, v_0) \xrightarrow{\tau_1, a_1} (\ell_1, v_1) \xrightarrow{\tau_2, a_2} \cdots$ and $T_n = \sum_{i\leq n} \tau_i$

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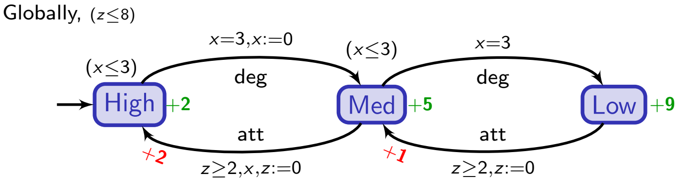
Going further 3: discounted-time cost optimization



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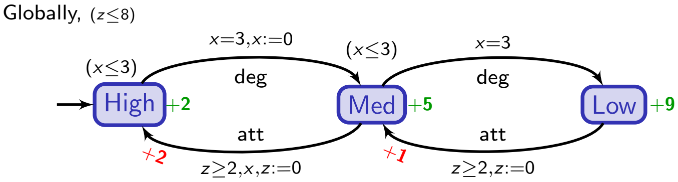

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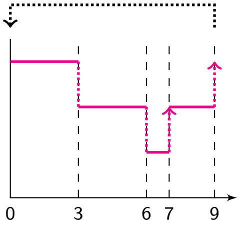
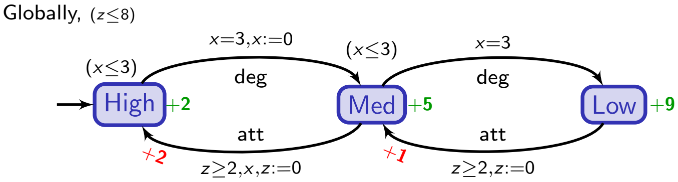


if $\lambda = e^{-1}$, the discounted cost of that infinite schedule is ≈ 2.16

[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.

Going further 3: discounted-time cost optimization



$\rightsquigarrow$ compute optimal infinite schedules that minimize
discounted cost over time

Theorem [FL08]

In weighted timed automata, the optimal discounted cost is computable in EXPTIME.

$\rightsquigarrow$ the corner-point abstraction can be used

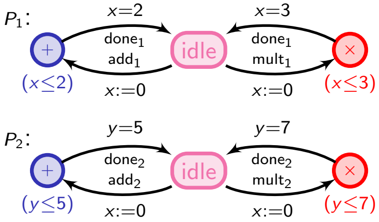
[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.

Outline

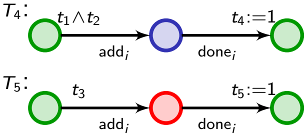
1. Timed automata
2. Weighted timed automata
3. Timed games
4. Weighted timed games
5. Conclusion

Modelling the task graph scheduling problem

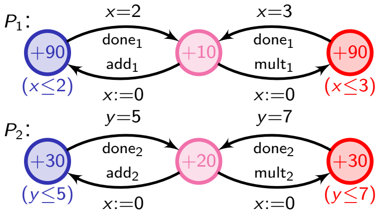
- Processors



- Tasks



- Modelling energy



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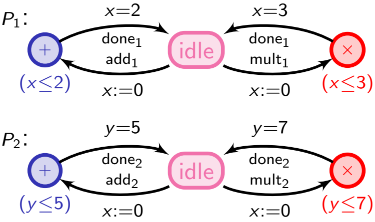

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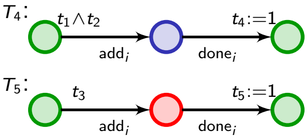

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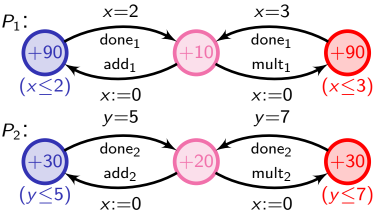
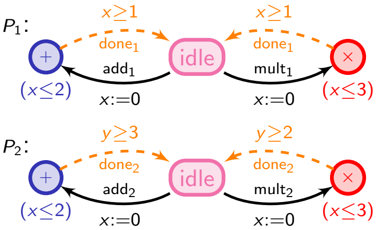


- Modelling uncertainty



Modelling the task graph scheduling problem

- Processors

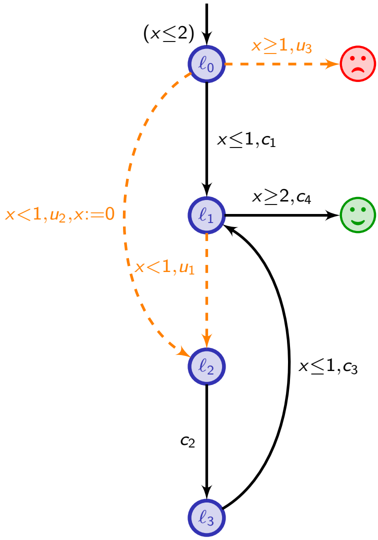
- Tasks

A (good) schedule is a strategy in the product game (with a low cost)

- Modelling energy

- Modelling uncertainty

An example of a timed game



Rule of the game

- Aim: avoid ☹ and reach ☺

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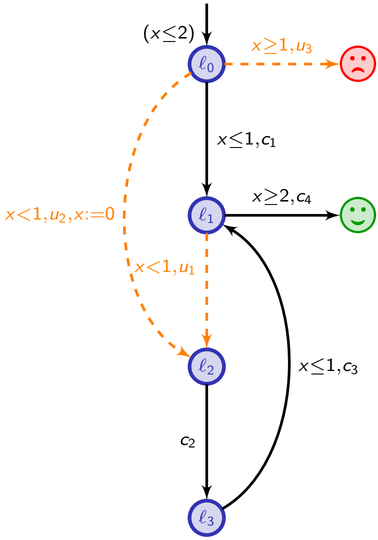

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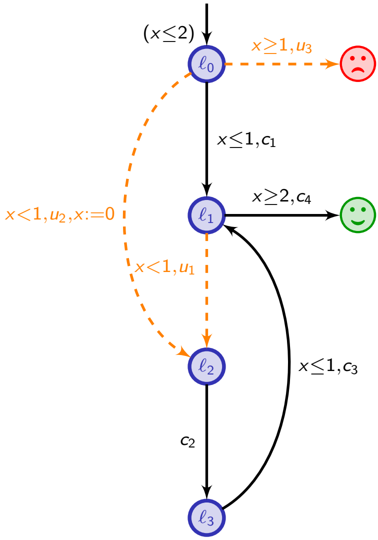

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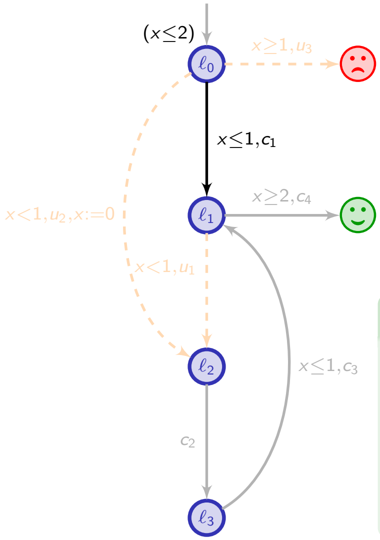

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- from $(\ell_0, 0)$, play $(0.5, c_1)$

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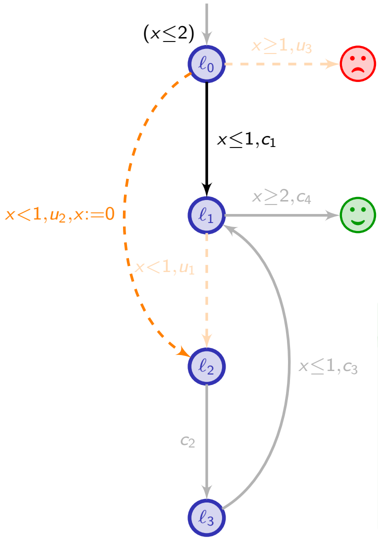

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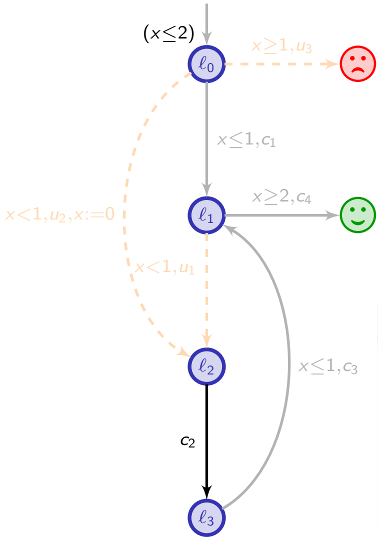

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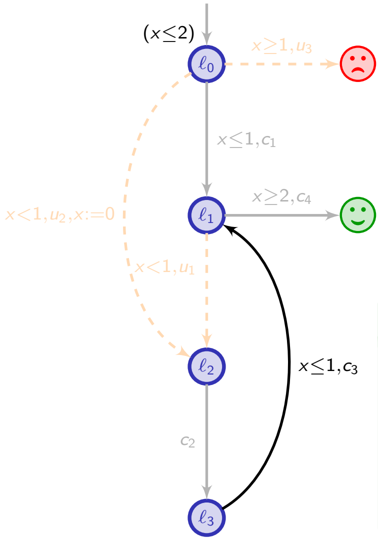

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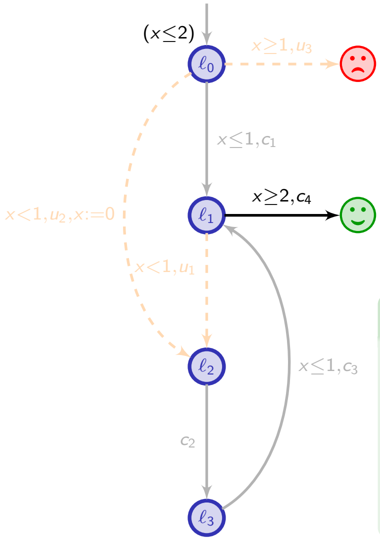

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- from $(\ell_3, 1)$, play $(0, c_3)$
- from $(\ell_1, 1)$, play $(1, c_4)$

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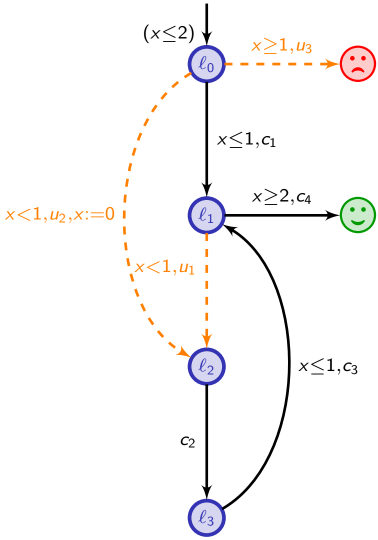

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Problems to be considered

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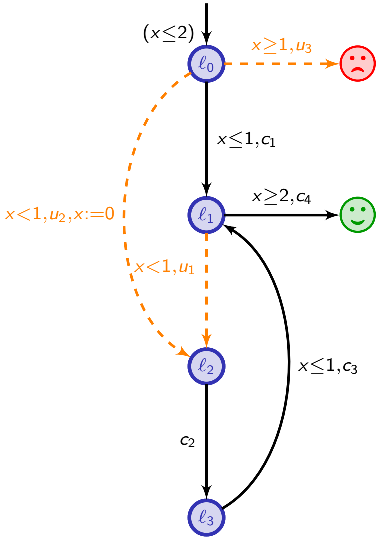

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Problems to be considered

- Does there exist a winning strategy?

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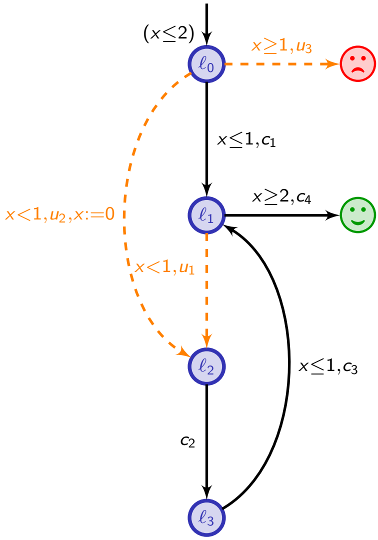

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Problems to be considered

- Does there exist a winning strategy?
- If yes, compute one (as simple as possible).

Decidability of timed games

Theorem [AMPS98,HK99]

Reachability and safety timed games are decidable and EXPTIME-complete. Furthermore memoryless and "region-based" strategies are sufficient.

[AMPS98] Asarin, Maler, Pnueli, Sifakis. Controller synthesis for timed automata *(SSC'98)*.
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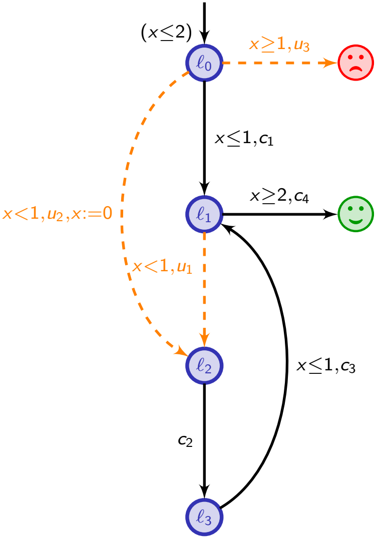
Theorem [AM99,BHPR07,JT07]

Optimal-time reachability timed games are decidable and EXPTIME-complete.

[AM99] Asarin, Maler. As soon as possible: time optimal control for timed automata *(HSCC'99)*.
[BHPR07] Brihaye, Henzinger, Prabhu, Raskin. Minimum-time reachability in timed games *(ICALP'07)*.
[JT07] Jurdziński, Trivedi. Reachability-time games on timed automata *(ICALP'07)*.

Back to the example: computing winning states



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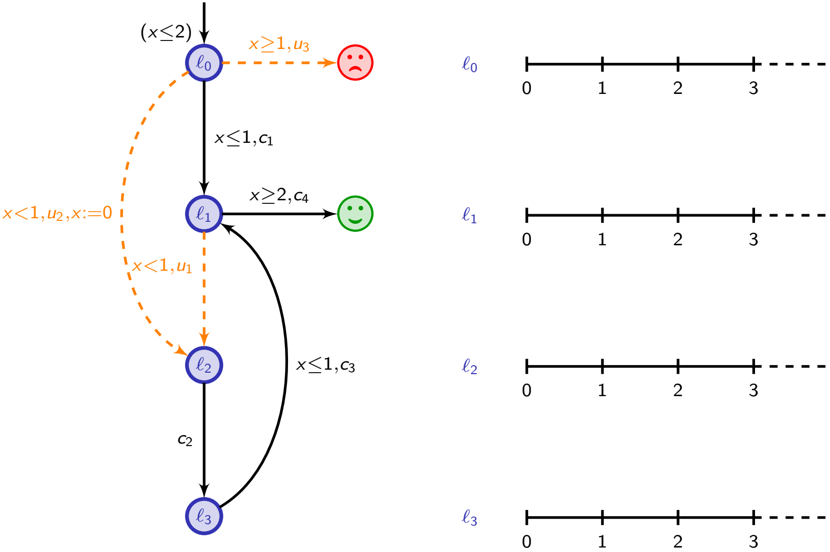

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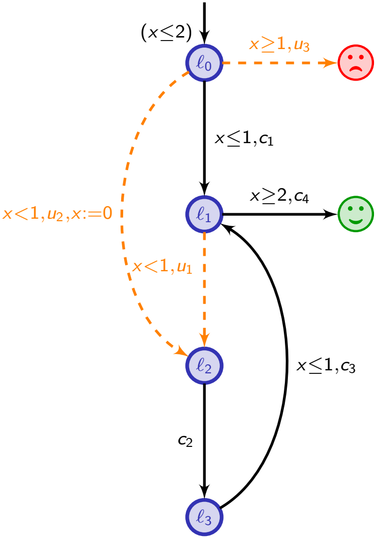

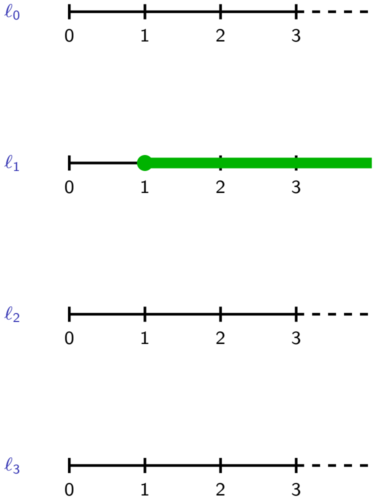

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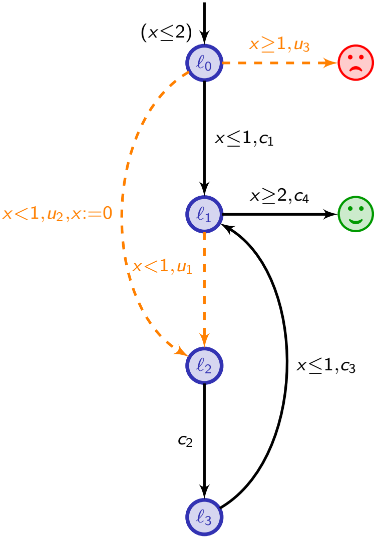

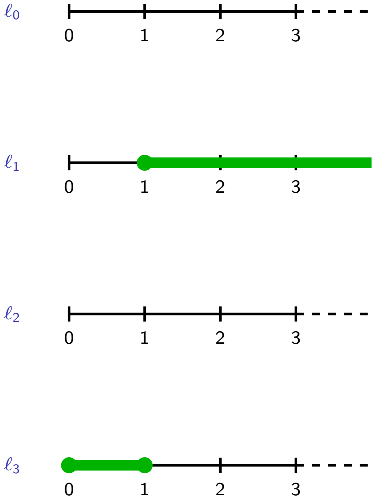

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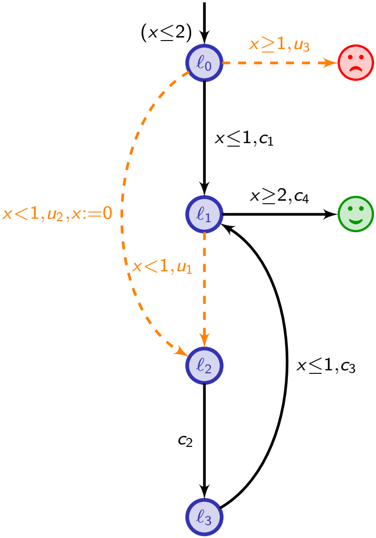

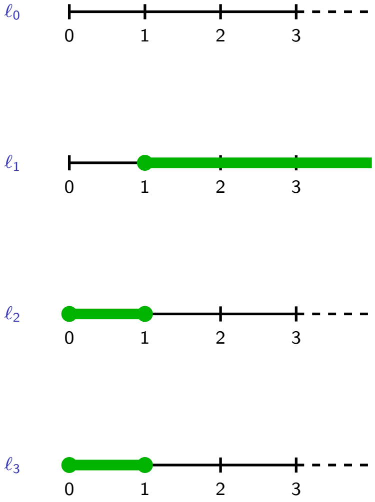

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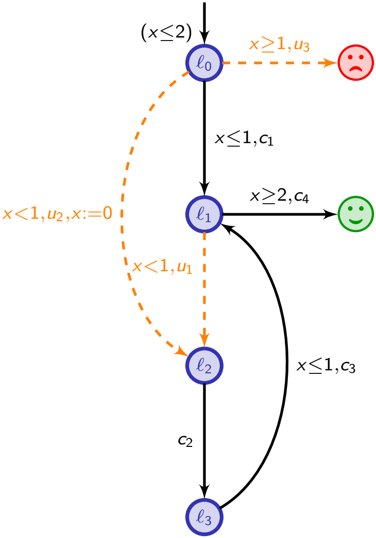

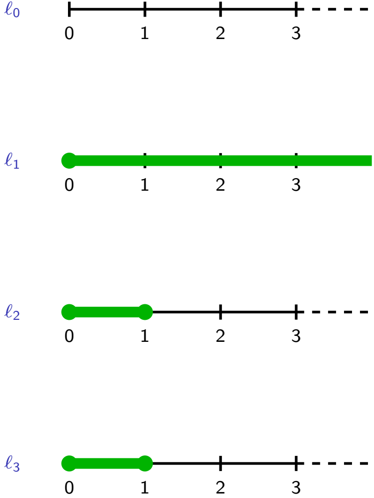

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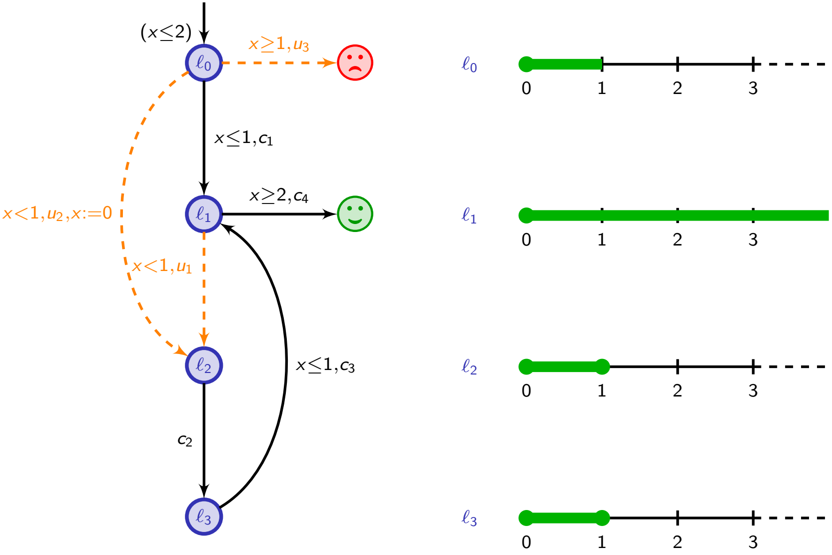

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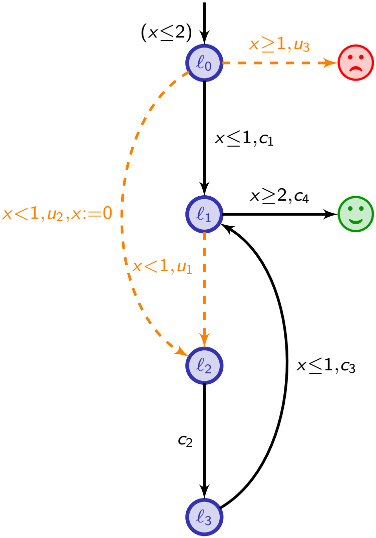

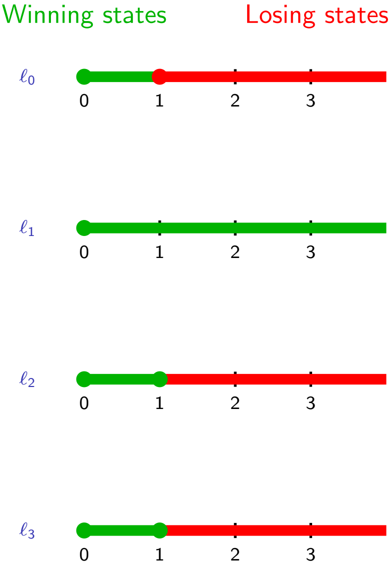

Decidability *via* attractors

Skip attractors

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Decidability *via* attractors

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$$\text{cPred}(X) = \bigcup_{a \text{ cont.}} \text{Pred}^a(X) \qquad \text{uPred}(X) = \bigcup_{a \text{ uncont.}} \text{Pred}^a(X)$$

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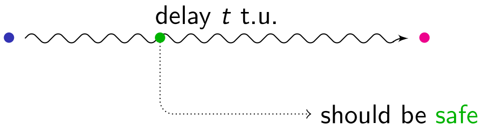

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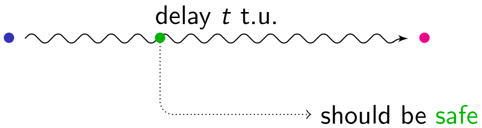

$$\mathrm{Pred}_\delta(X, \mathrm{Safe}) = \{\bullet \mid \exists t \geq 0,\ \bullet \xrightarrow{\delta(t)} \bullet$$
$$\text{and } \forall 0 \leq t' \leq t,\ \bullet \xrightarrow{\delta(t')} \bullet \in \mathrm{Safe}\}$$

Timed games with a reachability objective

We write:

$$\pi(X) = X \cup \mathsf{Pred}_\delta(\mathsf{cPred}(X), \neg\mathsf{uPred}(\neg X))$$

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- The states from which one can ensure ☺ in no more than 1 step is:

$$\mathsf{Attr}_1(☺) = \pi(☺)$$

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$$\begin{array}{rcl} \mathsf{Attr}_n(\smiley) & = & \pi(\mathsf{Attr}_{n-1}(\smiley)) \\ & = & \pi^n(\smiley) \end{array}$$

Stability w.r.t. regions

- if X is a union of regions, then:
  - $\mathsf{Pred}_a(X)$ is a union of regions,
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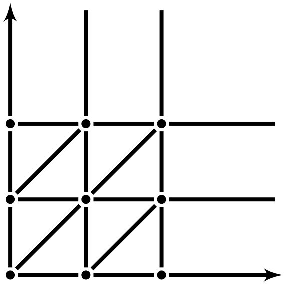

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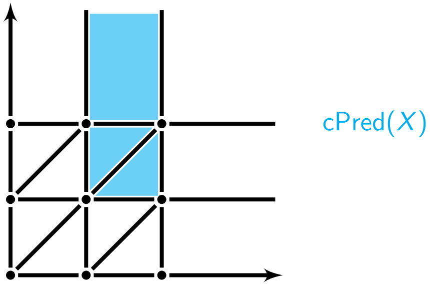

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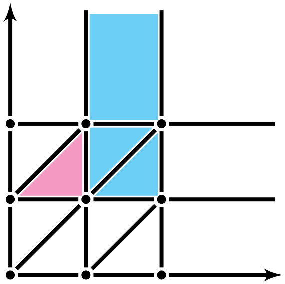

cPred(X)
uPred($\neg X$)

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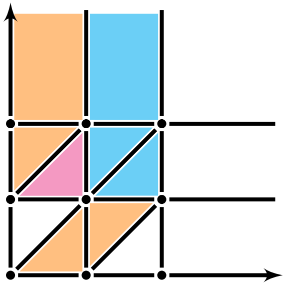

$\mathsf{cPred}(X)$

$\mathsf{uPred}(\neg X)$

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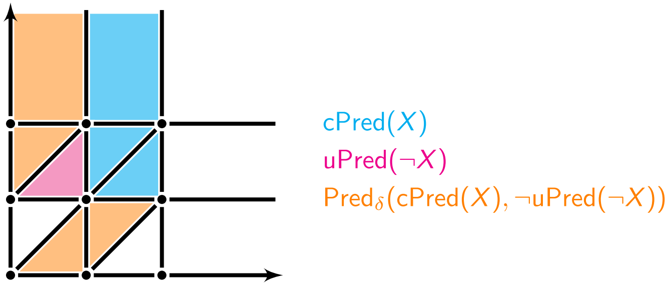

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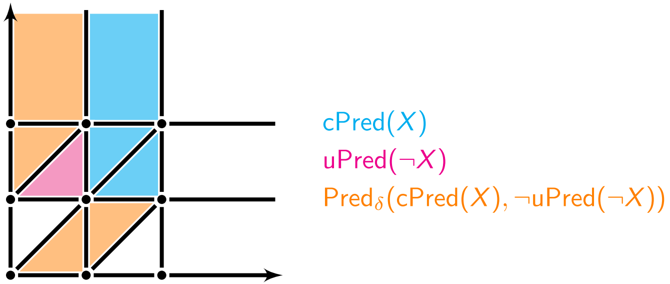

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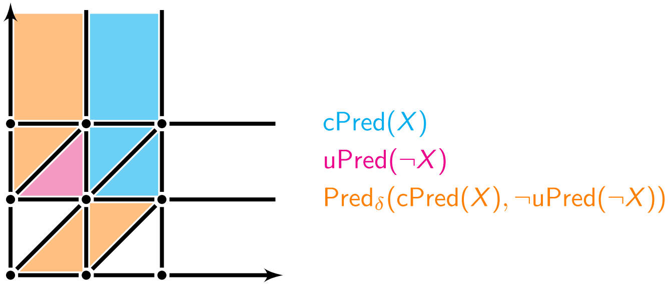

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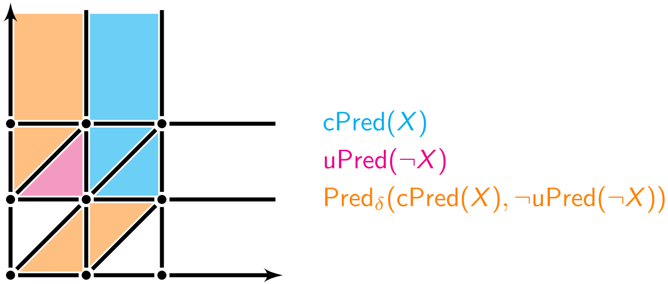

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... and is correct

Timed games with a safety objective

- We can use operator $\widetilde{\pi}$ defined by

$$\widetilde{\pi}(X) = \mathsf{Pred}_\delta(X \cap \mathsf{cPred}(X), \neg\mathsf{uPred}(\neg X))$$

instead of π, and compute $\widetilde{\pi}^*(\neg\text{☹})$

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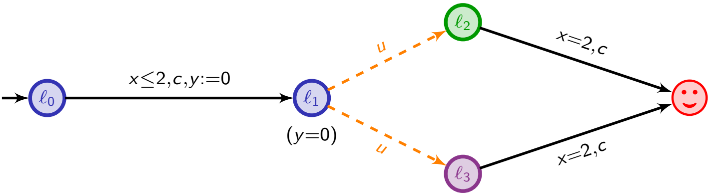
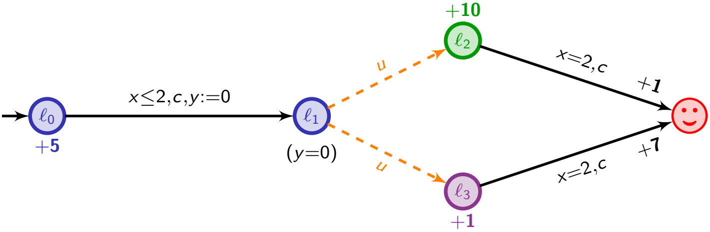
- It is also stable w.r.t. regions.

Outline

A simple timed game



A simple weighted timed game



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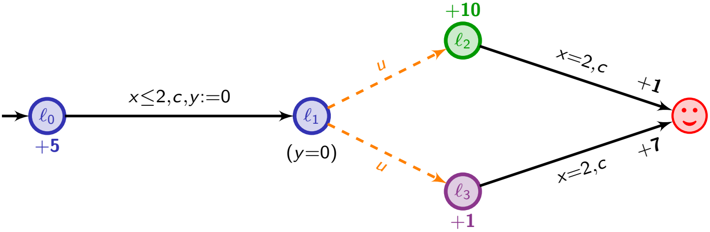

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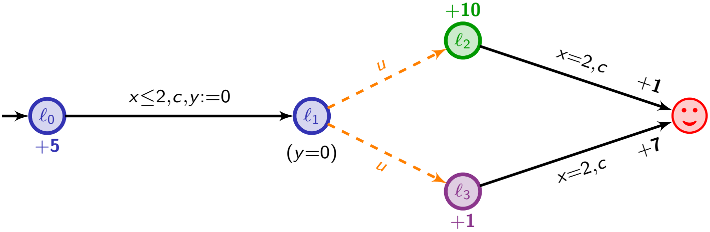
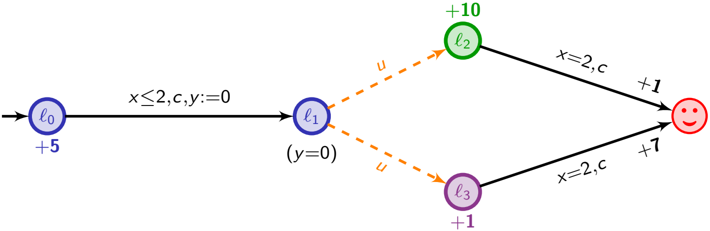


Question: what is the optimal cost we can ensure while reaching 🙂 ?

$$5t + 10(2-t) + 1$$

A simple weighted timed game



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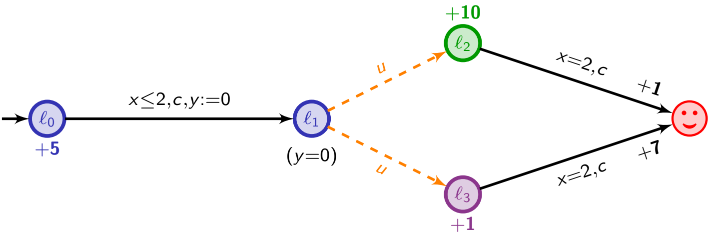
$$5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7$$

A simple weighted timed game



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$$\max\ (\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ )$$

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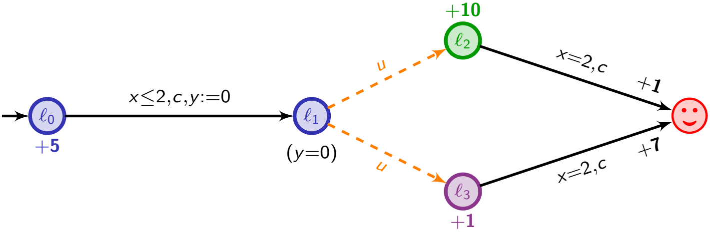

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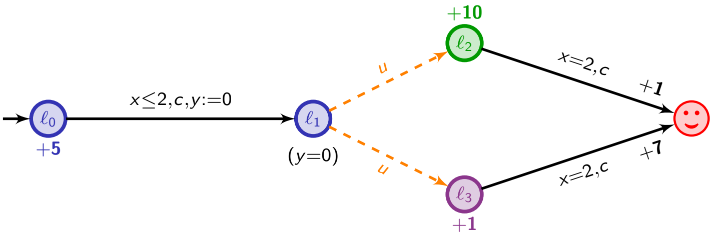
$$\inf_{0\leq t\leq 2} \max\left(\, 5t+10(2-t)+1 \,,\, 5t+(2-t)+7 \,\right) = 14+\frac{1}{3}$$

A simple weighted timed game



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$$\inf_{0\leq t\leq 2} \max\left(\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ \right) = 14 + \frac{1}{3}$$

$\rightsquigarrow$ *strategy:* wait in ℓ_0, and when $t = \frac{4}{3}$, go to ℓ_1

Optimal reachability in weighted timed games (1)

This topic has been fairly hot these last fifteen years...

[LMM02,ABM04,BCFL04,BBR05,BBM06,BLMR06,Rut11,HIM13,BGK+14]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.
[ABM04] Alur, Bernardsky, Madhusudan. Optimal reachability in weighted timed games *(ICALP'04)*.
[BCFL04] Bouyer, Cassez, Fleury, Larsen. Optimal strategies in priced timed game automata *(FSTTCS'04)*.
[BBR05] Brihaye, Bruyère, Raskin. On optimal timed strategies *(FORMATS'05)*.
[BBM06] Bouyer, Brihaye, Markey. Improved undecidability results on weighted timed automata *(Information Processing Letters)*.
[BLMR06] Bouyer, Larsen, Markey, Rasmussen. Almost-optimal strategies in one-clock priced timed automata *(FSTTCS'06)*.
[Rut11] Rutkowski. Two-player reachability-price games on single-clock timed automata *(QAPL'11)*.
[HIM13] Hansen, Ibsen-Jensen, Miltersen. A faster algorithm for solving one-clock priced timed games *(CONCUR'13)*.
[BGK+14] Brihaye, Geeraerts, Krishna, Manasa, Monmege, Trivedi. Adding Negative Prices to Priced Timed Games *(CONCUR'14)*.

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Tree-like weighted timed games can be solved in 2EXPTIME.

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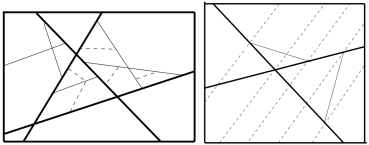
[LMM02]

Tree-like weighted timed games can be solved in 2EXPTIME.

[ABM04,BCFL04]

Depth-k weighted timed games can be solved in EXPTIME. There is a symbolic algorithm to solve weighted timed games **with a strongly non-Zeno cost**.



Optimal reachability in weighted timed games (2)

[BBR05,BBM06]

In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more.

Optimal reachability in weighted timed games (2)

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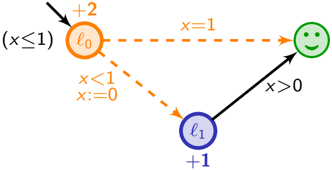
In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more.

[BLMR06,Rut11,HIM13,BGK+14]

Turn-based optimal timed games are decidable in EXPTIME (resp. PTIME) when automata have a single clock (resp. with two rates). They are PTIME-hard.

What is easier with a single clock?

- Memoryless strategies can be non-optimal...



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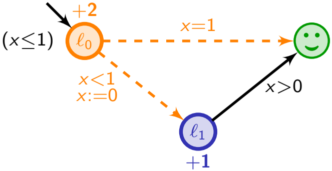

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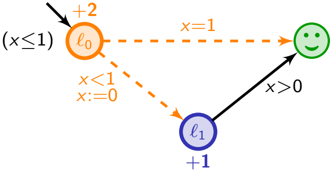

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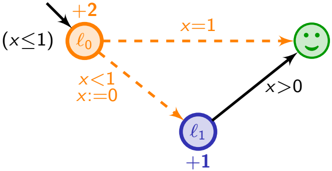

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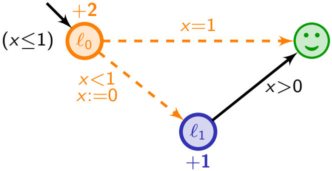

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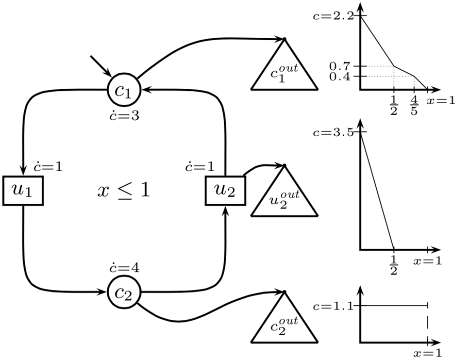
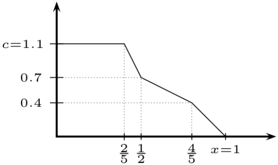
- Rather involved proofs of correctness





$$\sigma(c_2, x) = \begin{cases} c_2^{out} & \text{if } 0 \leq x < 2/5 \\ c_2 & \text{if } 2/5 \leq x < 1/2 \\ u_2 & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

Computing the optimal cost: why is that hard?

Given two clocks x and y, we can check whether $y = 2x$.

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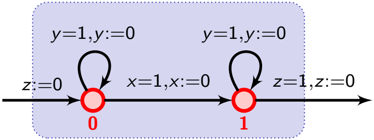
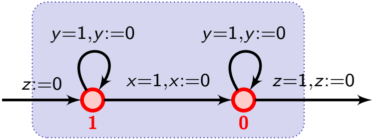
$\mathsf{Add}^+(x)$



The cost is increased by x_0

$\mathsf{Add}^-(x)$



The cost is increased by $1-x_0$

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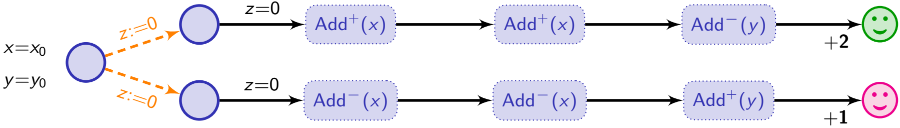

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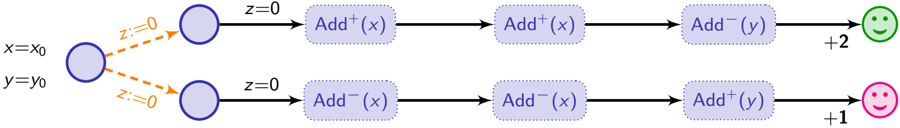

- In 🙂, cost $= 2x_0 + (1 - y_0) + 2$

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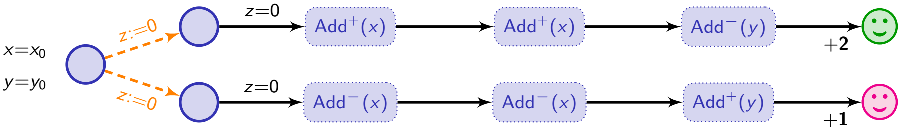

- In 🙂 (green), cost $= 2x_0 + (1 - y_0) + 2$

  In 🙂 (pink), cost $= 2(1 - x_0) + y_0 + 1$

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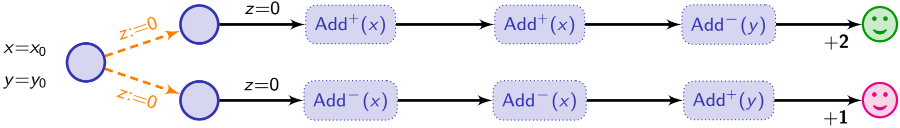

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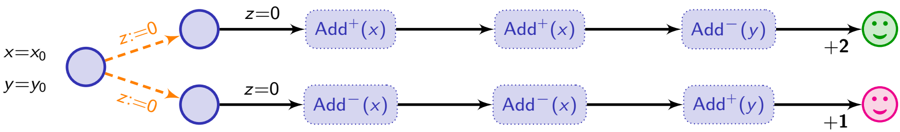

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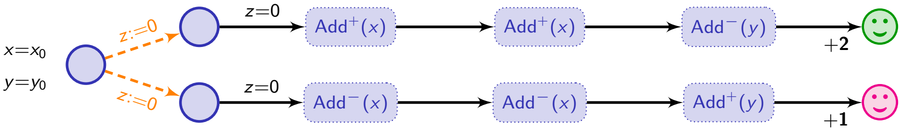

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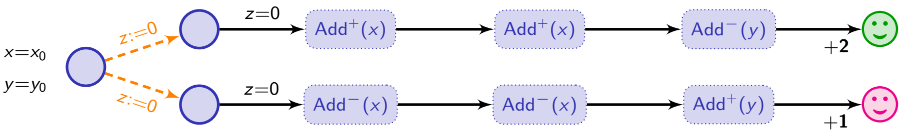

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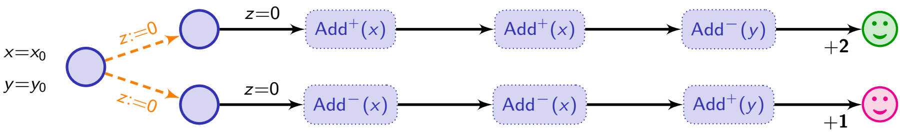

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  $\rightsquigarrow$ player 2 can enforce cost $3 + |y_0 - 2x_0|$
- Player 1 has a winning strategy with cost ≤ 3 iff $y_0 = 2x_0$

Computing the optimal cost: why is that hard?

Player 1 will simulate a two-counter machine:

- each instruction is encoded as a module;
- the counter values c_1 and c_2 are encoded by two clocks:

$$x = \frac{1}{2^{c_1}} \quad \text{and} \quad y = \frac{1}{2^{c_2}}$$

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The two-counter machine has a halting computation iff player 1 has a winning strategy to ensure a cost no more than 3.

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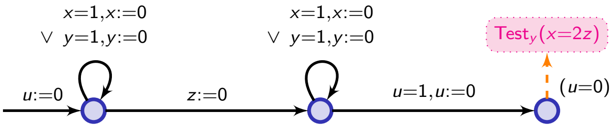
Globally, $(x \leq 1, y \leq 1, u \leq 1)$



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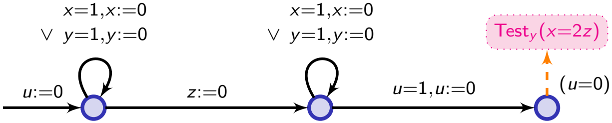

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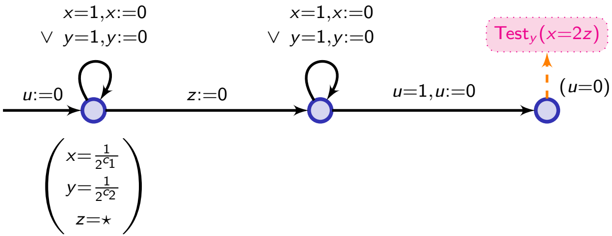

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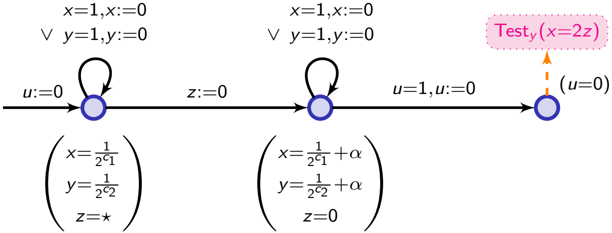

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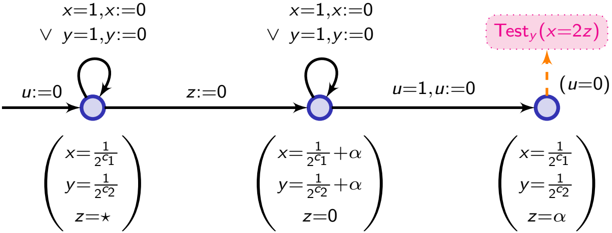

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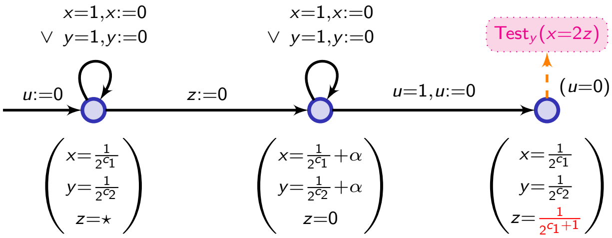

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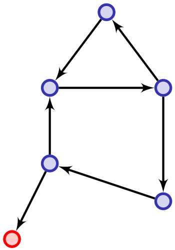

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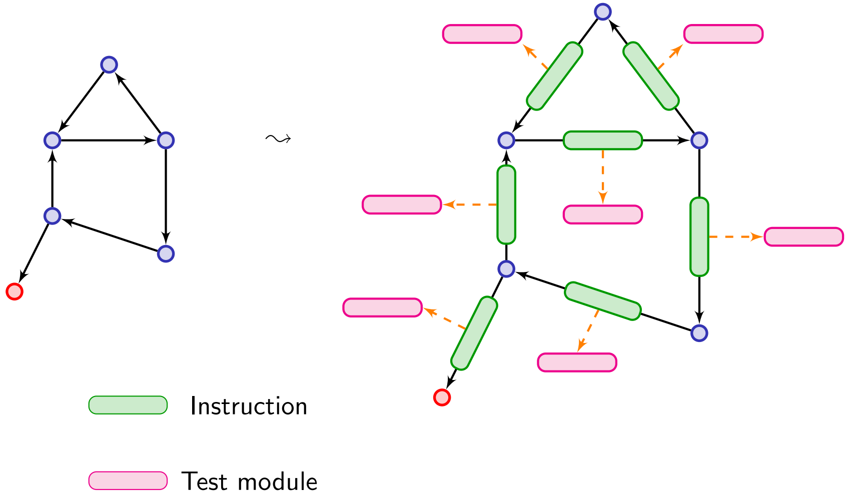

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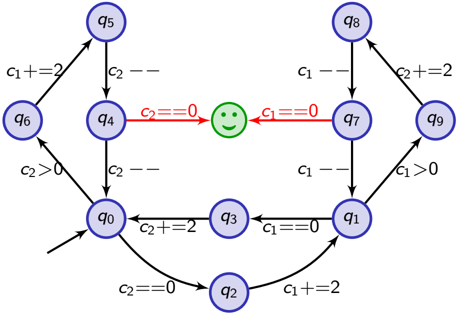

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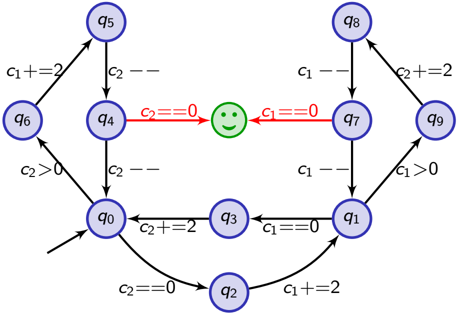

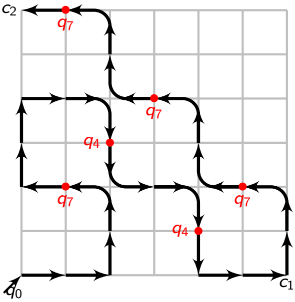

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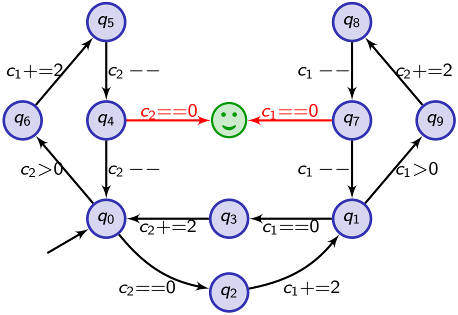

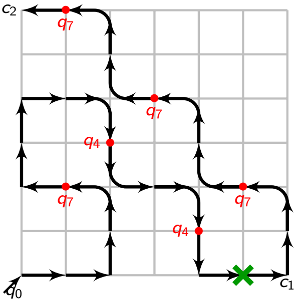

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[BMM14] Bouyer, Markey, Matteplackel. Averaging in LTL *(CONCUR'14)*.

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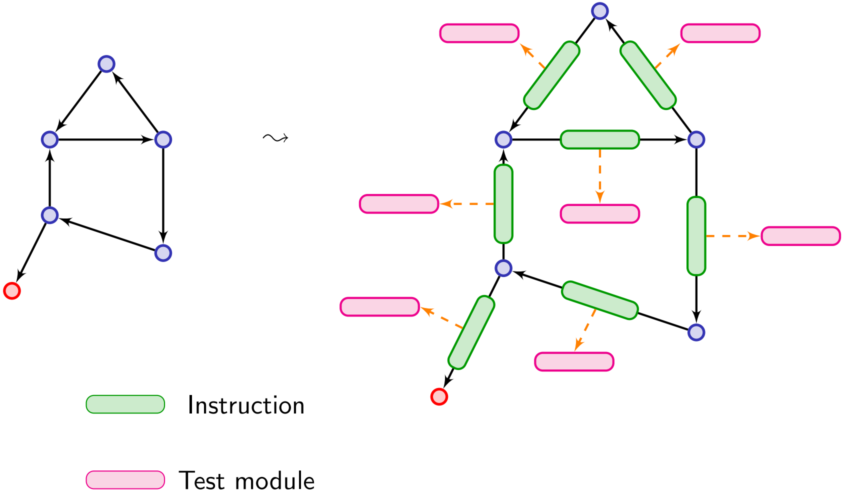
2. Propose an approximation algorithm for a large class of weighted timed games (that comprises the class of games used for proving the above undecidability)
   - Almost-optimality in practice should be sufficient
   - Even when we know how to compute the value, we are only able to synthesize almost-optimal strategies...

A snapshot on the undecidability proof



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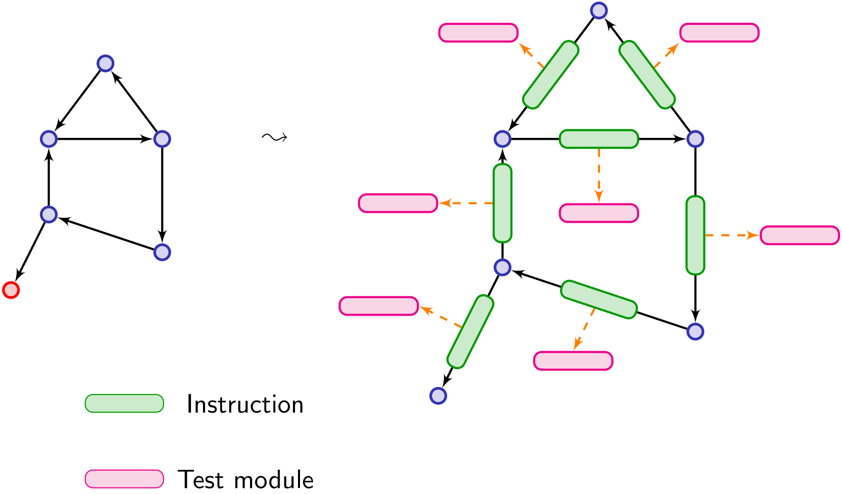

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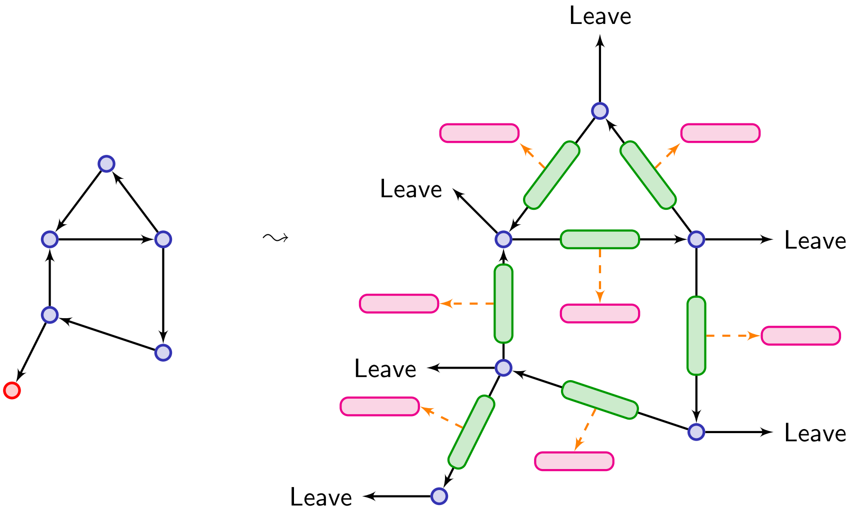

Leave with cost $3 + 1/2^n$ (n: length of the path)

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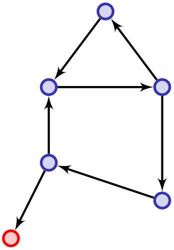
$\mathcal{M}$ does not halt iff the value of $\mathcal{G}_{\mathcal{M}}$ is 3



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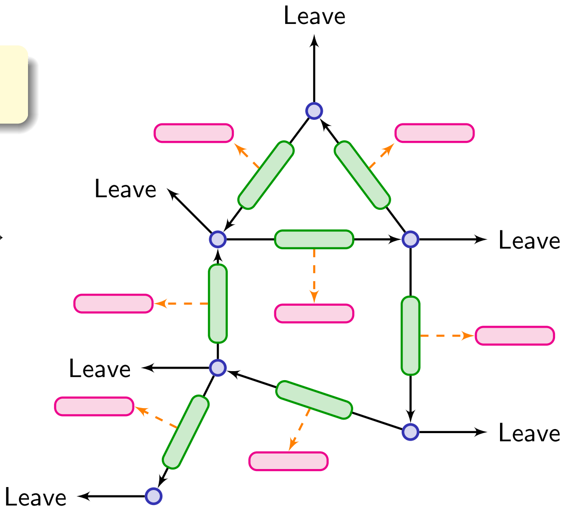

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Theorem [BJM15]

The value problem is undecidable in weighted timed games (with four clocks or more).

- Remark on the reduction:
  - Cost 0 within the core of the game
  - The rest of the game is acyclic

[BJM15] Bouyer, Jaziri, Markey. On the value problem in weighted timed games.

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... when cost is strongly non-zeno. [AM04,BCFL04]

That is, there exists $\kappa > 0$ such that for every region cycle C, for every real run ϱ read on C,

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Approximation of the optimal cost

Theorem

Let $\mathcal{G}$ be a weighted timed game, in which the cost is almost-strongly non-zeno. For every $\epsilon > 0$, one can compute:

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Skip approximation scheme

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$\leadsto$ This is not possible here
There might be runs with prefixes of arbitrary length and cost 0 (e.g. the game of the undecidability proof)

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Only partially unfold the game:

- Keep components with cost 0 untouched – we call it the kernel
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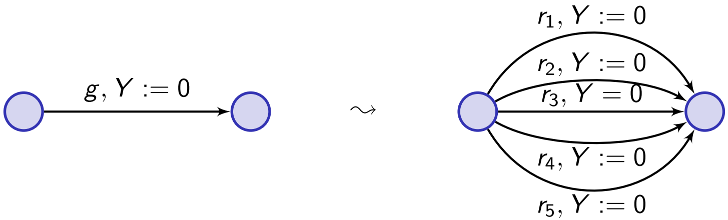
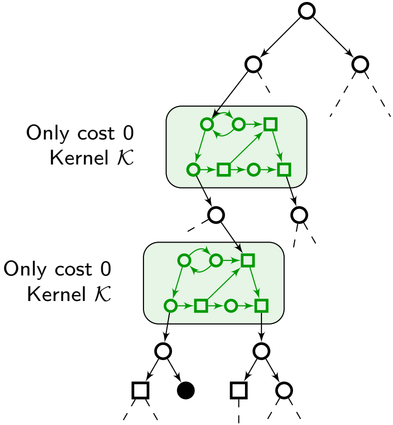
First: split the game along regions!



Semi-unfolding



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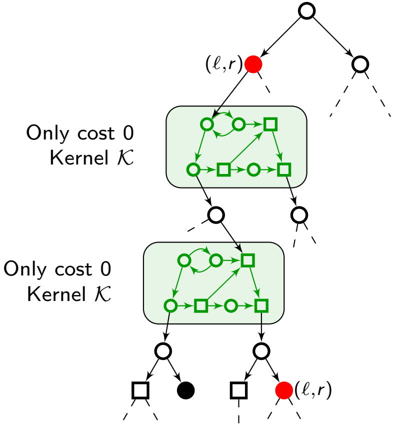

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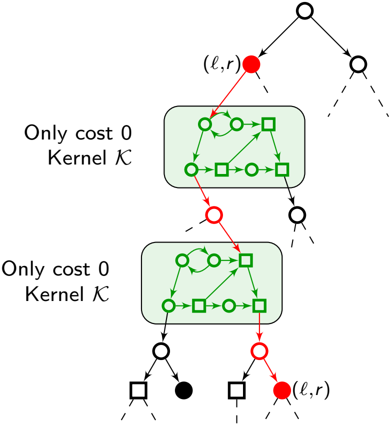

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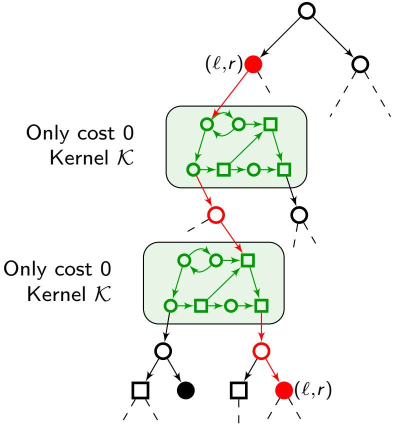

Hypothesis:
cost > 0 implies cost $\geq \kappa$

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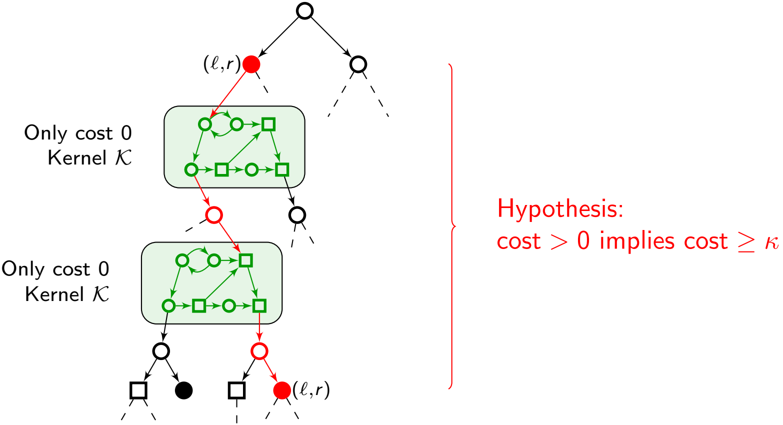
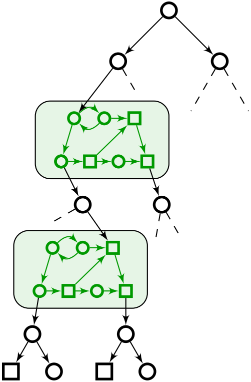


Conclusion: we can stop unfolding the game after N steps
(e.g. $N = (M+2) \cdot |\mathcal{R}(\mathcal{A})|$, where M is a pre-computed bound on $\text{optcost}_{\mathcal{G}}$)

Approximation scheme



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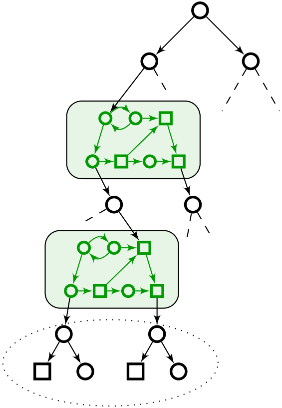

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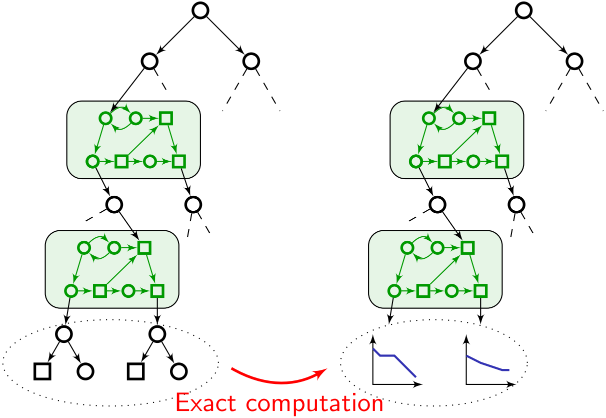

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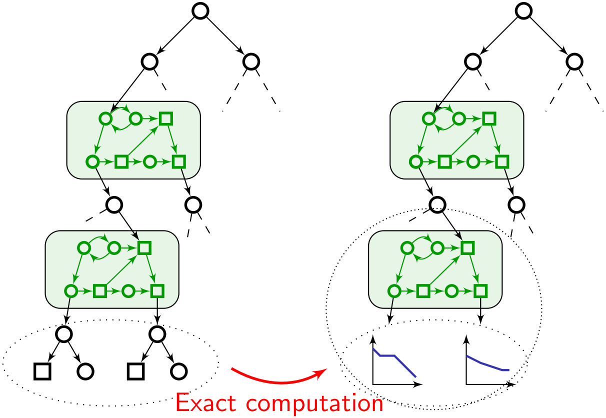

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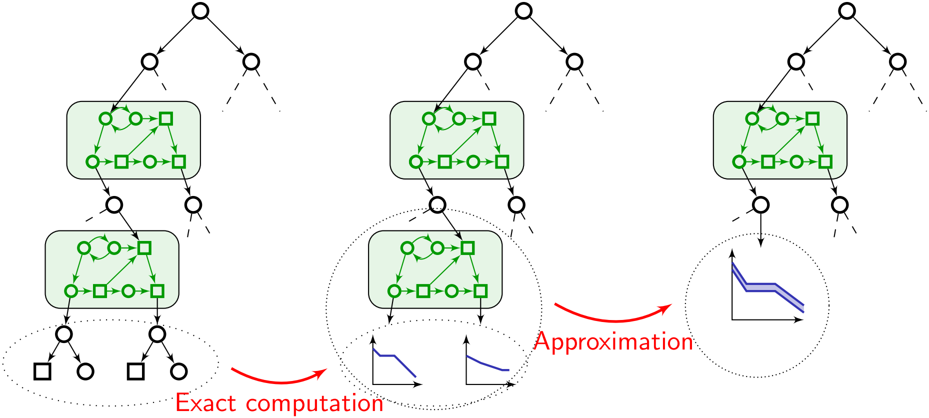


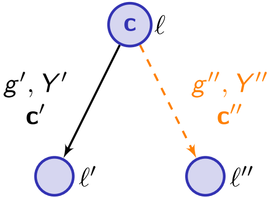
First step: Tree-like parts

$\rightsquigarrow$ Goes back to [LMM02]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

First step: Tree-like parts

$\rightsquigarrow$ Goes back to [LMM02]



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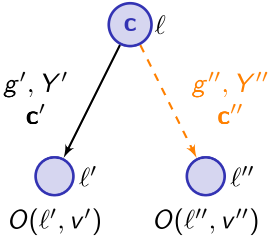

$O(\ell, v) =$

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

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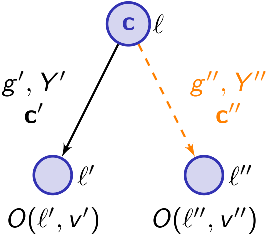

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'}$$

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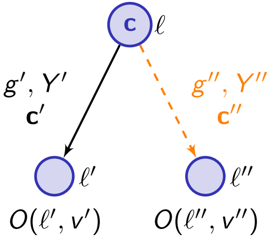

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max(\quad , \quad)$$

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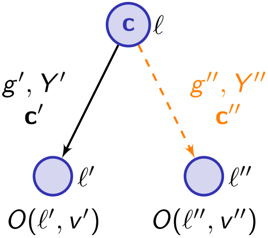

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max\big((\alpha), \qquad \big)$$

$$(\alpha) = t'\mathbf{c} + \mathbf{c}' + O(\ell', v')$$

$v' = [Y' \leftarrow 0](v+t')$

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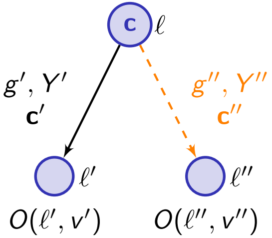
$\rightsquigarrow$ Goes back to [LMM02]



$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max\big((\alpha), (\beta)\big)$$

$$(\alpha) = t'\mathbf{c} + \mathbf{c}' + O(\ell', v')$$

$$(\beta) = \sup_{t'' \leq t' \mid v+t'' \models g''} t''\mathbf{c} + \mathbf{c}'' + O(\ell'', v'')$$

$$v' = [Y' \leftarrow 0](v+t')$$
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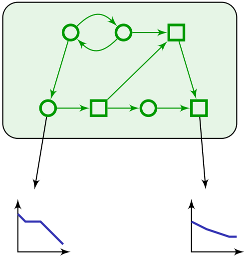
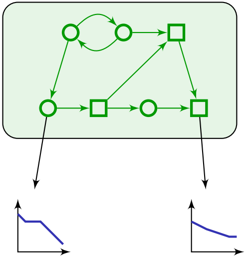
[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

Second step: Kernels



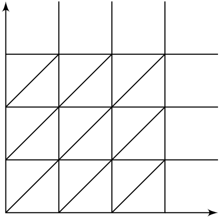
Output cost functions f

Second step: Kernels



Output cost functions f

1. Refine the regions such that f differs of at most ϵ within a small region



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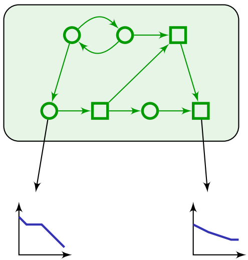

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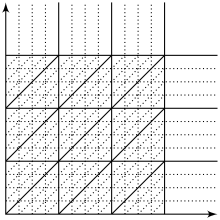

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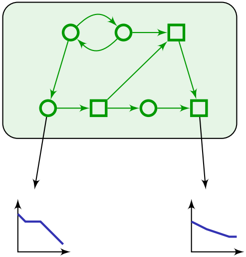

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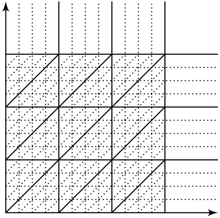

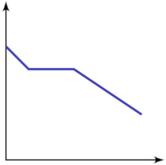

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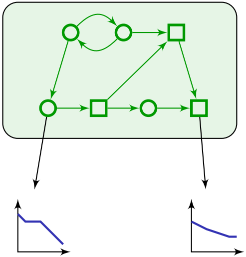

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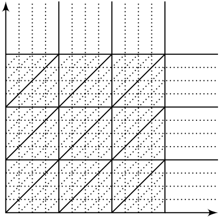
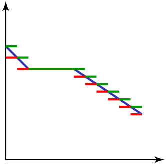
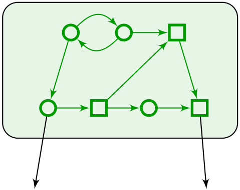


2. Under- and over-approximate by piecewise constant functions f_ϵ^- and f_ϵ^+



Second step: Kernels

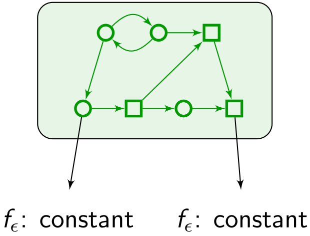
3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ



f_ϵ: constant f_ϵ: constant

Second step: Kernels

3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ
4. Since cost is 0 everywhere, the resulting game is nothing more than a reachability timed game with an order on target (output) edges (given by f_ϵ)



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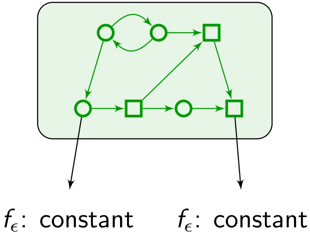
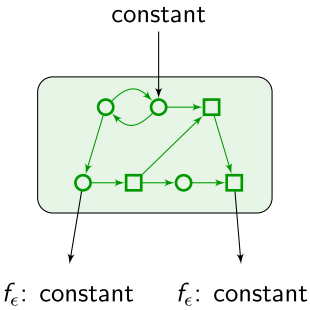


3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ
4. Since cost is 0 everywhere, the resulting game is nothing more than a reachability timed game with an order on target (output) edges (given by f_ϵ)
5. Those can be solved using standard technics based on attractors: small regions are sufficient, and the local optimal cost (for output f_ϵ) is constant within a small region

Second step: Kernels



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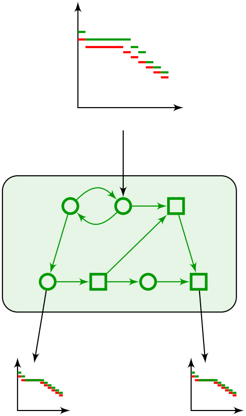

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$\rightsquigarrow$ We have computed ϵ-approximations of the optimal cost, which are constant within small regions. Corresponding strategies can be inferred

Outline

Conclusion

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- New insight into the value problem for this model

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- Understand the multiplayer setting