Let’s play!

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Thanks to Mickael Randour for his slides!
We are going to play…
We are going to play... and discover game theory!
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How will we proceed?
- We will play together...
We are going to play... and discover **game theory**!

**How will we proceed?**

- We will play together...
- ... and discover several key concepts!
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- ... and discover several key concepts!

What is game theory?

- General mathematical approach
- Interactions between agents or processes seen as games between several players
- Many applications: computer science, economy, biology, politics...
Historical context


- Movie *A beautiful mind* relating the life of John Nash!
Rationality

**Fundamental assumption:** players are rational!
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- They have no emotion, their choices are not influenced by friendships or social restrictions
Rationality

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Today, we will be selfish... that's just a game 😊
Un first example: the Nim game

Rules of the game

- Two players
- 13 matches
- At her turn, a player removes 1, 2 or 3 matches
- The player who has to take the last one loses!
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Who wins? Could the loser have won?  
Do you want to bet?
Nim game: if you bet, you should first know the following...

If $n$ is the number of matches, then:
- either the first player can win whatever the choices of the second player ($n \mod 4 \neq 1$)
- or the second player can win whatever the choices of the first player ($n \mod 4 = 1$)

Why that?
- If $n = 1$, the first player loses
- If $n = 2, 3$ or $4$, the first player removes 1, 2 or 3 matches, and the second player loses
- If $n = 5$, the first player loses
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- Even *Chess* are concerned with this kind of results

Zermelo theorem (1913) for chess

either white can force a win, or black can force a win, or both sides can force at least a draw
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Existence of a winning strategy does not mean it is easy to compute.

According to Claude Shannon, there are $10^{43}$ legal positions in Chess, so it will take an impossibly long time to compute a perfect strategy
Another game model

- Players choose their actions secretly and simultaneously.
- Each player earns (or loses) some amount of money (a payoff).
- Each player tries to maximize the amount of money (s)he earns.
An example: The prisoner dilemma

- Two suspects are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the same deal.

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How should they play? (remember they are rational...)
The prisoner dilemma: Analysis

- Notion of dominating strategy
  \[ \sim \text{whatever does the adversary, it is better to Betray...} \]
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- It is possible to enforce more cooperation by repeating the game and giving the possibility to players to punish the other player if they betrayed before
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The husband would prefer to attend the football game. The wife would rather go to the theater. Both would prefer to go to the same place rather than different ones.

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Where should they go? Recall that they are rational...
The battle of the sexes: Analysis

- Here, there is no dominating strategy (which would be better in any case)
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Playing according to an equilibrium requires some coordination between the players, but contrary to the prisoner’s dilemma, they know that the other player has no interest in deviating.
Rock/Paper/Scissors
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There is a unique Nash equilibrium, which is stochastic. Each player should play uniformly at random.
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What do you think the issue of the game will be?
What should the pirates do?
The pirate game: Analysis

Remember that everyone should be rational...

Finally, we have:

A : 98, B : 0, C : 1, D : 0, E : 1.

Surprising, no? But that's the only rational...
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- games with imperfect information
- coalition games
- repeated games
- games on graphs
- probabilistic games...
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- The theory is very rich. New results appear regularly
Some critical systems cannot tolerate bugs!
Application in computer science: fiability of critical systems

- Some critical systems cannot tolerate bugs!

- One needs to check that they behave correctly: they have to be somehow resilient to (unexpected) actions from the environment (e.g. lightnings)
... using games

- Model interactions between the system and its environment using a game
- The system is one player which aims at behaving in a correct manner (with no bug)
- The environment is another player with the opposite goal (zero-sum)
... using games

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- The environment is another player with the opposite goal (zero-sum)

- If one finds a winning strategy for the system (remember the Nim game), then we know how to control the system and react to actions by the environment while ensuring the safety of the system
- Game theory offers a mathematical framework to prove formally that a system is correct, and to synthesize correct controllers
Wants to go further?

... and many more!