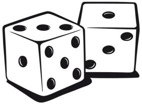

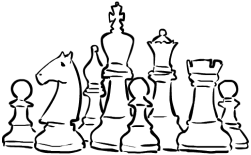

Let's play!

Patricia Bouyer-Decitre

LSV, CNRS & ENS Paris-Saclay, France

Thanks to Mickael Randour for his slides!

We are going to play...

We are going to play... and discover game theory!

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How will we proceed?

- We will play together...

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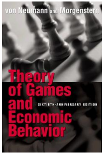
What is game theory?

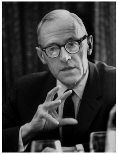
- General mathematical approach
- Interactions between agents or processes seen as games between several players
- Many applications: computer science, economy, biology, politics...

Historical context

- Bases of game theory in *The Theory of Games and Economic Behavior* (1944).

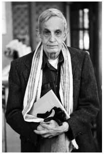
John von Neumann

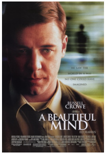




Oskar Morgenstern

- Movie *A beautiful mind* relating the life of John Nash!





Rationality

Fundamental assumption: players are rational!

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Today, we will be selfish... that's just a game ☺

Un first example: the Nim game

Rules of the game

- Two players
- 13 matches
- At her turn, a player removes 1, 2 or 3 matches
- The player who has to take the last one loses!

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- Existence of a winning strategy does not mean it is easy to compute...
  - According to Claude Shannon, there are 10^{43} legal positions in *Chess*, so it will take an impossibly long time to compute a perfect strategy

Another game model

- Players choose their actions secretly and simultaneously
- Each player earns (or loses) some amount of money (a payoff).
- Each player tries to maximize the amount of money (s)he earns.

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How should they play? (remember they are rational...)

The prisoner dilemna: Analysis

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- For some applications (e.g. in computer science), it is fine. However for some others (e.g. in economy), this is not so clear...
- It is possible to enforce more cooperation by repeating the game and giving the possibility to players to punish the other player in case (s)he betrayed before

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Where should they go? Recall that they are rational...

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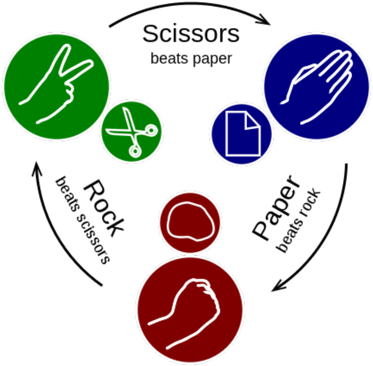
- Playing according to an equilibrium requires some coordination between the players, but contrary to the prisoner's dilemna, they know that the other player has no interest in deviating.

Rock/Paper/Scissors



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Does this game have an equilibrium?

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Does this game have an equilibrium?

- There is a unique Nash equilibrium, which is stochastic
- Each player should play uniformly at random

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What do you think the issue of the game will be?
What should the pirates do?

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  - If the five pirates are there, then A can keep 98 gold coins and offer one coin to C and one coin to E: it's better for both of them, since otherwise they will get 0! Hence they accept.

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- Finally, we have:

$$A : 98, \;\; B : 0, \;\; C : 1, \;\; D : 0, \;\; E : 1.$$

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- Surprising, no? But that's the only rational issue. . .

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  - coalition games
  - repeated games
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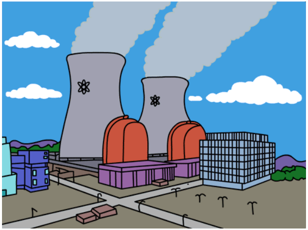
- The theory is very rich. New results appear regularly

Application in computer science: fiability of critical systems

- Some critical systems cannot tolerate bugs!



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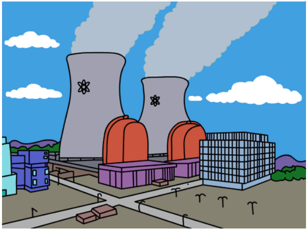

- One needs to check that they behave correctly: they have to be somehow resilient to (unexpected) actions from the environment (e.g. lightnings)

… using games

- Model interactions between the system and its environment using a game
- The system is one player which aims at behaving in a correct manner (with no bug)
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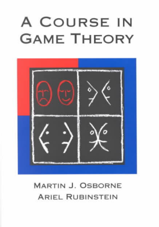
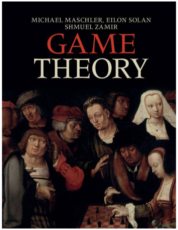
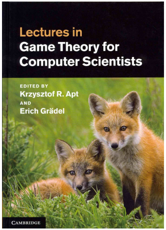
- If one finds a winning strategy for the system (remember the Nim game), then we know how to control the system and react to actions by the environment while ensuring the safety of the system
- Game theory offers a mathematical framework to prove formally that a system is correct, and to synthesize correct controllers

Wants to go further?







... and many more!