Nash equilibria in games on graphs with public signal monitoring

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What this talk is about

- pure Nash equilibria in game graphs
- imperfect information monitoring
- public signals
- computability of Nash equilibria
Two-player turn-based games

- Game graph $G = (V, E)$
- $V$ partitioned into $V\Diamond$ and $V\Box$
Two-player turn-based games

- **Game graph** $G = (V, E)$
- $V$ partitioned into $V_{\diamond}$ and $V_{\Box}$

- **Strategy for player $i$:**
  $$\sigma_i : V^* V_i \to V$$
  s.t. $(\text{last}(h), \sigma_i(h)) \in E$ if $\text{last}(h) \in V_i$

  - **Memoryless** if $\sigma_i(h) = \sigma_i(h')$ if $\text{last}(h) = \text{last}(h')$, that is:
    $$\sigma_i : V_i \to V$$

  - Given $(\sigma_{\diamond}, \sigma_{\Box})$, unique outcome

$$\sigma_{\diamond}(v_0) = v_3, \quad \sigma_{\diamond}(v_2) = \sigma_{\diamond}(v_4) = \bigcirc$$

is a memoryless strategy for $\diamond$
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- Zero-sum hyp.:
  $$\text{payoff}\Box(\rho) = -\text{payoff}\Diamond(\rho)$$

$\sigma\Diamond(v_0) = v_3$, $\sigma\Diamond(v_2) = \sigma\Diamond(v_4) = \text{);}$

is a memoryless strategy for $\Diamond$

Goal of $\Diamond$: Reach 😊
Goal of $\Box$: Avoid 😋
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- **Goal of $\diamond$: Reach $\smiley$**
- **Goal of $\Box$: Avoid $\frown$**
- $\text{payoff}_\diamond(\rho) \in \{-1, 1\}$
- $\text{payoff}_\diamond(\rho) = 1$ iff $\rho$ visits $\smiley$

- **Zero-sum hyp.:**
  $$\text{payoff}_\Box(\rho) = -\text{payoff}_\diamond(\rho)$$

- $\sigma_\diamond(v_0) = v_3$, $\sigma_\diamond(v_2) = \sigma_\diamond(v_4) = \smiley$
  is a memoryless strategy for $\diamond$
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Goal of $\Diamond$: Reach 😊
Goal of $\Box$: Avoid 😬

payoff$\Diamond(\rho) \in \{-1, 1\}$

payoff$\Diamond(\rho) = 1$ iff $\rho$ visits 😊

$\sigma\Diamond$ ensures payoff $+1$ for $\Diamond$: it is a winning strategy
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- Zero-sum hyp.:
  $\text{payoff}_\Box(\rho) = -\text{payoff}_\Diamond(\rho)$

Goal of $\Diamond$: Reach ☺
 Goal of $\Box$: Avoid ☹
payoff$\Diamond(\rho) \in \{0, 1\}$
payoff$\Diamond(\rho) = 1$ iff $\rho$ visits ☺

$\sigma\Diamond$ ensures payoff +1 for $\Diamond$: it is a winning strategy
Non-zero-sum multiplayer games

- Several players $\text{Agt} = \{A_1, \ldots, A_N\}$
Non-zero-sum multiplayer games

- Several players $A_{gt} = \{A_1, \ldots, A_N\}$
- Each player $A$ plays according to a strategy $\sigma_A$
Non-zero-sum multiplayer games

- Several players $\text{Agt} = \{A_1, \ldots, A_N\}$
- Each player $A$ plays according to a strategy $\sigma_A$
- Each player $A$ has a payoff function

$$\text{payoff}_A : V^\omega \rightarrow \mathbb{R}$$
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\text{payoff}_A : V^\omega \rightarrow \mathbb{R}
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- Non-zero-sum...
- Selfishness hypothesis: each player wants to maximize her own payoff!
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- Need of solution concepts to describe the kind of interactions between the players
Non-zero-sum multiplayer games

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- Non-zero-sum...
- Selfishness hypothesis: each player wants to maximize her own payoff!
- Need of solution concepts to describe the kind of interactions between the players
- The simplest: Nash equilibria
A strategy profile \((\sigma_A)_{A \in \text{Agt}}\) is a **Nash equilibrium** if no player can improve her payoff by unilaterally changing her strategy.
Nash equilibria in turn-based games

Nash equilibrium

A strategy profile \((\sigma_A)_{A \in \text{Agt}}\) is a Nash equilibrium if no player can improve her payoff by unilaterally changing her strategy.

is a Nash equilibrium with payoff \((0, 1, 0)\)
Nash equilibria in turn-based games

**Nash equilibrium**

A strategy profile \((\sigma_A)_{A \in \text{Agt}}\) is a **Nash equilibrium** if no player can improve her payoff by unilaterally changing her strategy.

is not a Nash equilibrium
Characterization of Boolean Nash equilibria in turn-based games

\[ \psi_A : \text{objective of player } A \]

Player \( A_1 \) loses along that play.
Characterization of Boolean Nash equilibria in turn-based games

$\psi_A$: objective of player $A$

Player $A_1$ loses along that play

Player $A_1$ should lose

Main outcomes of Boolean Nash equilibria in turn-based games can be characterized by an LTL formula:

$$\Phi_{NE} = \bigwedge_{A \in \text{Agt}} (\neg \psi_A \Rightarrow G (p_A \Rightarrow X \neg W_A))$$

where $p_A$ labels $A$-states and $W_A$ is the set of winning states for $A$ against the coalition of the other players (should be precomputed).
Characterization of Boolean Nash equilibria in turn-based games

\[ \psi_A : \text{objective of player } A \]

Coalition \[ \{ A_2, A_3 \} \]

prevents \( A_1 \) from winning

Player \( A_1 \) loses along that play

Main outcomes of Boolean Nash equilibria in turn-based games can be characterized by an LTL formula:

\[ \Phi_{NE} = \bigwedge_{A \in \text{Agt}} (\neg \psi_A \Rightarrow G (p_A \Rightarrow X \neg W_A)) \]

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Characterization of Boolean Nash equilibria in turn-based games

\[ \Phi_{NE} = \bigwedge_{A \in \text{Agt}} \left( \neg \psi_A \Rightarrow \mathbf{G}(p_A \Rightarrow \mathbf{X}\neg W_A) \right) \]

where \( p_A \) labels \( A \)-states and \( W_A \) is the set of winning states for \( A \) against the coalition of the other players (should be precomputed).
Existing results in the framework of turn-based games

[UW11,Umm11]

- There always exists a Nash equilibrium for Boolean $\omega$-regular objectives.
Existing results in the framework of turn-based games

[UW11, Umm11]

- There always exists a Nash equilibrium for Boolean $\omega$-regular objectives
- One can decide the constrained existence of a Nash equilibrium (and compute one!)
Existing results in the framework of turn-based games

There always exists a Nash equilibrium for Boolean $\omega$-regular objectives

One can decide the constrained existence of a Nash equilibrium (and compute one!)

One cannot decide the existence of a mixed (i.e. stochastic) Nash equilibrium

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[Umm11] Ummels. Stochastic multiplayer games: theory and algorithms (RWTH Aachen University)
Existing results in the framework of turn-based games

[**UW11**, **Umm11**]

- There always exists a Nash equilibrium for Boolean $\omega$-regular objectives
- One can decide the constrained existence of a Nash equilibrium (and compute one!)
- One cannot decide the existence of a mixed (i.e. stochastic) Nash equilibrium

~ this is why we restrict to pure equilibria
What about concurrent games?

concurrent games [BBMU15]

What about concurrent games?

The matching-penny game:

\[ \langle a, a \rangle, \langle b, b \rangle \rightarrow v_1 \quad (1,0) \]
\[ \langle a, b \rangle, \langle b, a \rangle \rightarrow v_2 \quad (0,1) \]

What about concurrent games?

**concurrent games [BBMU15]**

The matching-penny game:

- \((a,a), (b,b)\) leads to \(v_1\) with payoff \((1,0)\)
- \((a,b), (b,a)\) leads to \(v_2\) with payoff \((0,1)\)

There is no pure Nash eq.
What about concurrent games?

Invisible actions in concurrent games [BBMU15]

The matching-penny game:

\[
\langle a, a \rangle, \langle b, b \rangle \quad \rightarrow \quad (1, 0)
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\langle a, b \rangle, \langle b, a \rangle \rightarrow v_2 \quad (0,1)
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There is no pure Nash eq.

\[
\text{susp}\left( (v_0, v_3), \text{○○○} \right) = \{A_1\}
\]
What about concurrent games?

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The matching-penny game:

\[ \langle a,a \rangle, \langle b,b \rangle \rightarrow v_1 \quad (1,0) \]
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There is no pure Nash eq.

\[ \text{susp} \left( (v_0, v_3), \bigcirc \bigcirc \right) = \{A_1\} \]
\[ \text{susp} \left( (v_0, v_2), \bigcirc \bigcirc \right) = \{A_2, A_3\} \]
What about concurrent games?

**Invisible actions in concurrent games [BBMU15]**

The matching-penny game:

\[(a,a), (b,b) \rightarrow v_1 \quad (1,0)\]

\[(a,b), (b,a) \rightarrow v_2 \quad (0,1)\]

There is no pure Nash eq.

\[\text{susp}((v_0, v_3), \bigcirc \bigcirc) = \{A_1\}\]

\[\text{susp}((v_0, v_2), \bigcirc \bigcirc) = \{A_2, A_3\}\]

Solution via the suspect game abstraction, a structure to track suspect players

What about concurrent games?

Invisible actions in concurrent games [BBMU15]

The matching-penny game:

\[ \langle a, a \rangle, \langle b, b \rangle \quad (1, 0) \]

\[ \langle a, b \rangle, \langle b, a \rangle \quad (0, 1) \]

There is no pure Nash eq.

\[ \text{susp} \left( (v_0, v_3), \bigcirc \bigcirc \right) = \{ A_1 \} \]

\[ \text{susp} \left( (v_0, v_2), \bigcirc \bigcirc \right) = \{ A_2, A_3 \} \]

Solution via the suspect game abstraction, a structure to track suspect players

Can we add more partial information to that framework?

Concurrent games with signals

Concurrent games with signals

- Signal for player $A_1$: $\bullet$ and $\bullet$
- On playing $a$, player $A_1$ will receive $\bullet$
- On playing $b$, player $A_1$ will receive $\bullet$

Concurrent games with signals

- Signal for player $A_1$: and
- Signal for player $A_2$: , and

- On playing $a$, player $A_1$ will receive
- On playing $b$, player $A_1$ will receive
- On playing $a$, player $A_2$ will receive either or
- On playing $b$, player $A_2$ will receive

Concurrent games with signals

Signal for player $A_1$: $\bullet$ and $\bullet$
Signal for player $A_2$: $\bullet$, $\bullet$, $\bullet$, $\bullet$ and $\bullet$

- On playing $a$, player $A_1$ will receive $\bullet$
- On playing $b$, player $A_1$ will receive $\bullet$
- On playing $a$, player $A_2$ will receive either $\bullet$ or $\bullet$
- On playing $b$, player $A_2$ will receive $\bullet$

Public signal
Same signal to every player!

A concurrent game with signals is a tuple

\[ G = \langle V, v_{\text{init}}, \text{Agt}, \text{Act}, \Sigma, \text{Allow}, \text{Tab}, (\ell_A)_{A \in \text{Agt}}, (\text{payoff}_A)_{A \in \text{Agt}} \rangle \]

where:

- \( V \) is a finite set of vertices,
- \( v_{\text{init}} \in V \) is the initial vertex,
- \( \text{Agt} \) is a finite set of players,
- \( \text{Act} \) is a finite set of actions,
- \( \Sigma \) is a finite alphabet,
- \( \text{Allow} : V \times \text{Agt} \rightarrow 2^{\text{Act} \setminus \{\emptyset\}} \) is a mapping indicating the actions available to a given player in a given state,
- \( \text{Tab} : V \times \text{Act}^{\text{Agt}} \rightarrow V \) associates, with a given state and a given move of the players (i.e., an element of \( \text{Act}^{\text{Agt}} \)), the state resulting from that move,
- for every \( A \in \text{Agt}, \ell_A : (\text{Act}^{\text{Agt}} \times V) \rightarrow \Sigma \) is a signal,
- for every \( A \in \text{Agt}, \text{payoff}_A : V \times (\text{Act}^{\text{Agt}} \times V)^\omega \rightarrow \mathbb{D} \) is a payoff function for player \( A \).
Strategies

- What player A sees from history $h = v_0 \xrightarrow{m_0} v_1 \xrightarrow{m_1} \ldots \xrightarrow{m_{k-1}} v_k$:

$$
\pi_A(h) = v_0 \cdot m_0(A) \cdot \ell_A(m_0, v_1) \cdot m_1(A) \ldots m_{k-1}(A) \cdot \ell_A(m_{k-1}, v_k)
$$

$\sim$ perfect recall hypothesis
Strategies

- What player $A$ sees from history $h = v_0 \xrightarrow{m_0} v_1 \xrightarrow{m_1} \ldots \xrightarrow{m_{k-1}} v_k$:
  \[
  \pi_A(h) = v_0 \cdot m_0(A) \cdot \ell_A(m_0, v_1) \cdot m_1(A) \ldots m_{k-1}(A) \cdot \ell_A(m_{k-1}, v_k)
  \]

  $\sim$ perfect recall hypothesis

- Undistinguishability relation for player $A$:
  \[
  h \sim_A h' \text{ iff } \pi_A(h) = \pi_A(h')
  \]
Strategies

- What player $A$ sees from history $h = v_0 \xrightarrow{m_0} v_1 \xrightarrow{m_1} \dots \xrightarrow{m_{k-1}} v_k$:
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  $\sim$ perfect recall hypothesis

- Undistinguishability relation for player $A$:
  \[ h \sim_A h' \iff \pi_A(h) = \pi_A(h') \]

- A strategy for player $A$ is a (partial) function:
  \[ \sigma_A : V \cdot \left( \text{Act}^\text{Agt} \cdot V \right)^* \to \text{Act} \]
  such that $h \sim_A h'$ implies $\sigma_A(h) = \sigma_A(h')$. 

Strategies

- What player $A$ sees from history $h = v_0 \xrightarrow{m_0} v_1 \xrightarrow{m_1} \ldots \xrightarrow{m_{k-1}} v_k$:

$$\pi_A(h) = v_0 \cdot m_0(A) \cdot \ell_A(m_0, v_1) \cdot m_1(A) \ldots m_{k-1}(A) \cdot \ell_A(m_{k-1}, v_k)$$

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- **Undistinguishability relation** for player $A$:

$$h \sim_A h' \iff \pi_A(h) = \pi_A(h')$$

- A **strategy** for player $A$ is a (partial) function:

$$\sigma_A : V \cdot (\text{Act}^\text{Agt} \cdot V)^* \rightarrow \text{Act}$$

such that $h \sim_A h'$ implies $\sigma_A(h) = \sigma_A(h')$.

- A **strategy profile** is a tuple $\sigma_{\text{Agt}} = (\sigma_A)_{A \in \text{Agt}}$ where $\sigma_A$ is a strategy for player $A$. 
Discussion on the perfect-recall assumption

In most existing frameworks, strategies are defined through observation maps

\[ O_A : V \rightarrow \Sigma \quad \sigma_A : \Sigma^* \rightarrow \text{Act} \]
Discussion on the perfect-recall assumption

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\[ O_A : V \rightarrow \Sigma \quad \sigma_A : \Sigma^* \rightarrow \text{Act} \]

- This choice is suitable for distributed synthesis and Nash equilibria (for instance)...

Example (Subgame-perfect equilibrium)

\[ v_0 \quad v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \]

\[ \langle a, a \rangle, \langle b, a \rangle \]

\[ \langle a, - \rangle, \langle a, - \rangle \]

\[ \langle b, - \rangle, \langle b, b \rangle \]

\[ \langle a, - \rangle, \langle b, - \rangle \]
Discussion on the perfect-recall assumption

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- but I think this choice is not suitable in general
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Example (Subgame-perfect equilibrium)

- \((v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)\)
- \(v_0 \rightarrow v_1, v_2, v_3\)
- \(v_1 \rightarrow v_1, v_4\)
- \(v_2 \rightarrow v_4, v_5\)
- \(v_3 \rightarrow v_6, v_7\)
- \(v_4 \rightarrow (a, \neg)\)
- \(v_5 \rightarrow (b, \neg)\)
- \(v_6 \rightarrow (a, \neg)\)
- \(v_7 \rightarrow (b, \neg)\)

\(v_0 \rightarrow (a, a) \rightarrow b, a \rightarrow (a, b) \rightarrow v_2\)

\(v_1 \rightarrow (a, a) \rightarrow b, a \rightarrow (a, b) \rightarrow v_4\)

\(v_2 \rightarrow (a, b) \rightarrow v_4\)

\(v_3 \rightarrow (b, b) \rightarrow (b, \neg) \rightarrow v_7\)

\(v_4 \rightarrow (a, \neg) \rightarrow v_6\)
Discussion on the perfect-recall assumption

In most existing frameworks, strategies are defined through observation maps

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- This choice is suitable for distributed synthesis and Nash equilibria (for instance)...
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Example (Subgame-perfect equilibrium)
**Digression on payoff functions**

**Payoff functions**

- Payoff function for player $A$ ($\mathbb{D} \subseteq \mathbb{R}$):

  \[
  \text{payoff}_A : V \cdot \left( \text{Act}^{\text{Agt}} \cdot V \right) \omega \rightarrow \mathbb{D}
  \]
Payoff functions

- Payoff function for player $A$ ($\mathcal{D} \subseteq \mathbb{R}$):

  $$\text{payoff}_A : V \cdot \left( \text{Act}^A \cdot V \right)^\omega \rightarrow \mathcal{D}$$

- $\text{payoff}_A$ is privately visible whenever

  $$\pi_A(\rho) = \pi_A(\rho') \implies \text{payoff}_A(\rho) = \text{payoff}_A(\rho')$$
Digression on payoff functions

**Payoff functions**

- Payoff function for player $A$ ($\mathbb{D} \subseteq \mathbb{R}$):

  $$\text{payoff}_A : V \cdot \left( \text{Act}^\text{Agt} \cdot V \right) \omega \rightarrow \mathbb{D}$$

- $\text{payoff}_A$ is **privately visible** whenever

  $$\pi_A(\rho) = \pi_A(\rho') \text{ implies } \text{payoff}_A(\rho) = \text{payoff}_A(\rho')$$

- If signal $\ell$ is public ($\ell_A = \ell$ for every $A$), $\text{payoff}_A$ is **publicly visible** whenever

  $$\ell(\rho) = \ell(\rho') \text{ implies } \text{payoff}_A(\rho) = \text{payoff}_A(\rho')$$
Some payoff functions

- **Boolean \( \omega \)-regular payoff function (for \( \Omega \)):**

  \[
  \text{payoff}(\rho) = \begin{cases} 
  1 & \text{if } \rho \in \Omega \\
  0 & \text{otherwise}
  \end{cases}
  \]
Digression on payoff functions (cont’d)

Some payoff functions

- **Boolean ω-regular payoff function (for Ω):**

  \[
  \text{payoff}(\rho) = \begin{cases} 
  1 & \text{if } \rho \in \Omega \\
  0 & \text{otherwise}
  \end{cases}
  \]

- **Mean-payoff (limsup or liminf) w.r.t. weight function } w:**

  \[
  \begin{align*}
  \underline{MP}_w(\rho) &= \lim \inf_{n \to \infty} \sum_{i=0}^{n} w(v_i \xrightarrow{m_i} v_{i+1}) \\
  \overline{MP}_w(\rho) &= \lim \sup_{n \to \infty} \sum_{i=0}^{n} w(v_i \xrightarrow{m_i} v_{i+1})
  \end{align*}
  \]

For public visibility, we will assume that atomic propositions/atomic weights are defined w.r.t. the signal alphabet Σ.
Some payoff functions

- Boolean $\omega$-regular payoff function (for $\Omega$):

\[
\text{payoff}(\rho) = \begin{cases} 
1 & \text{if } \rho \in \Omega \\ 
0 & \text{otherwise}
\end{cases}
\]

- Mean-payoff (limsup or liminf) w.r.t. weight function $w$:

\[
\begin{align*}
\overline{\text{MP}}_w(\rho) &= \limsup_{n \to \infty} \sum_{i=0}^{n} w(v_i \xrightarrow{m_i} v_{i+1}) \\
\underline{\text{MP}}_w(\rho) &= \liminf_{n \to \infty} \sum_{i=0}^{n} w(v_i \xrightarrow{m_i} v_{i+1})
\end{align*}
\]

For public visibility, we will assume that atomic propositions/atomic weights are defined w.r.t. the signal alphabet $\Sigma$. 
An example

- Three players concurrent game with public signal
An example

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- Consider the (partial) strategy profile $\sigma_{Agt}$. Can we complete it into a Nash equilibrium?
An example

- Three players concurrent game with public signal
- Consider the (partial) strategy profile $\sigma_{\text{Agt}}$. Can we complete it into a Nash equilibrium?
- This is an $A_2$-deviation, which is invisible to both $A_1$ and $A_3$. $A_1$ has to play $a$ and cannot deviate to $2,0,0$. 

The diagram shows the game with transitions and payoffs. Each node represents a strategy profile, and the arrows indicate possible transitions with associated payoffs. The yellow nodes represent $A_2$'s perspective, and the green nodes represent $A_1$ and $A_3$'s perspectives.
Three players concurrent game with public signal

Consider the (partial) strategy profile $\sigma_{A_{gt}}$. Can we complete it into a Nash equilibrium?

This is an $A_2$-deviation, which is invisible to both $A_1$ and $A_3$. $A_1$ has to play $a$ and cannot deviate to $2,0,0$.

This is a non-profitable $A_1$-deviation.
An example

- Three players concurrent game with public signal
- Consider the (partial) strategy profile $\sigma_{\text{Agt}}$. Can we complete it into a Nash equilibrium?
- This is an $A_2$-deviation, which is invisible to both $A_1$ and $A_3$. $A_1$ has to play $a$ and cannot deviate to $2,0,0$.
- This is a non-profitable $A_1$-deviation.
- No one (alone) can deviate to $v_3$. 
An example

Three players concurrent game with public signal

Consider the (partial) strategy profile $\sigma_{\text{Agt}}$. Can we complete it into a Nash equilibrium?

This is an $A_2$-deviation, which is invisible to both $A_1$ and $A_3$. $A_1$ has to play $a$ and cannot deviate to $2,0,0$.

This is a non-profitable $A_1$-deviation.

No one (alone) can deviate to $v_3$.

$A_1$ can deviate to $v_4$ and $A_3$ can deviate to $v_5$: $A_2$ knows there has been a deviation, but (s)he doesn’t know whether $A_1$ or $A_3$ did so, and whether the game proceeds to $v_4$ or $v_5$. On the other hand, both $A_1$ and $A_3$ know!
An example

- Three players concurrent game with public signal
- Consider the (partial) strategy profile $\sigma_{Agt}$. Can we complete it into a Nash equilibrium?
- This is an $A_2$-deviation, which is invisible to both $A_1$ and $A_3$. $A_1$ has to play $a$ and cannot deviate to $2,0,0$.
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- $A_1$ can deviate to $v_4$ and $A_3$ can deviate to $v_5$: $A_2$ knows there has been a deviation, but (s)he doesn’t know whether $A_1$ or $A_3$ did so, and whether the game proceeds to $v_4$ or $v_5$. On the other hand, both $A_1$ and $A_3$ know! But if the game proceeds to $v_4$, $A_3$ can help $A_2$ punishing $A_1$, and if the game proceeds to $v_5$, $A_1$ can help $A_2$ punishing $A_3$. 
Three players concurrent game with public signal

Consider the (partial) strategy profile $\sigma_{\text{Agt}}$. Can we complete it into a Nash equilibrium?

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How to systematically track all undistinguishable behaviours and all individual deviations? Is that always possible?
First undecidability results

One cannot decide the existence problem in games with signals with three players and publicly visible qualitative $\omega$-regular payoff functions.

by reduction from the distributed synthesis problem (construction for reachability properties taken in [BK10])

[BK10] Berwanger, Kaiser. Information Tracking in Games on Graphs (Journal of Logic, Language and Information)
[DDG+10] Degorre, Doyen, Gentilini, Raskin, Toruńczyk. Energy and Mean-Payoff Games with Imperfect Information (CSL’10)
First undecidability results

One cannot decide the existence problem in games with signals with three players and publicly visible qualitative $\omega$-regular payoff functions.

$\leadsto$ by reduction from the distributed synthesis problem (construction for reachability properties taken in [BK10])

One cannot decide the constrained existence of a Nash equilibrium in a game with public signals, for a mixture of limsup and liminf mean-payoff functions which are privately visible. Even for two players.

$\leadsto$ by reduction from blind mean-payoff games (proven undecidable in [DDG+10])

[BK10] Berwanger, Kaiser. Information Tracking in Games on Graphs (Journal of Logic, Language and Information)
[DDG+10] Degorre, Doyen, Gentilini, Raskin, Toruńczyk. Energy and Mean-Payoff Games with Imperfect Information (CSL ’10)
Proof idea for the second undecidability result

\[ a \rightarrow w(e_1) \]

\[ a \rightarrow w(e_2) \]
Proof idea for the second undecidability result

\[ G \rightarrow a \wedge w(e_1) \]

\[ H \rightarrow a \rightarrow w(e_2) \]

\[ \langle a, a \rangle \]

\[ \langle a, b \rangle (a \neq b) \]

\[ \langle - , - \rangle \]

\[ (0, -W - 1) \]

\[ (0, -w(e_1)) \]

\[ (0, -w(e_2)) \]

the public signal only reveals lost
Proof idea for the second undecidability result

\[ G \xrightarrow{a} w(e_1) \]
\[ G \xrightarrow{a} w(e_2) \]

\[ \langle a, a \rangle \]

\[ \langle a, b \rangle (a \neq b) \]

\[ \langle -, - \rangle (0, -W - 1) \]

\[ \langle a, a \rangle \]

\[ \langle -, a \rangle \]

\[ (0, -W(e_2)) \]

\[ (0, -w(e_1)) \]

\[ \langle a, a \rangle \]

\[ \langle a, a \rangle \]

The public signal only reveals lost but player \( A_2 \) has full information.
Proof idea for the second undecidability result

has a winning strategy in $G$ ensuring $MP > 0$

iff

there is an NE in $H$ such that player $A_2$ has a payoff $< 0$
Proof idea for the second undecidability result

\[ \text{is blind} \]

\[ G \]

\[ \langle a, a \rangle \]

\[ \langle a, b \rangle \quad (a \neq b) \]

\[ \text{lost} \]

\[ \langle -, - \rangle \]

\[ (0, -W - 1) \]

\[ w(e_1) \]

\[ w(e_2) \]

\[ \langle a \rangle \]

\[ \langle a, a \rangle \]

\[ \langle a, - \rangle \]

\[ (0, -w(e_1)) \]

\[ (0, -w(e_2)) \]

\[ (0, -W) \]

the public signal only reveals lost but player A_2 has full information
The epistemic game abstraction

Inspired by:

- the standard powerset construction [Rei84]
- the epistemic unfolding for coordination/distributed synthesis [BKP11]
- the suspect game [BBMU15]
- the deviator game [Bre16]
The epistemic game abstraction

Inspired by:

- the standard powerset construction [Rei84]
- the epistemic unfolding for coordination/distributed synthesis [BKP11]
- the suspect game [BBMU15]
- the deviator game [Bre16]

The idea is to track all possible undistinguishable behaviours, including the single-player deviations

[Rei84] Reif. The complexity of two-player games of incomplete information (J. Comp. and Syst. Sc.)
[BBMU15] Pure Nash equilibria in concurrent games (Log. Meth. in Comp. Sc.)
[Bre16] Brenguier. Robust equilibria in mean-payoff games (FoSSaCS’16)
Epistemic states (type-1)

- $s$ (vertex the game is in if no deviation)
- $V_{A_1}, A_1$ (vertices the game might be in if $A_2$ has deviated)
- $V_{A_2}, A_2$
- $V_{A_3}, A_3$

Captures set of histories that some of the players do not distinguish. $A_i$ cannot distinguish between the normal outcome (no deviation) and deviations of other players leading to some $v \in V_{A_j}$ with $j \neq i$.
Epistemic states (type-2)

Captures set of histories that some of the players do not distinguish. $A_i$ cannot distinguish between the possible deviations of other players (but he knows there has been a deviation).
The example again
The example again

```
⟨a, a, a⟩ → v1
  ⟨a, b, a⟩ → v2
    ⟨b, a, a⟩ → v3
      ⟨b, b, a⟩ → v4
        ⟨∗, *, b⟩ → v5
          ⟨∗, ∗, b⟩

⟨a, a, a⟩ → v1
  ⟨a, b, a⟩ → v2
    ⟨b, −, −⟩ → 0, 1, 0
      ⟨b, −, −⟩
        ⟨∗, −, −⟩ → 3, 3, 3
          ⟨∗, b, a⟩
            ⟨b, a, a⟩ → v4
              ⟨a, a, a⟩, ⟨b, a, b⟩
                ⟨a, a, a⟩
                  ⟨a, a, a⟩
                    ⟨a, a, a⟩
                      ⟨∗, ∗, b⟩
                        ⟨∗, ∗, b⟩
                          ⟨∗, ∗, b⟩
                ⟨b, −, −⟩ → 0, 0, 0
                  ⟨b, a, a⟩
                    ⟨b, a, a⟩
                      ⟨b, a, a⟩
                        ⟨b, a, a⟩
                          ⟨b, a, a⟩
                            ⟨b, a, a⟩
      ⟨a, −, −⟩ → 1, 1, 0
        ⟨a, −, −⟩
          ⟨a, −, −⟩
            ⟨a, −, −⟩
              ⟨a, −, −⟩
                ⟨a, −, −⟩
                  ⟨a, −, −⟩
                    ⟨a, −, −⟩
                      ⟨a, −, −⟩
                        ⟨a, −, −⟩
                          ⟨a, −, −⟩
                ⟨b, −, −⟩ → 2, 0, 0
                  ⟨b, −, −⟩
                    ⟨b, −, −⟩
                      ⟨b, −, −⟩
                        ⟨b, −, −⟩
                          ⟨b, −, −⟩
```

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Example of construction

\[
\begin{align*}
\text{s}_0 & : v_0, \perp \\
& : \emptyset, A_1 \\
& : \emptyset, A_2 \\
& : \emptyset, A_3 \\
\end{align*}
\]
Example of construction

\[s_0\]

\[v_0, \perp\]
\[\emptyset, A_1\]
\[\emptyset, A_2\]
\[\emptyset, A_3\]

\[\langle a, b, a\rangle\] → \[s_0, \langle a, b, a\rangle\]
Example of construction
Example of construction
Example of construction

\[
\begin{align*}
S_0 & \quad \langle a, b, a \rangle \\
\langle a, b, a \rangle & \quad \langle a, b, a \rangle \\
S_1 & \quad \langle a, b, a \rangle \\
\langle a, b, a \rangle & \quad \langle a, b, a \rangle
\end{align*}
\]
Example of construction
Example of construction
Example of construction

\[ s_0, \langle a, b, a \rangle \rightarrow s_0, \langle a, b, a \rangle \rightarrow v_2, \bot \rightarrow s_1, \langle a, \_, \_ \rangle \rightarrow s_1, \langle a, \_, \_ \rangle \rightarrow (1, 1, 0), \bot \]

\[ s_1, \langle b, \_, \_ \rangle \rightarrow s_1, \langle b, \_, \_ \rangle \rightarrow \{ (0, 1, 0) \}, A_1 \rightarrow \{ (1, 1, 0) \}, A_2 \rightarrow \emptyset, A_3 \]

\[ s_0, \emptyset, A_1 \rightarrow s_0, \emptyset, A_2 \rightarrow s_0, \emptyset, A_3 \]

\[ \langle a, b, a \rangle \rightarrow \langle a, b, a \rangle \rightarrow \langle a, a, b \rangle \rightarrow \langle a, a, b \rangle \rightarrow \langle b, a, a \rangle \rightarrow \langle b, a, a \rangle \rightarrow (1, 1, 0) \]
Example of construction

\[
\begin{align*}
\text{Example of construction} & \quad \text{Example of construction} \\
\text{Example of construction} & \quad \text{Example of construction}
\end{align*}
\]
Example of construction

\[ s_0 \]

\[ s_1 \]

\[ s_2 \]
Example of construction
Example of construction

\[
\begin{align*}
Eve\text{-state:} & \\
\langle a, b, a \rangle & \rightarrow s_0, \langle a, b, a \rangle & \rightarrow s_0, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_2, \langle b, a, a \rangle & \rightarrow s_1, \langle b, a, a \rangle & \rightarrow s_1, \langle b, a, a \rangle & \rightarrow s_1, \langle b, a, a \rangle & \rightarrow s_1, \langle b, a, a \rangle & \rightarrow s_1, \langle b, a, a \rangle & \rightarrow s_1, \langle b, a, a \rangle & \rightarrow (0,1,0), \perp & \{ (1,1,0) \}, A_1 & \{ (2,0,0) \}, A_2 & \emptyset, A_3 \\
\langle a, b, a \rangle & \rightarrow s_0, \langle a, b, a \rangle & \rightarrow s_0, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow (1,1,0), \perp & \{ (0,1,0) \}, A_1 & \{ (1,1,0) \}, A_2 & \emptyset, A_3 \\
\langle a, b, a \rangle & \rightarrow s_0, \langle a, b, a \rangle & \rightarrow s_0, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow s_1, \langle a, b, a \rangle & \rightarrow (0,0,0), (1,0,3) \}, A_1 & \emptyset, A_2 & \{ (1,0,0), (3,0,1) \}, A_3 \\
\end{align*}
\]
Properties of the epistemic game

- To every history $H$ in the epistemic game, one can associate sets:
  - $concrete_\bot(H)$: at most one concrete real history (if no deviation)
  - $concrete_A(H)$: all possible $A$-deviations
  - $concrete(H) = \bigcup_{A \in \text{Agt} \cup \{\bot\}} concrete_A(H)$
Properties of the epistemic game

- To every history $H$ in the epistemic game, one can associate sets
  - $concrete_{\perp}(H)$: at most one concrete real history (if no deviation)
  - $concrete_A(H)$: all possible $A$-deviations
  - $concrete(H) = \bigcup_{A \in Agt \cup \{\perp\}} concrete_A(H)$

$H$ history in the epistemic game. For every $h_1 \neq h_2 \in concrete(H)$,

$$h_1 \sim_A h_2 \iff h_1, h_2 \notin concrete_A(H)$$
Properties of the epistemic game (cont’d)

Winning condition for Eve

A strategy $\sigma_{\text{Eve}}$ is said winning for payoff $p \in \mathbb{R}^{\text{Agt}}$ from $s_0$ whenever payoff($\text{concrete}_\bot(\text{out}_\bot(\sigma_{\text{Eve}}, s_0))$) = $p$, and for every $R \in \text{out}(\sigma_{\text{Eve}}, s_0)$, for every $A \in \text{Agt}$, for every $\rho \in \text{concrete}_A(R)$, payoff$_A(\rho) \leq p_A$. 

Proposition

There is a Nash equilibrium in $G$ with payoff $p$ from $v_0$ if and only if Eve has a winning strategy for $p$ in $E_G$ from $s_0$. 

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Winning condition for Eve (publicly visible payoffs)

A strategy $\sigma_{Eve}$ is said winning for $p$ from $s_0$ whenever 
payoff$'$(out$_\bot$($\sigma_{Eve}, s_0$)) = $p$, and for every $R \in \text{out}(\sigma_{Eve}, s_0)$, for every $A \in \text{susp}(R)$, payoff$'_A(R) \leq p_A$. 

Proposition

There is a Nash equilibrium in $G$ with payoff $p$ from $v_0$ if and only if Eve has a winning strategy for $p$ in $E_G$ from $s_0$. 

Properties of the epistemic game (cont’d)

**Winning condition for Eve (publicly visible payoffs)**

A strategy $\sigma_{\text{Eve}}$ is said winning for $p$ from $s_0$ whenever $\text{payoff}'(\text{out}_\bot(\sigma_{\text{Eve}}, s_0)) = p$, and for every $R \in \text{out}(\sigma_{\text{Eve}}, s_0)$, for every $A \in \text{susp}(R)$, $\text{payoff}'_A(R) \leq p_A$.

**Proposition**

There is a Nash equilibrium in $\mathcal{G}$ with payoff $p$ from $v_0$ if and only if Eve has a winning strategy for $p$ in $\mathcal{E}_\mathcal{G}$ from $s_0$. 
Application to $\omega$-regular objectives

Player $A_1$ loses along that play

$\psi_A$: objective of player $A$
Application to $\omega$-regular objectives

$\psi_A$: objective of player $A$

Player $A_1$ should lose

Player $A_1$ loses along that play
Application to $\omega$-regular objectives

$\psi_A$: objective of player $A$

Coalition $\{A_2, A_3\}$ prevents $A_1$ from winning

Player $A_1$ loses along that play
Application to $\omega$-regular objectives

$\psi_A$: objective of player $A$

Players $A_1, A_2$ lose along that $\perp$-play

$A_3$ will not deviate
his aim is to potentially punish both $A_1$ and $A_2$

$A_2$ and $A_3$ will not deviate
their aim is to punish $A_1$
Application to $\omega$-regular objectives (cont’d)

- This amounts to solving two-player turn-based games with generalized (i.e. conjunctions of) $\omega$-regular objectives.

[CDHR07] Chatterjee, Doyen, Henzinger, Raskin. Algorithms for $\omega$-regular games with imperfect information (LMCS)
Application to $\omega$-regular objectives (cont’d)

- This amounts to solving two-player turn-based games with generalized (i.e. conjunctions of) $\omega$-regular objectives.

**Theorem**

One can decide the (constrained) existence of a Nash equilibrium in a game with public signal and publicly visible payoff functions associated with parity conditions in EXPSPACE. It is EXPTIME-hard.

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Application to $\omega$-regular objectives (cont’d)

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One can decide the (constrained) existence of a Nash equilibrium in a game with public signal and publicly visible payoff functions associated with parity conditions in EXPSPACE. It is EXPTIME-hard.

- EXPTIME-hardness: same proof as for the distributed synthesis problem [CDHR07]

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Application to \(\omega\)-regular objectives (cont’d)

- This amounts to solving two-player turn-based games with generalized (i.e. conjunctions of) \(\omega\)-regular objectives

**Theorem**

One can decide the (constrained) existence of a Nash equilibrium in a game with public signal and publicly visible payoff functions associated with parity conditions in EXPSPACE. It is EXPTIME-hard.

- EXPTIME-hardness: same proof as for the distributed synthesis problem [CDHR07]
- Can be extended to (finite) preorders over such objectives

[CDHR07] Chatterjee, Doyen, Henzinger, Raskin. Algorithms for \(\omega\)-regular games with imperfect information (LMCS)
Application to $\omega$-regular objectives (cont’d)

- This amounts to solving two-player turn-based games with generalized (i.e. conjunctions of) $\omega$-regular objectives

**Theorem**

One can decide the (constrained) existence of a Nash equilibrium in a game with public signal and publicly visible payoff functions associated with parity conditions in EXPSPACE. It is EXPTIME-hard.

- EXPTIME-hardness: same proof as for the distributed synthesis problem [CDHR07]
- Can be extended to (finite) preorders over such objectives
- May even probably be extended to privately visible or invisible payoff functions (needs to be checked)

[CDHR07] Chatterjee, Doyen, Henzinger, Raskin. Algorithms for $\omega$-regular games with imperfect information (LMCS)
Application to (publicly visible) mean-payoff payoff functions

The mean-payoff payoff publicly visible functions can be used in the epistemic game, and the winning condition for Eve rewrites as:

A strategy for Eve is said winning for payoff $p \in \mathbb{R}^{\text{Agt}}$ from $s_0$ whenever $\text{MP}(\text{out}_\perp(\sigma_{\text{Eve}}, s_0)) = p$, and for every $\rho \in \text{out}(\sigma_{\text{Eve}}, s_0)$, for every $A \in \text{susp}(\rho)$, $\text{MP}_A(\rho) \leq p_A$. 

Inspired by [Bre16], we can reduce the constrained existence problem of a Nash equilibrium to the polyhedron problem [BR15].
Application to (publicly visible) mean-payoff payoff functions

The mean-payoff payoff publicly visible functions can be used in the epistemic game, and the winning condition for Eve rewrites as:

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Inspired by [Bre16], we can reduce the constrained existence problem of a Nash equilibrium to the polyhedron problem [BR15].
In a multi-dimensional mean-payoff two-player turn-based game, the polyhedron problem asks, given a polyhedron $\pi$, whether there is a strategy for Eve which ensures a payoff vector which belongs to $\pi$.

\[
\text{value}_G = \{ v \in \mathbb{R}^d \mid \exists \sigma \forall \rho \in \text{out}(\sigma), \forall i, \text{MP}_i(\rho) \geq v_i \}\]
The polyhedron problem

In a multi-dimensional mean-payoff two-player turn-based game, the polyhedron problem asks, given a polyhedron $\pi$, whether there is a strategy for Eve which ensures a payoff vector which belongs to $\pi$.

- [BR15]: if there is a solution, there is one solution with a payoff of polynomial size.
The polyhedron problem

In a multi-dimensional mean-payoff two-player turn-based game, the polyhedron problem asks, given a polyhedron $\pi$, whether there is a strategy for Eve which ensures a payoff vector which belongs to $\pi$.

- **[BR15]**: if there is a solution, there is one solution with a payoff of polynomial size.
- **[BR15]**: the polyhedron problem is $\Sigma_2^P$-complete ($\Sigma_2^P = \text{NP}^{\text{NP}}$)
The polyhedron problem

In a multi-dimensional mean-payoff two-player turn-based game, the polyhedron problem asks, given a polyhedron $\pi$, whether there is a strategy for Eve which ensures a payoff vector which belongs to $\pi$.

- [BR15]: if there is a solution, there is one solution with a payoff of polynomial size.
- [BR15]: the polyhedron problem is $\Sigma_2^P$-complete ($\Sigma_2^P = \text{NP}^{\text{NP}}$)

Theorem

One can decide the (constrained) existence of a Nash equilibrium in a game with public signal and publicly visible mean-payoff payoff functions, in $\text{NP}$, with a co$\text{NEXPTIME}$ oracle. This in particular can be solved in $\text{EXPSPACE}$. It is $\text{EXPTIME}$-hard.
Application to mean-payoff payoff functions (cont’d)

Original weight functions: $w_A$

New weight functions: $u_i$, $u_{N+i}$, $u_{2N+i}$

$$\beta u_i = 2 w_A$$

$$u_i = u_{N+i} = 2 W$$

$$u_{2N+i} = -2 w_A$$

if $A_i$ suspect

otherwise
Application to mean-payoff payoff functions (cont’d)

Original weight functions: $w_{A_i}$
New weight functions: $u_i, u_{N+i}, u_{2N+i}$
Application to mean-payoff payoff functions (cont’d)

Original weight functions: $w_{A_i}$
New weight functions: $u_i, u_{N+i}, u_{2N+i}$

\[
\begin{align*}
\begin{cases}
  u_i &= 2w_{A_i} \\
  u_{N+i} &= u_{2N+i} = -2w_{A_i}
\end{cases}
\end{align*}
\]
Application to mean-payoff payoff functions (cont’d)

Original weight functions: $w_{A_i}$
New weight functions: $u_i, u_{N+i}, u_{2N+i}$

\[
\begin{align*}
\{ u_i &= 2w_{A_i} \\
u_{N+i} &= u_{2N+i} = -2w_{A_i} \}
\end{align*}
\]

\[
\begin{align*}
\{ u_i &= u_{N+i} = 2W \\
u_{2N+i} &= \begin{cases} -2w_{A_i} & \text{if } A_i \text{ suspect} \\ 2W & \text{otherwise} \end{cases} \}
\end{align*}
\]
There is a Nash equilibrium in the original game with payoff $p$ if and only if there is a strategy for Eve in the epistemic game such that for every outcome $\rho$, for every $1 \leq i \leq N$,

$$\begin{align*}
\text{MP}_{u_i}(\rho) & \geq p_{A_i} \\
\text{MP}_{u_{N+i}}(\rho) & \geq -p_{A_i} \\
\text{MP}_{u_{2N+i}}(\rho) & \geq -p_{A_i}
\end{align*}$$
Application to mean-payoff payoff functions (cont’d)

There is a Nash equilibrium in the original game with a payoff $\nu \leq p \leq \nu'$ ($\nu$ and $\nu'$ are fixed thresholds) if and only if there is a strategy for Eve in the epistemic game solving the polyhedron problem for the polyhedron

$$\bigwedge_{1 \leq i \leq N} \left( x_i = -x_{N+i} = -x_{2N+i} \right) \land \bigwedge_{1 \leq i \leq N} \left( \nu_i \leq x_i \leq \nu'_i \right)$$

Original weight functions: $w_{A_i}$
New weight functions: $u_i, u_{N+i}, u_{2N+i}$
Conclusion

We have:

- proposed a framework for games over graphs with a public signal monitoring
  Note: framework inspired by [Tom98]
- proposed an abstraction called the epistemic game abstraction, which allows to characterize Nash equilibria in the original game
- used it to propose several decidability results.

Conclusion

We have:

- proposed a framework for games over graphs with a public signal monitoring
  Note: framework inspired by [Tom98]
- proposed an abstraction called the epistemic game abstraction, which allows to characterize Nash equilibria in the original game
- used it to propose several decidability results.

We want to:

- work out the precise complexities
- understand whether one can extend the approach to other communication architectures ([RT98]??)
- understand whether the current approach is specific to Nash equilibria or if it can be extended to more expressive languages (like fragments of Strategy Logic)