Staying Alive as Cheaply as Possible

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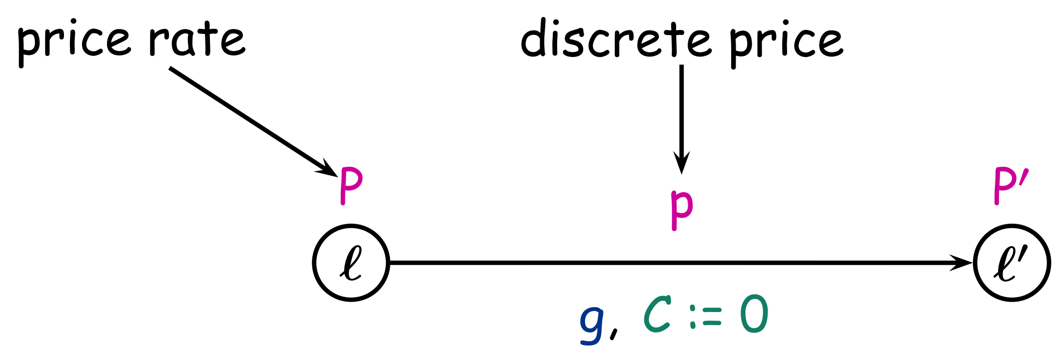
Model of Priced Timed Automata



[BFH+01a,BFH+01b,LBB+01,ALTP01]

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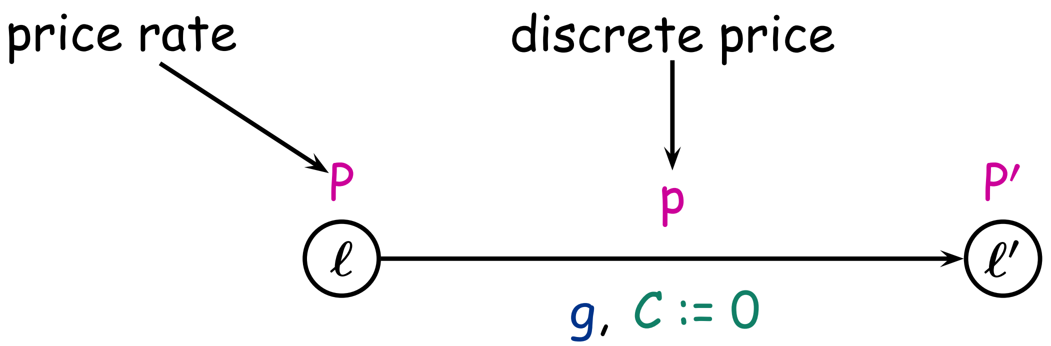

✔ **previous problem:** reachability problem with an optimization criterium on the price

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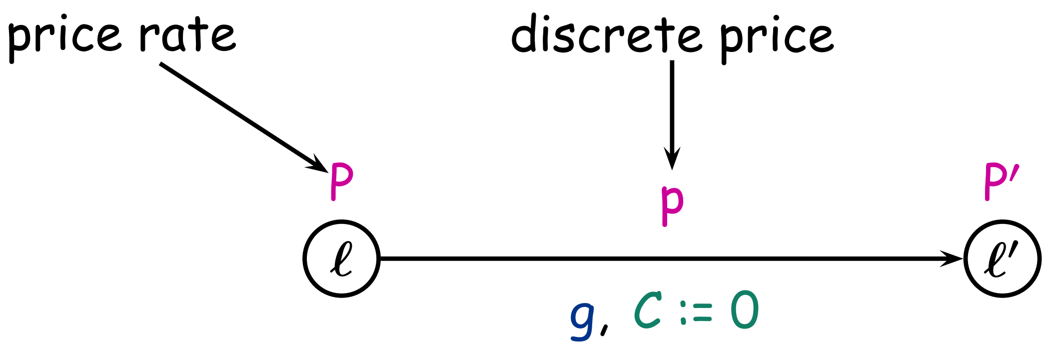

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- ✔ what if infinite paths?
  - price per unit of time?
  - price per transition?
  - ...

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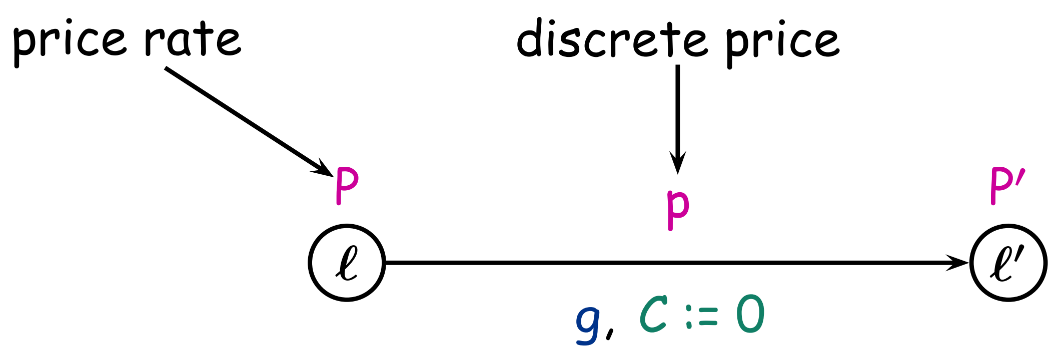

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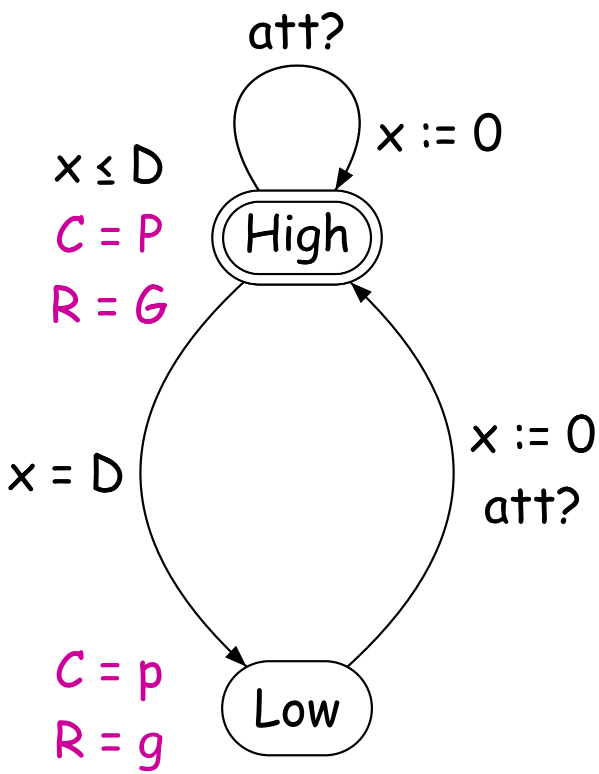
➜ optimal stationary behaviours

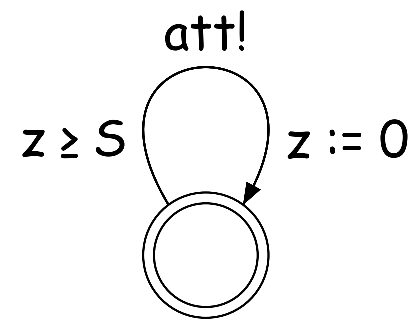
[BFH+01a,BFH+01b,LBB+01,ALTP01]

An Example

A production system:



Single machine M(D, G, P, g, p)



Operator O(S)

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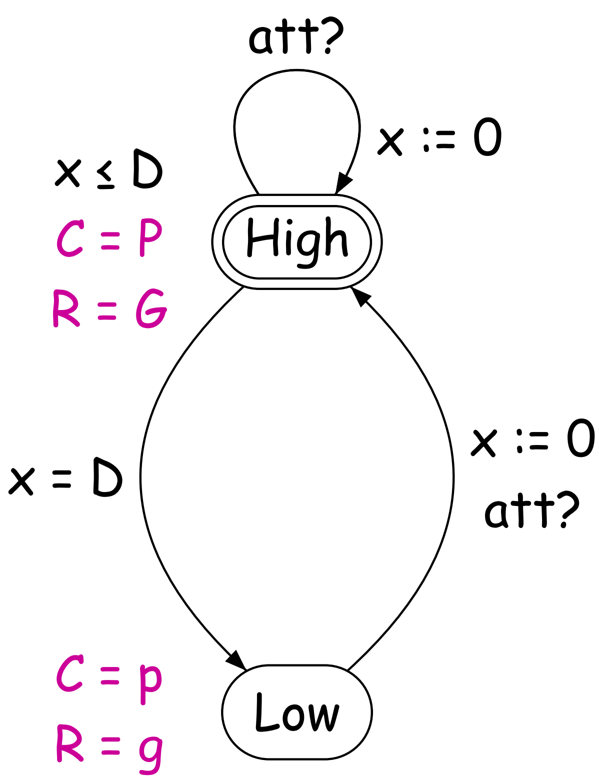
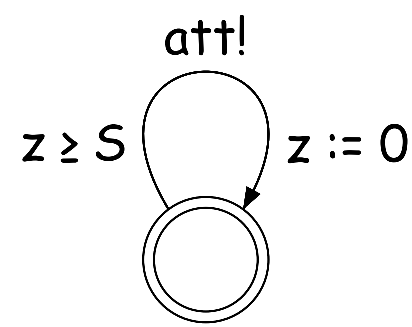


Single machine M(D,G,P,g,p)

Question: How to minimize

$$\lim_{n \to +\infty} \frac{\text{accumulated cost}(n)}{\text{accumulated reward}(n)} ?$$



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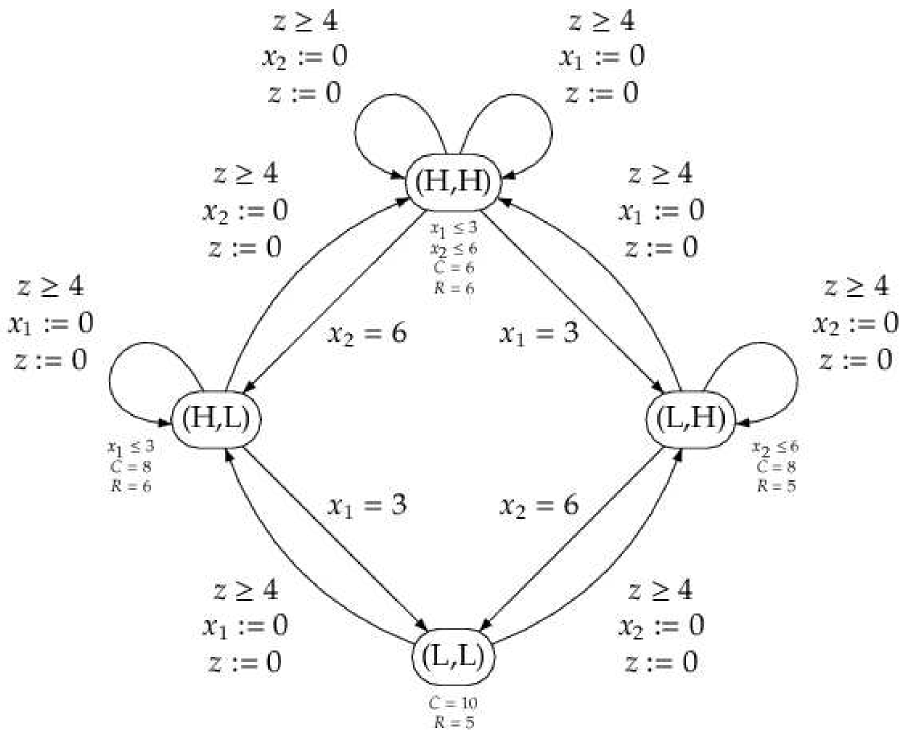
An Example (cont'd)

Two machines $M_1(D = 3, P = 3, G = 4, p = 5, g = 3)$, $M_2(D = 6, P = 3, G = 2, p = 5, g = 2)$ and an Operator $O(4)$.



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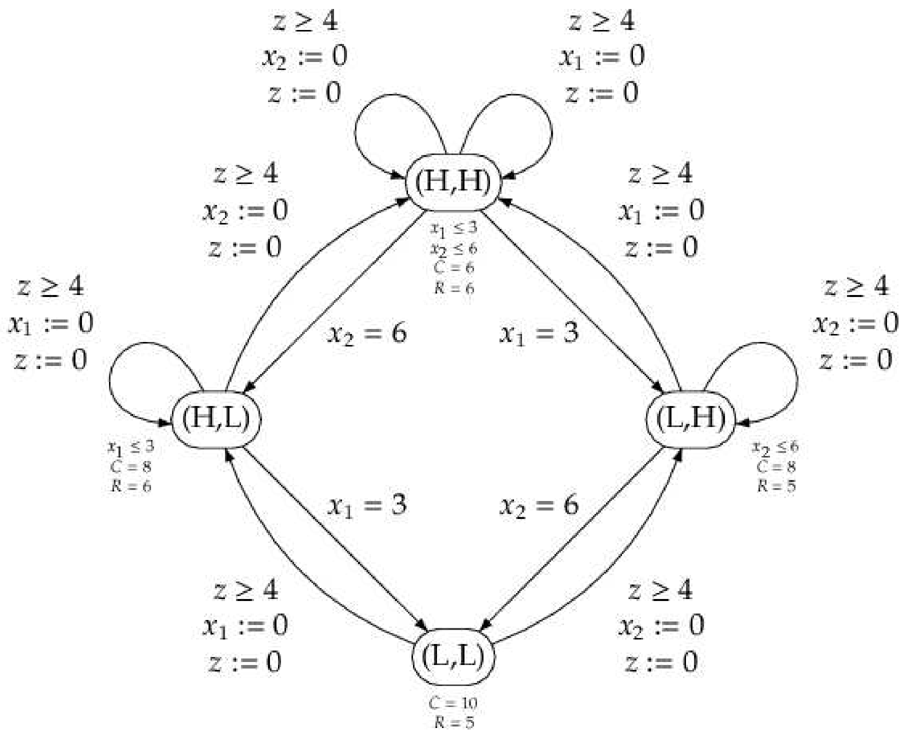

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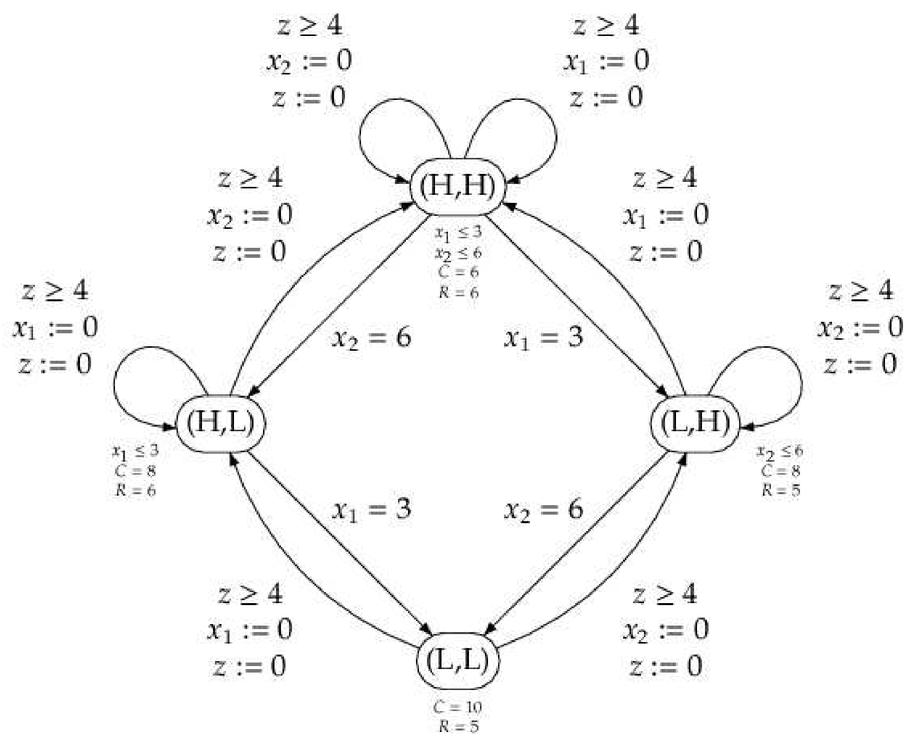

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$\xrightarrow{18,18}$ $((L, H), x_1 = x_2 = z = 3)$

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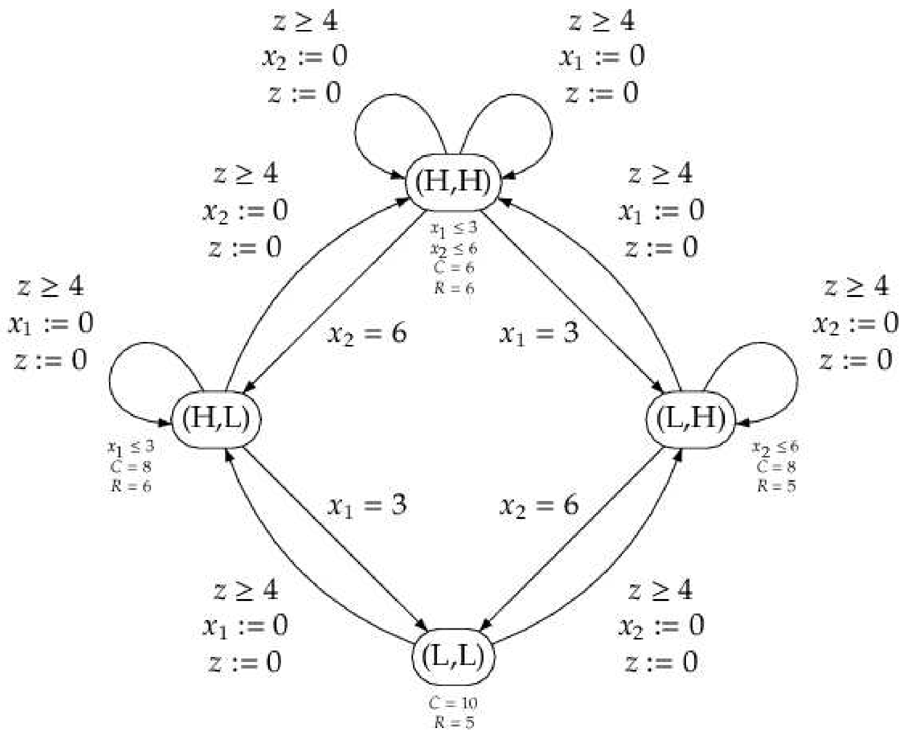

$$((H,H), x_1 = x_2 = z = 0)$$
$$\xrightarrow{18,18} ((L,H), x_1 = x_2 = z = 3)$$
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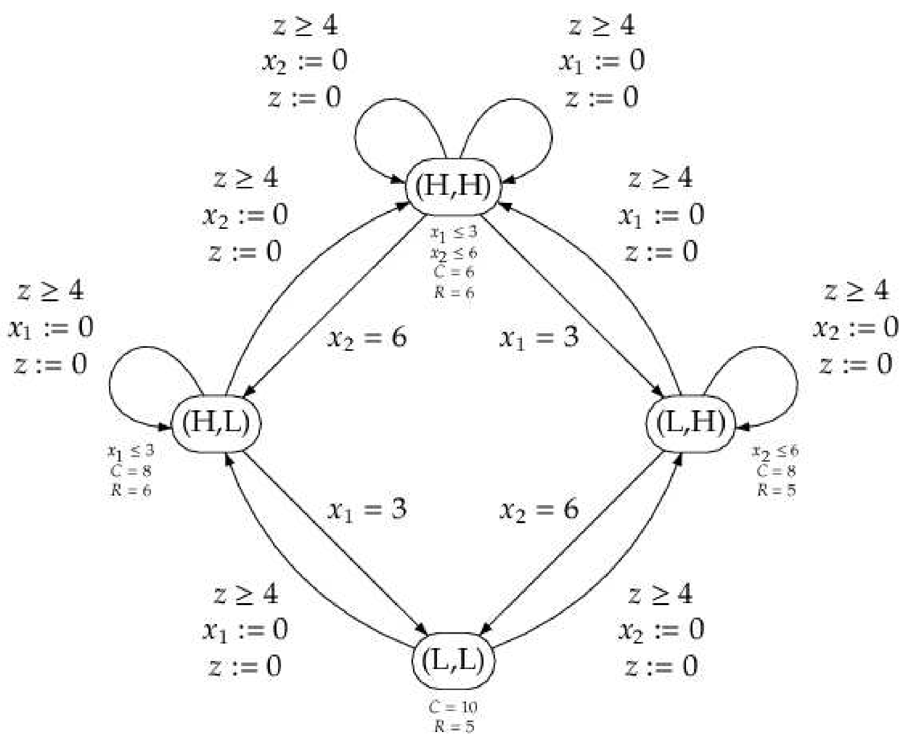

$$
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 & ((H, H), x_1 = x_2 = z = 0) \\
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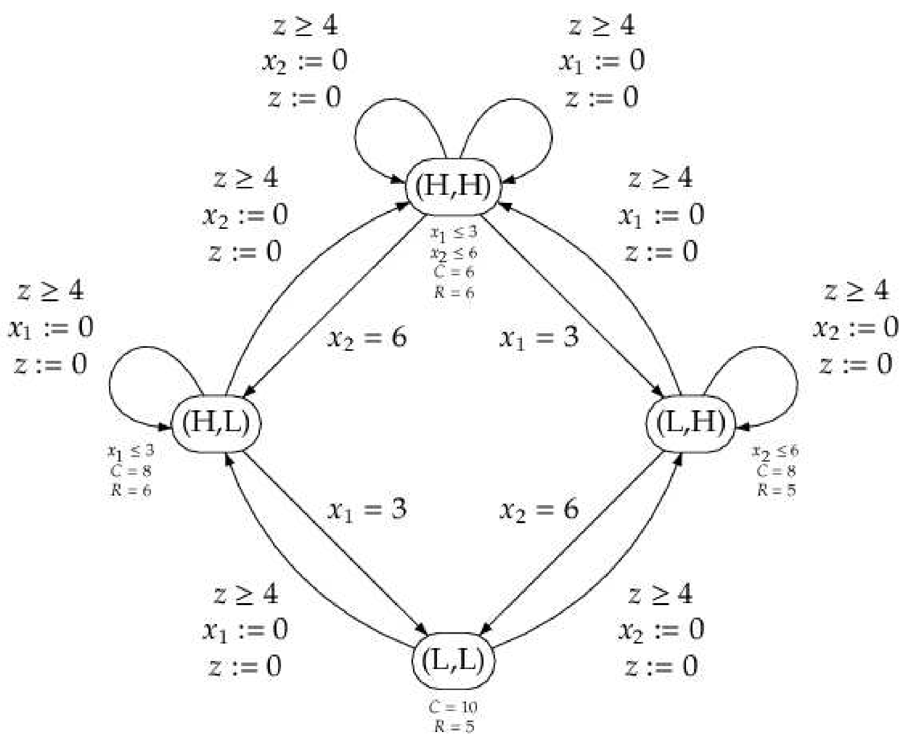

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\xrightarrow{8,6} & ((L,L), x_1 = z = 3, x_2 = 7) & \\
\xrightarrow{10,5} & ((L,L), x_1 = z = 4, x_2 = 8) & \\
\longrightarrow & ((H,L), x_1 = z = 0, x_2 = 8) & \\
\xrightarrow{24,18} & ((L,L), x_1 = z = 3, x_2 = 11) & \\
\xrightarrow{10,5} & ((L,L), x_1 = z = 4, x_2 = 12) & \\
\longrightarrow & ((L,H), x_1 = 4, x_2 = z = 0) & \\
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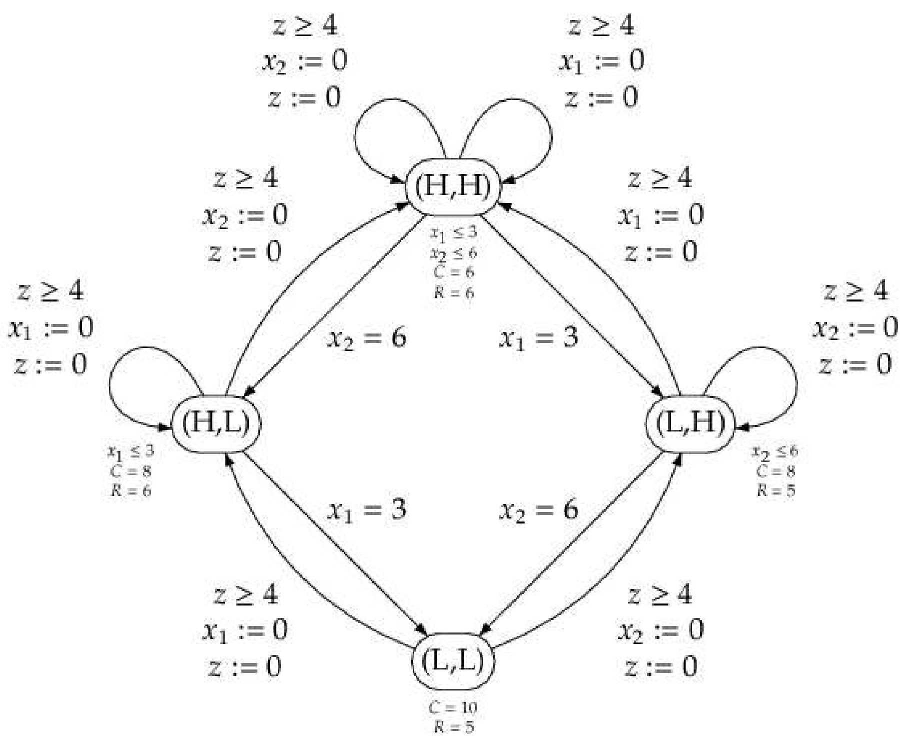

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$$\text{limit } \frac{\text{cost}}{\text{reward}} = \frac{96}{66} \simeq 1{,}455$$

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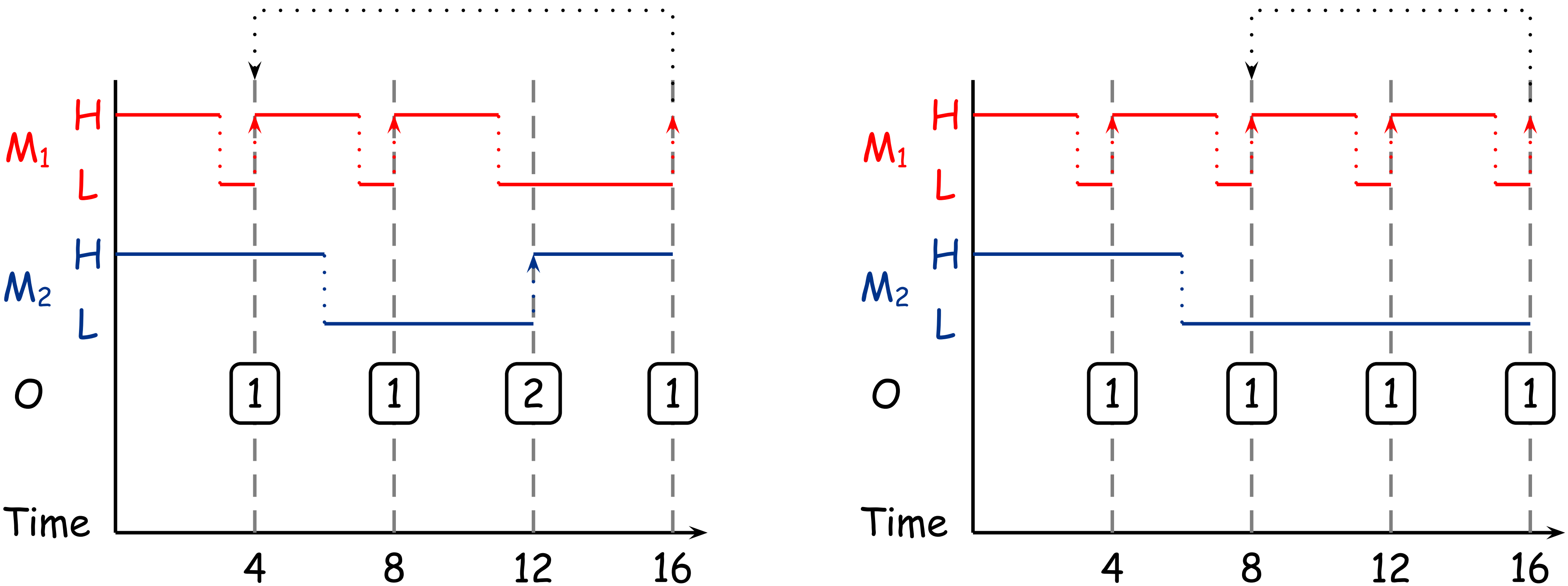
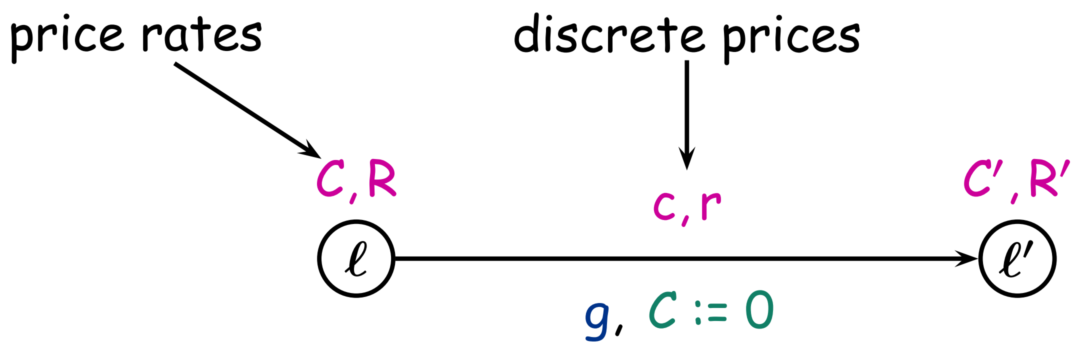


(a) Schedule with ratio 1,455

(b) Schedule with ratio 1,478

Our Model: Double Priced Timed Automata



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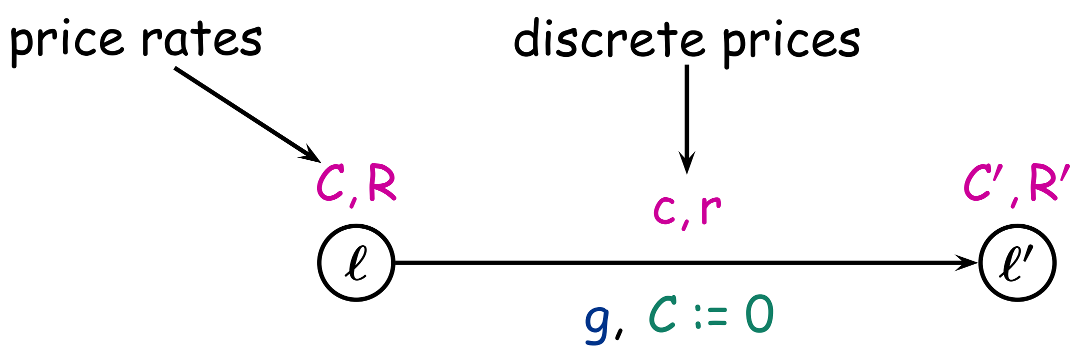

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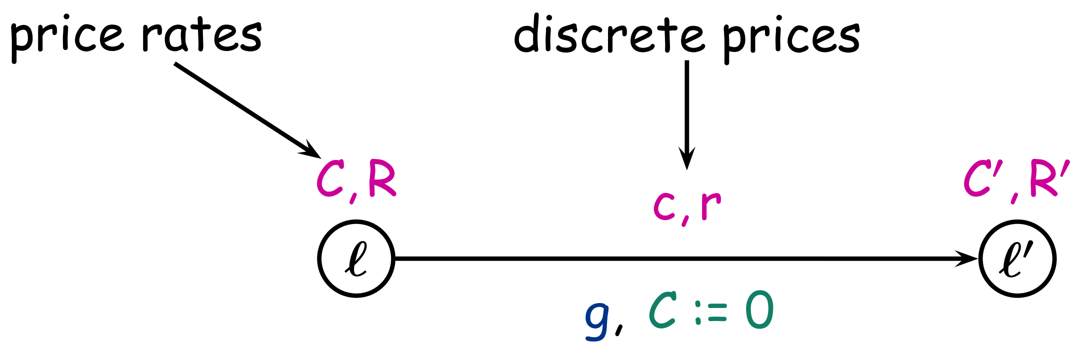

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notation: μ^*

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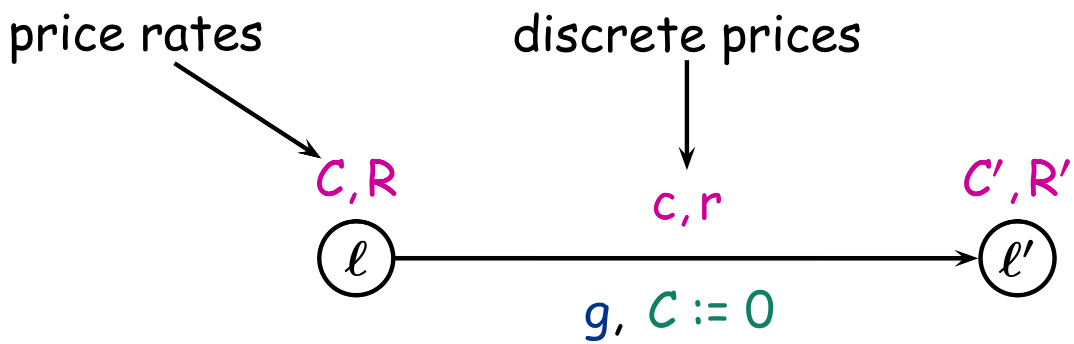

hyp: positive strongly diverging reward

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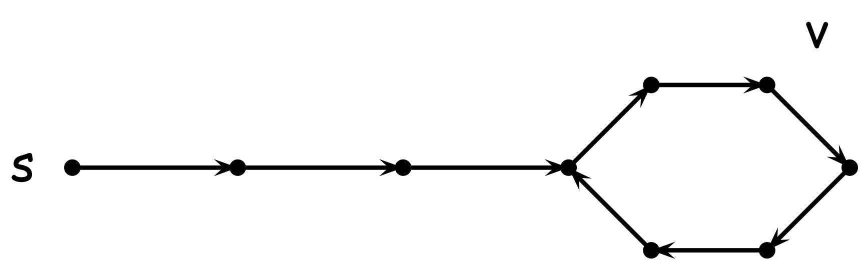
The Discrete Case

Karp's and Howard's theorems/algorithms: optimal paths are cycles.

$$\mu^* = \min_{v \in V} \max_{0 \le k \le n-1} \frac{\delta_n(s,v) - \delta_k(s,v)}{n-k}$$

[Karp78,DG98,DIG99]



Back to the Timed Framework

Idea: reduction to the discrete case

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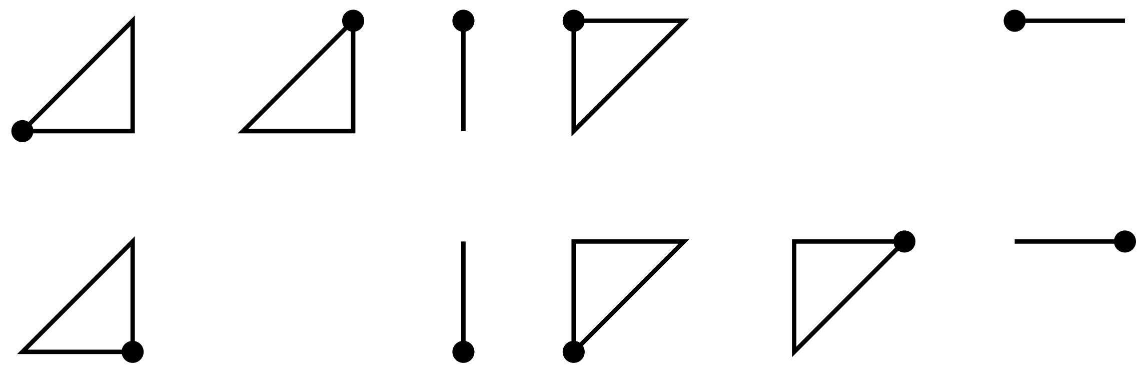

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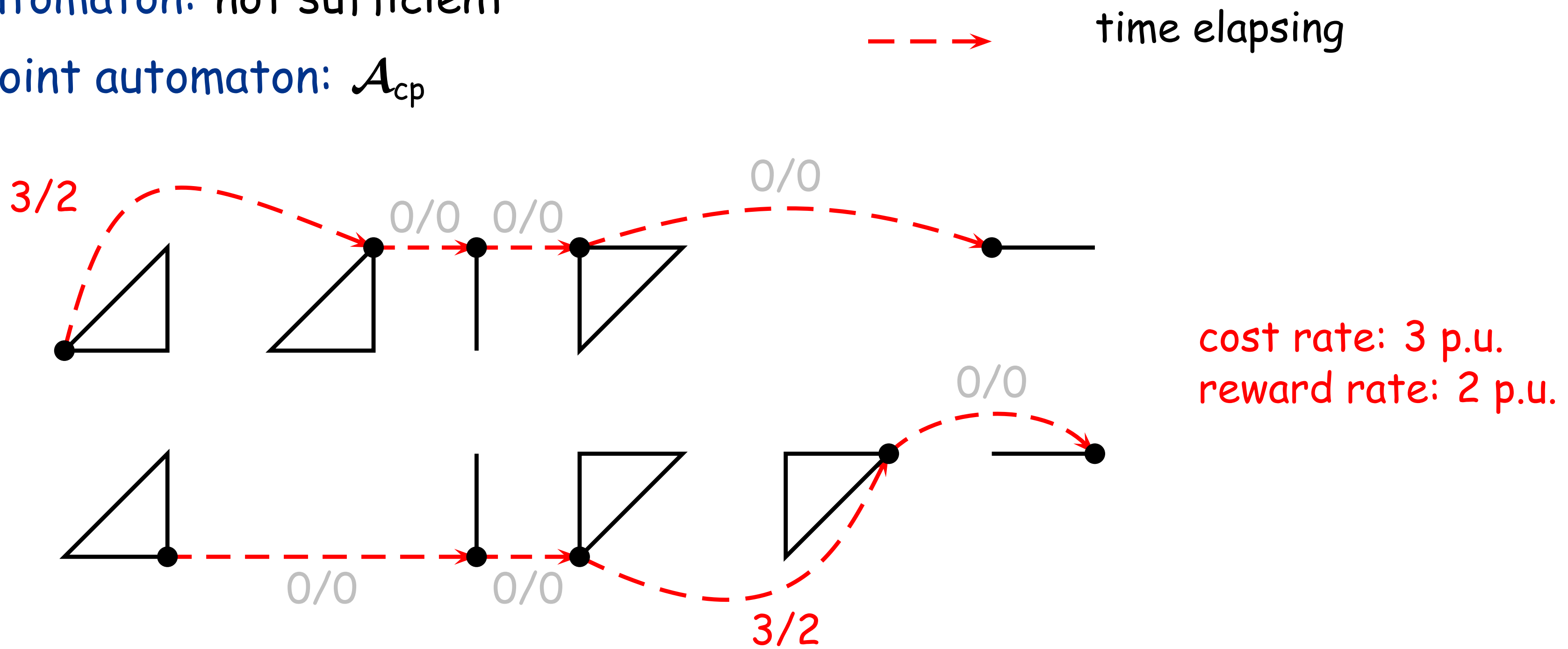

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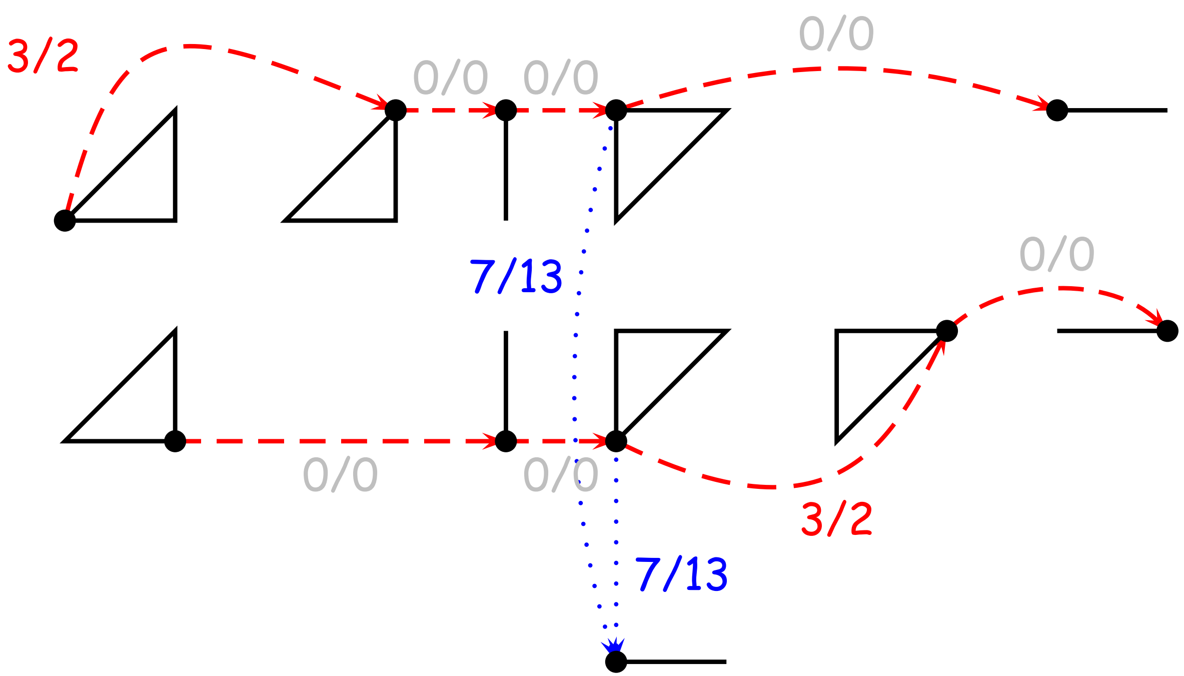
reset to 0

time elapsing



cost rate: 3 p.u.
reward rate: 2 p.u.
discrete cost: 7
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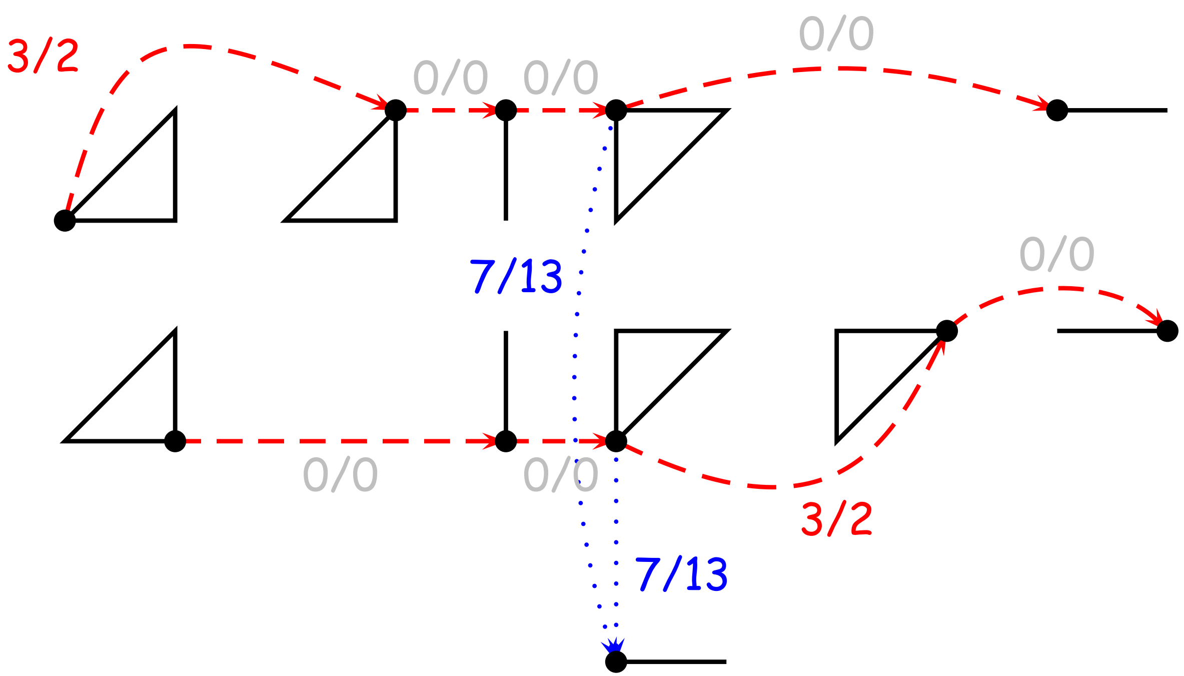

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Aim: prove that $\mu^*(\mathcal{A}_{cp}) = \mu^*(\mathcal{A})$

From Timed to Discrete Behaviours (1)

- ✔ **Finite behaviours:** based on the following property

Lemma. Let Z be a bounded zone and f be a function

$$f : (t_1, \ldots, t_n) \mapsto \frac{\sum_{i=1}^{n} c_i t_i + c}{\sum_{i=1}^{n} r_i t_i + r}$$

well-defined on $\overline{Z}$. Then $\inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

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➜ for any finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{cp}$ such that

$$\text{ratio}(\Pi) \leq \text{ratio}(\pi)$$

[Π is a "corner-point projection" of π]

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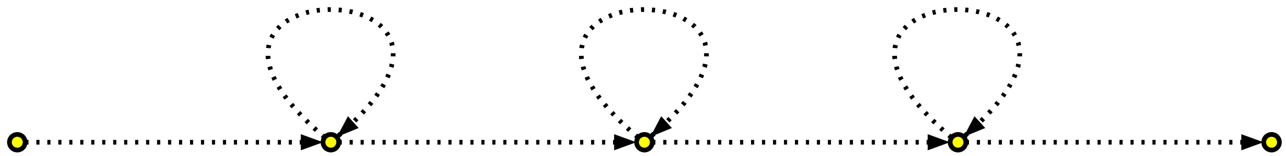
☞ optimal finite behaviours are not prefix-closed

From Timed to Discrete Behaviours (2)

✔ **Infinite behaviours:** decompose each sufficiently long projection into cycles



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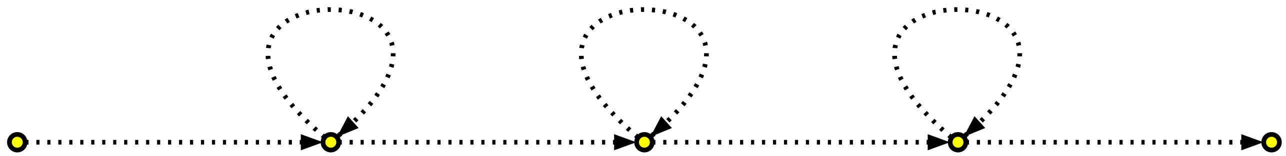

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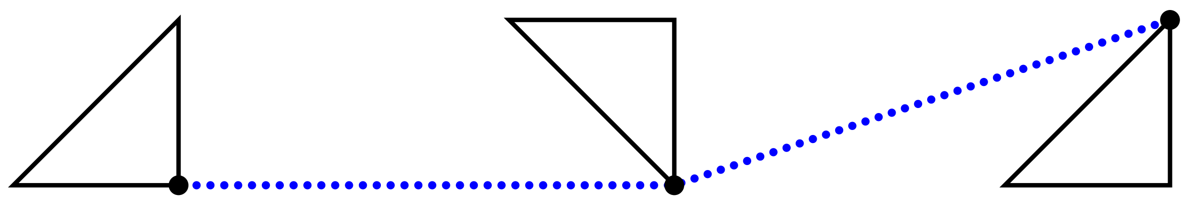
➜ the optimal cycle of $\mathcal{A}_{cp}$ is better than any infinite path of $\mathcal{A}$

From Discrete to Timed Behaviours

Approximation of abstract paths:



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From Discrete to Timed Behaviours

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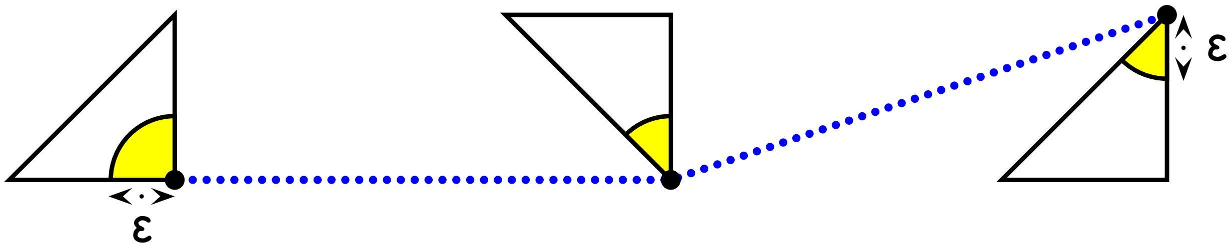

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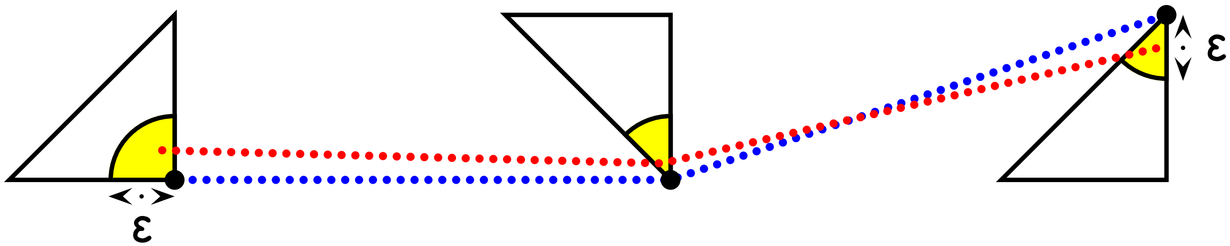

For any path Π of $\mathcal{A}_{cp}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

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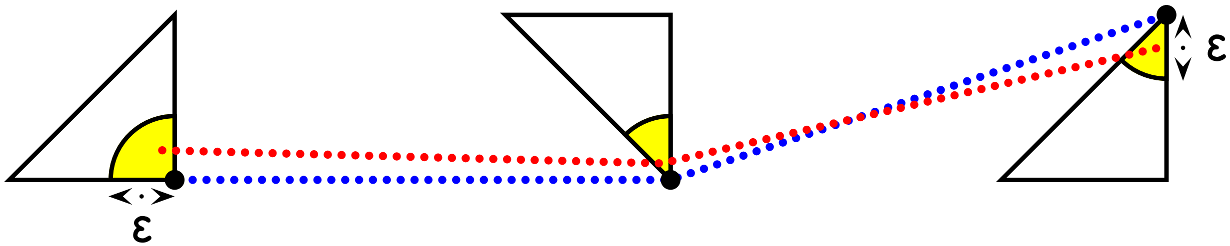

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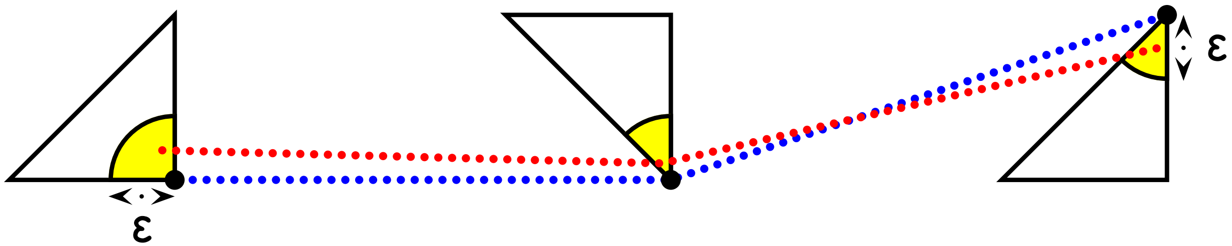

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For every $\eta > 0$, there exists $\varepsilon > 0$ s.t.

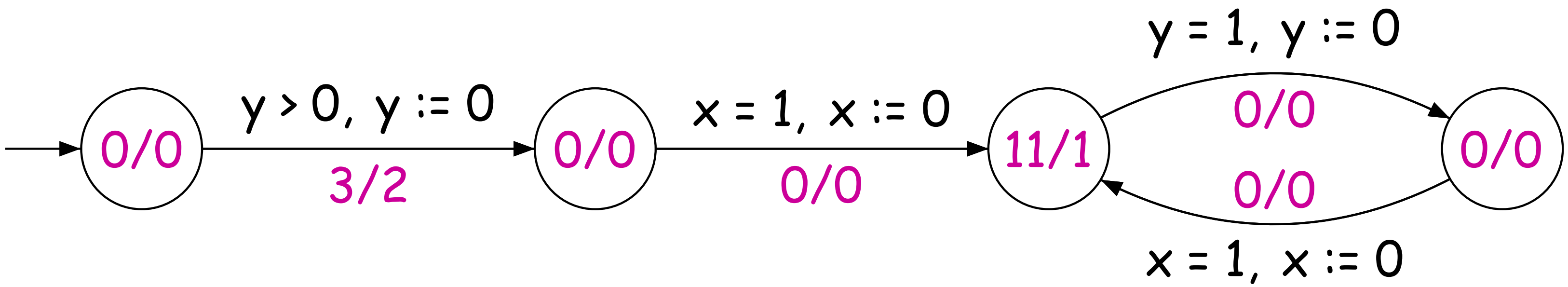
$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon \Rightarrow |\text{ratio}(\Pi) - \text{ratio}(\pi_\varepsilon)| < \eta$$

Main Theorem

Theorem. Under the positive strongly diverging reward hypothesis, optimal infinite schedules in timed automata are computable.

➜ **Complexity:** PSPACE-complete

Without the Hypothesis, What's Wrong?



$\pi_{d,n}$: path s.t. the first transition is taken at date d and the loop is taken n times.

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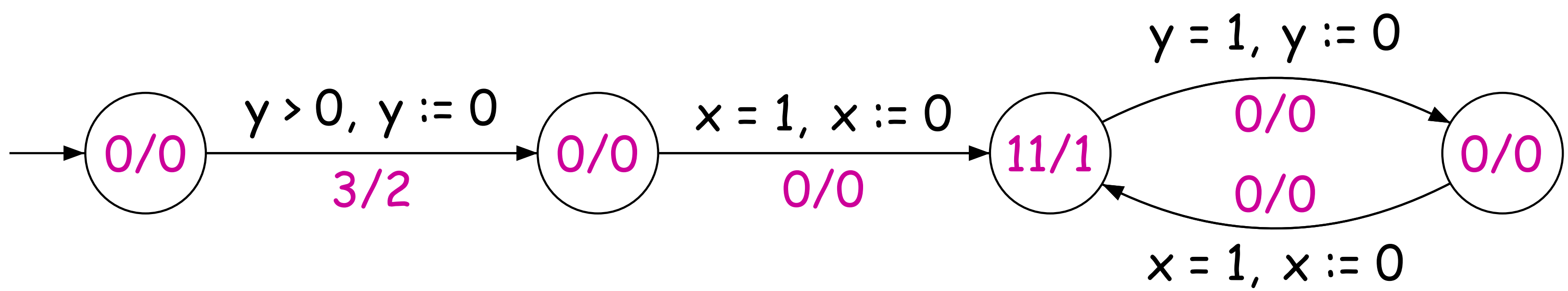

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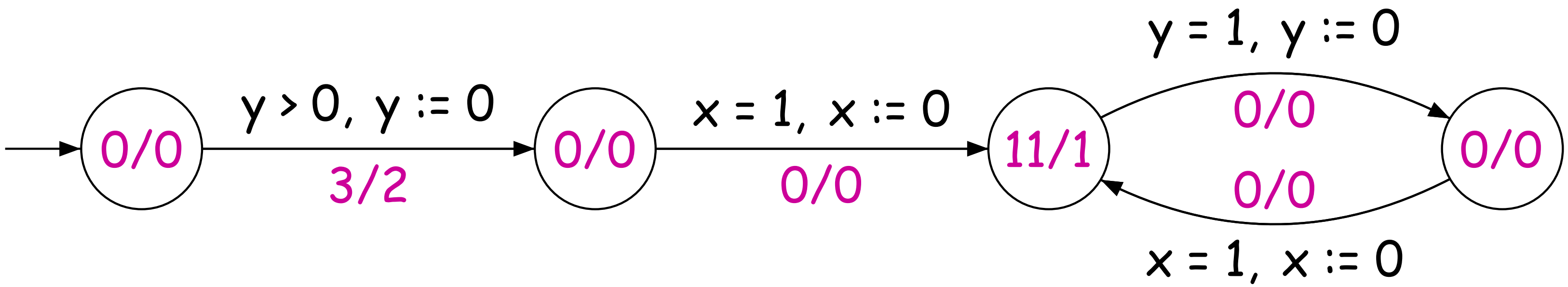

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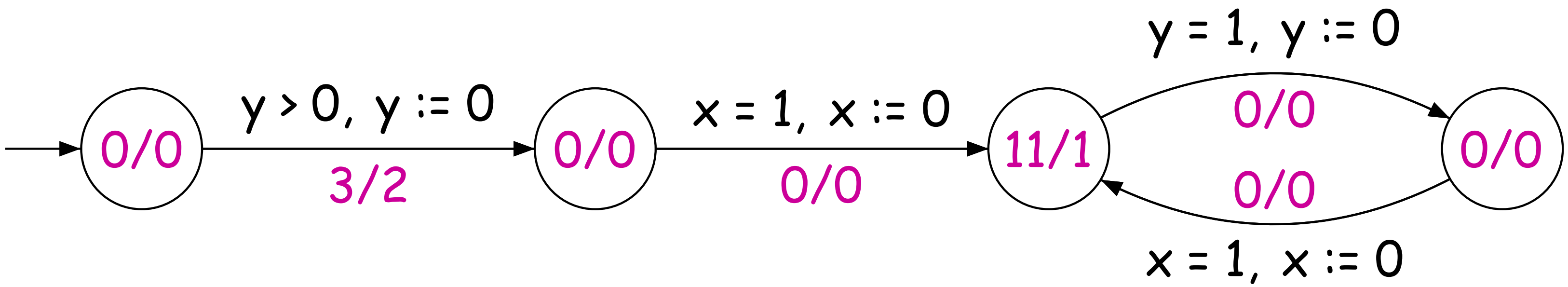

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reward($\pi_{d,n}$) = 2 + d.n and cost($\pi_{d,n}$) = 3 + 11d.n

For any real infinite path π_d, ratio(π_d) = 11 but ratio(π_0) = $\frac{3}{2}$.

➜ this automaton is not strongly reward diverging

Conclusion & Further Work

Conclusion

- ✔ computability of optimal infinite paths in double priced timed automata

Further work

- ✔ implementation
- ✔ **control**, **games**: what if an opponent? what if uncontrollable actions?

☞ the computation is not modular...

untimed case: [ZP96,BSV04]