On the value problem in weighted timed games

Patricia Bouyer-Decitre

LSV, CNRS & ENS Cachan, France

Joint work with Samy Jaziri and Nicolas Markey
An example: The task graph scheduling problem

Compute \( D \times (C \times (A+B)) +(A+B)+(C \times D) \) using two processors:

**\( P_1 \) (fast):**

<table>
<thead>
<tr>
<th>Time</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>2 picoseconds</td>
</tr>
<tr>
<td>×</td>
<td>3 picoseconds</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Energy</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>idle</td>
<td>10 Watt</td>
</tr>
<tr>
<td>in use</td>
<td>90 Watts</td>
</tr>
</tbody>
</table>

**\( P_2 \) (slow):**

<table>
<thead>
<tr>
<th>Time</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>5 picoseconds</td>
</tr>
<tr>
<td>×</td>
<td>7 picoseconds</td>
</tr>
</tbody>
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</tr>
</thead>
<tbody>
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<td>20 Watts</td>
</tr>
<tr>
<td>in use</td>
<td>30 Watts</td>
</tr>
</tbody>
</table>

**Diagram:**

```
A ———+——— B ———+——— C ———+——— D
       |      |      |      |
       ×——— T1 ———×——— T2 ———×——— T3 ———×——— T4
       |      |      |      |
       ×——— T5 ———×——— T6 ———×——— T7
```
An example: The task graph scheduling problem

Compute \( D \times (C \times (A + B)) + (A + B) + (C \times D) \) using two processors:

\[
P_1 \text{ (fast)}:
\begin{array}{|c|c|}
\hline
\text{time} & 2 \text{ picoseconds} \\
\hline
\times & 3 \text{ picoseconds} \\
\hline
\end{array}
\]

\[
P_2 \text{ (slow)}:
\begin{array}{|c|c|}
\hline
\text{time} & 5 \text{ picoseconds} \\
\hline
\times & 7 \text{ picoseconds} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{energy} & \text{idle} 10 \text{ Watt} \\
\hline
\text{in use} & 90 \text{ Watts} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{energy} & \text{idle} 20 \text{ Watts} \\
\hline
\text{in use} & 30 \text{ Watts} \\
\hline
\end{array}
\]
An example: The task graph scheduling problem

Compute \( D \times (C \times (A+B)) + (A+B) + (C \times D) \) using two processors:

\( P_1 \) (fast):

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\( P_2 \) (slow):

<table>
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<th>5 picoseconds</th>
<th>7 picoseconds</th>
</tr>
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<td>20 Watts</td>
<td></td>
</tr>
<tr>
<td>in use</td>
<td>30 Watts</td>
<td></td>
</tr>
</tbody>
</table>

\( T_1 + T_2 \times T_3 - T_4 + T_5 \times T_6 \)
An example: The task graph scheduling problem

Compute \( D \times (C \times (A+B)) + (A+B) + (C \times D) \) using two processors:

\( P_1 \) (fast):

- **Time**
  - +: 2 picoseconds
  - ×: 3 picoseconds

- **Energy**
  - Idle: 10 Watt
  - In use: 90 Watts

\( P_2 \) (slow):

- **Time**
  - +: 5 picoseconds
  - ×: 7 picoseconds

- **Energy**
  - Idle: 20 Watts
  - In use: 30 Watts

Diagram:

- Task graph with nodes labeled A to D
- Edges indicating addition (+) and multiplication (×)
- Time and energy values for each task
- Schedule diagrams for \( P_1 \) and \( P_2 \) showing task durations and overall times for Sch1, Sch2, and Sch3
The model of timed automata
The model of timed automata

safe \xrightarrow{23} \text{safe} \quad \text{problem, } x:=0 \quad \text{alarm} \xrightarrow{15.6} \text{alarm} \quad \text{delayed, } y:=0 \quad \text{failsafe}

\begin{align*}
 x & : 0 \quad 23 \quad 0 \quad 15.6 \quad 15.6 \quad \ldots \\
 y & : 0 \quad 23 \quad 23 \quad 38.6 \quad 0 \\
\end{align*}
Modelling the task graph scheduling problem
Modelling the task graph scheduling problem

- Processors

\[ P_1: \]
\[
\begin{align*}
&x \leq 2 \\
&x := 0 \\
&x = 2 \\
&\text{add}_1 \\
&\text{done}_1
\end{align*}
\]
\[
\begin{align*}
&x = 3 \\
&x := 0 \\
&\text{done}_1 \\
&\text{mult}_1 \\
&\times
\end{align*}
\]
\[
\begin{align*}
&P_2: \\
&y \leq 5 \\
&x := 0 \\
&y = 5 \\
&\text{done}_2 \\
&\text{add}_2
\end{align*}
\]
\[
\begin{align*}
&P_2: \\
&y \leq 7 \\
&x := 0 \\
&y = 7 \\
&\text{done}_2 \\
&\text{mult}_2 \\
&\times
\end{align*}
\]
Modelling the task graph scheduling problem

- **Processors**
  - $P_1$: 
    - $x = 2$ (idle) 
    - $x = 3$ (idle) 
    - $x = 0$ (idle)
  - $P_2$: 
    - $y = 5$ (idle) 
    - $y = 7$ (idle) 
    - $x = 0$ (idle)

- **Tasks**
  - $T_4$: 
    - $t_1 \land t_2$ (add) 
    - $t_4 := 1$ (done)
  - $T_5$: 
    - $t_3$ (add) 
    - $t_5 := 1$ (done)

A schedule is a path in the product automaton
Modelling the task graph scheduling problem

- **Processors**
  - $P_1$:
    - $(x \leq 2)$
    - $x = 2$
    - done$_1$
    - add$_1$
    - $x = 3$
    - done$_1$
    - mult$_1$
    - $x = 0$
    - $(x \leq 3)$
    - done$_1$
    - $x = 0$
    - mult$_1$
    - $x = 0$
    - $x = 0$
  - $P_2$:
    - $(y \leq 5)$
    - $y = 5$
    - done$_2$
    - add$_2$
    - $y = 7$
    - done$_2$
    - mult$_2$
    - $x = 0$
    - $(y \leq 7)$
    - done$_2$
    - $x = 0$
    - mult$_2$
    - $x = 0$

- **Tasks**
  - $T_4$:
    - $t_1 \land t_2$
    - $t_4 := 1$
    - add$_i$
    - done$_i$
    - $t_3$
    - $t_5 := 1$
    - add$_i$
    - done$_i$

- **Modelling energy**
  - $P_1$:
    - $(x \leq 2)$
    - $x = 2$
    - done$_1$
    - add$_1$
    - $+90$
    - $x = 3$
    - done$_1$
    - mult$_1$
    - $+90$
    - $x = 0$
    - $(x \leq 3)$
    - done$_1$
    - $x = 0$
    - mult$_1$
    - $x = 0$
    - $x = 0$
    - $x = 0$
  - $P_2$:
    - $(y \leq 5)$
    - $y = 5$
    - done$_2$
    - add$_2$
    - $+30$
    - $y = 7$
    - done$_2$
    - mult$_2$
    - $+30$
    - $x = 0$
    - $(y \leq 7)$
    - done$_2$
    - $x = 0$
    - mult$_2$
    - $x = 0$

A good schedule is a path in the product automaton with a low cost.
Modelling the task graph scheduling problem

- **Processors**
  - \( P_1 \):
    - \( x = 2 \rightarrow \text{idle} \) \( \text{add}_1 \) \( \text{mult}_1 \)
    - \( x = 3 \rightarrow \times \)
    - \( x \geq 1 \rightarrow \times \)
  - \( P_2 \):
    - \( y = 5 \rightarrow \text{idle} \) \( \text{add}_2 \) \( \text{mult}_2 \)
    - \( y = 7 \rightarrow \times \)
    - \( y \geq 3 \rightarrow \times \)

- **Tasks**
  - \( T_4 \):
    - \( t_1 \wedge t_2 \rightarrow t_4 = 1 \)
  - \( T_5 \):
    - \( t_3 \rightarrow t_5 = 1 \)

- **Modelling energy**
  - \( P_1 \):
    - \( +90 \)
    - \( x = 2 \rightarrow \text{idle} \) \( \text{add}_1 \) \( \text{mult}_1 \)
    - \( x = 3 \rightarrow \times \)
    - \( x \geq 1 \rightarrow \times \)
  - \( P_2 \):
    - \( +30 \)
    - \( y = 5 \rightarrow \text{idle} \) \( \text{add}_2 \) \( \text{mult}_2 \)
    - \( y = 7 \rightarrow \times \)
    - \( y \geq 2 \rightarrow \times \)

- **Modelling uncertainty**
  - \( P_1 \):
    - \( x \leq 2 \)
    - \( x = 0 \rightarrow \text{idle} \) \( \text{add}_1 \) \( \text{mult}_1 \)
    - \( x = 3 \rightarrow \times \)
    - \( x \geq 1 \rightarrow \times \)
  - \( P_2 \):
    - \( y \leq 5 \)
    - \( y = 0 \rightarrow \text{idle} \) \( \text{add}_2 \) \( \text{mult}_2 \)
    - \( y = 7 \rightarrow \times \)
    - \( y \geq 2 \rightarrow \times \)
Modelling the task graph scheduling problem

- **Processors**
  - $P_1$: $x = 2 \land (x \leq 2)$
  - $P_2$: $y = 5 \land (y \leq 5)$

- **Tasks**
  - $T_4$: $t_1 \land t_2 \land t_4$ (idle)
  - $T_5$: $t_3 \land t_5$ (mult)

- **Modelling energy**
  - $P_1$: $+90 \land (+10 \land +90)$
  - $P_2$: $+30 \land (+20 \land +30)$

- **Modelling uncertainty**
  - $P_1$: $x \geq 1 \land x \leq 2 \land x = 0$ (x:0)
  - $P_2$: $y \geq 3 \land y \leq 5 \land y = 0$ (y:0)

A (good) schedule is a strategy in the product game (with a low cost)
Weighted/priced timed automata [ALP01,BFH+01]

\[ \ell_0 \xrightarrow{+5} \ell_1 \xrightarrow{u} \ell_2 \xrightarrow{+10} \ell_3 \xrightarrow{u} \ell_2 \xrightarrow{x=2,c} +1 \]

That can be generalized!


Weighted/priced timed automata

\[ \ell_0 \xrightarrow{+5} \ell_1 \overset{x\leq 2, c, y:=0}{\rightarrow} \ell_0 \overset{1.3}{\rightarrow} \ell_0 \overset{c}{\rightarrow} \ell_1 \overset{u}{\rightarrow} \ell_3 \overset{0.7}{\rightarrow} \ell_3 \overset{c}{\rightarrow} \text{\(\smile\)} \]

\[
\begin{array}{c|ccc|ccc}
\ell & \ell_0 & \ell_0 & c & \ell_1 & u & \ell_3 & c \\
x & 0 & 1.3 & c & 1.3 & u & 1.3 & 0.7 \\
y & 0 & 1.3 & 0 & 0 & 2 & 0.7 & \\
\end{array}
\]

That can be generalized!
Weighted/priced timed automata

\[\ell_0 \xrightarrow{+5} \ell_1 \quad \ell_1 \xrightarrow{u} (y=0) \quad \ell_1 \xrightarrow{u} \ell_2 \quad \ell_1 \xrightarrow{u} \ell_3 \quad \ell_2 \xrightarrow{x=2,c} +1 \quad \ell_3 \xrightarrow{x=2,c} +1 \quad \ell_3 \xrightarrow{c} +1 \]

\[\ell_0 \xrightarrow{1.3} \ell_0 \xrightarrow{c} \ell_1 \xrightarrow{u} \ell_3 \xrightarrow{0.7} \ell_3 \xrightarrow{c} +1\]

\[
x 0 \quad 1.3 \quad 1.3 \quad 1.3 \quad 2 \quad y 0 \quad 1.3 \quad 0 \quad 0 \quad 0.7 \]

cost : 14.2
Weighted/priced timed automata

\[ \ell_0 \xrightarrow{+5} \ell_1 \xrightarrow{u} \ell_2 \xrightarrow{+10} \ell_3 \xrightarrow{+1} \text{(happy face)} \]

\[ x \leq 2, c, y := 0 \]

Cost:

\[
\begin{align*}
\ell_0 & \quad 1.3 \\
\ell_0 & \quad c \quad \ell_1 \\
\ell_1 & \quad u \quad \ell_3 \\
\ell_3 & \quad 0.7 \quad \ell_3 \\
\ell_3 & \quad c \quad \text{(happy face)} \\
\end{align*}
\]

\[
\begin{align*}
x & \quad 0 \quad 1.3 \quad 1.3 \quad 1.3 \quad 2 \\
y & \quad 0 \quad 1.3 \quad 0 \quad 0 \quad 0.7 \\
\end{align*}
\]

cost : 6.5

That can be generalized!
Weighted/priced timed automata

\[
\begin{align*}
\ell_0 & \xrightarrow{+5} \ell_0 & x \leq 2, c, y := 0 \\
\ell_0 & \xrightarrow{c} \ell_1 & (y = 0) \\
\ell_1 & \xrightarrow{u} \ell_2 & x = 2, c \\
\ell_1 & \xrightarrow{u} \ell_3 & x = 2, c \\
\ell_2 & \xrightarrow{+10} \ell_2 \\
\ell_3 & \xrightarrow{+1} \ell_3 \\
\ell_3 & \xrightarrow{c} \ast \\
\end{align*}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
 & \ell_0 & \ell_0 & \ell_1 & \ell_1 & \ell_3 & \ell_3 & \ell_3 & \ast \\
\hline
x & 0 & 1.3 & 1.3 & 1.3 & 0 & 0 & 0 & 0 \\
y & 0 & 1.3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

cost : 6.5 + 0
Weighted/priced timed automata

\[
\begin{align*}
\ell_0 & \xrightarrow{\text{c}} \ell_1 & \ell_0 & \xrightarrow{\text{x} \leq 2, c, y:=0} \ell_0 & \xrightarrow{\text{c}} \ell_1 & \ell_1 & \xrightarrow{u} \ell_2 & \ell_2 & \xrightarrow{+10} \ell_3 & \ell_3 & \xrightarrow{+1} \ell_2 & \ell_3 & \xrightarrow{+7} \ell_3 & \ell_3 & \xrightarrow{c} \text{smiley face} \\
\ell_0 & \xrightarrow{5} & (y=0) & \ell_2 & \xrightarrow{x=2, c} & +1 & \ell_3 & \xrightarrow{x=2, c} & +1 \\
\end{align*}
\]

\[
\begin{array}{c|ccc|ccc|ccc|c}
& \ell_0 & \ell_0 & \ell_1 & \ell_2 & \ell_3 & \ell_3 & \ell_3 & \text{smiley face} \\
\hline
x & 0 & 1.3 & 1.3 & 1.3 & 2 & 0 & 0 & \\
y & 0 & 1.3 & 0 & 0 & 0.7 & \\
\end{array}
\]

cost : 6.5 + 0 + 0

That can be generalized!
Weighted/priced timed automata

\[\ell_0 \xrightarrow{+5} \ell_1 \] \(x \leq 2, c, y := 0\)

\(\ell_1 \xrightarrow{u} \ell_2 \) \((y = 0)\)

\(\ell_2 \xrightarrow{+10} \ell_3 \)

\(\ell_3 \xrightarrow{c} \ell_1 \)

\(\ell_3 \xrightarrow{+1} \) \(x = 2, c\)

\(\ell_3 \xrightarrow{+7} \) \(x = 2, c\)

\(\ell_0 \xrightarrow{1.3} \ell_0 \) \(\ell_0 \xrightarrow{c} \ell_1 \) \(\ell_1 \xrightarrow{u} \ell_3 \) \(\ell_3 \xrightarrow{0.7} \ell_3 \) \(\ell_3 \xrightarrow{c} \)

\(x\) \(0\) \(1.3\) \(1.3\) \(1.3\) \(2\) \(0.7\)

\(y\) \(0\) \(1.3\) \(0\) \(0\) \(0.7\)

\[\text{cost : } 6.5 + 0 + 0 + 0 + 0.7\]
Weighted/priced timed automata

\[
\begin{align*}
\ell_0 & \xrightarrow{+5} \ell_1 \\
x \leq 2, c, y := 0 \\n\ell_1 & \xrightarrow{u} \ell_2 \\
y = 0 \\n\ell_2 & \xrightarrow{u} \ell_3 \\
x = 2, c \\n\ell_3 & \xrightarrow{u} \ell_0 \\
x = 2, c \\
\ell_0 & \xrightarrow{\ell_0} \ell_0 \\
\ell_0 & \xrightarrow{c} \ell_1 \\
\ell_1 & \xrightarrow{u} \ell_3 \\
\ell_3 & \xrightarrow{c} \ell_3 \\
\ell_3 & \xrightarrow{c} \smiley
\end{align*}
\]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\ell_0$</th>
<th>$\ell_1$</th>
<th>$\ell_2$</th>
<th>$\ell_3$</th>
<th>$\ell_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Cost:

\[
\text{cost} : \quad 6.5 + 0 + 0 + 0.7 + 7
\]
Weighted/priced timed automata

That can be generalized!

cost : 6.5 + 0 + 0 + 0.7 + 7 = 14.2
Weighted/priced timed automata

Question: what is the optimal cost for reaching 😊?
Weighted/priced timed automata

Question: what is the optimal cost for reaching 🎉?

$$5t + 10(2 - t) + 1$$
Weighted/priced timed automata

Question: what is the optimal cost for reaching 😊?

\[ 5t + 10(2 - t) + 1, \quad 5t + (2 - t) + 7 \]
Weighted/priced timed automata

Question: what is the optimal cost for reaching 🌻?

\[
\min \left( 5t + 10(2 - t) + 1, 5t + (2 - t) + 7 \right)
\]
Question: what is the optimal cost for reaching ☺️?

\[
\inf_{0 \leq t \leq 2} \min ( 5t + 10(2 - t) + 1, 5t + (2 - t) + 7 ) = 9
\]
Weighted/priced timed automata

Question: what is the optimal cost for reaching 😊?

\[
\inf_{0 \leq t \leq 2} \min \left( 5t + 10(2 - t) + 1 , \ 5t + (2 - t) + 7 \right) = 9
\]

≈ \textit{strategy:} leave immediately \( l_0 \), go to \( l_3 \), and wait there 2 t.u.
Weighted/priced timed automata

\[ \ell_0 \xrightarrow{+5} \ell_1 \xrightarrow{u} \ell_2 \xrightarrow{x=2,c} \ell_3 \xrightarrow{+1} \ell_f \]

\[ y = 0 \]

Question: what is the optimal cost for reaching \( \ell_f \)?

\[ \inf_{0 \leq t \leq 2} \min (5t + 10(2 - t) + 1, 5t + (2 - t) + 7) = 9 \]

\( \leadsto \) strategy: leave immediately \( \ell_0 \), go to \( \ell_3 \), and wait there 2 t.u.

That can be generalized!


A simple timed game

\[
x \leq 2, c, y := 0
\]

\[
(x = 0)
\]

\[
x = 2, c
\]

\[
x = 2, c
\]

\[
(\ell_0, \ell_1, \ell_2, \ell_3)
\]

\[
\text{strategy: wait in } \ell_0, \text{ and when } t = 4/3, \text{ go to } \ell_1
\]

\[
\inf_{0 \leq t \leq 2} \max (5t + 10(2 - t) + 1, 5t + (2 - t) + 7) = 14 + 1/3;
\]
A simple weighted timed game

\begin{align*}
\ell_0 & \xrightarrow{x \leq 2, c, y := 0} \ell_1 \\
\ell_1 & \xrightarrow{(y = 0)} \ell_3 \\
\ell_3 & \xrightarrow{u} \ell_2 \\
\ell_2 & \xrightarrow{x = 2, c} \text{goal} \\
\end{align*}

\text{inf}_{0 \leq t \leq 2} \max (5t + 10(2 - t) + 1, 5t + (2 - t) + 7) = 14 + 1/3;
A simple weighted timed game

Question: what is the optimal cost we can ensure while reaching 😊?
A simple weighted timed game

**Question:** what is the optimal cost we can ensure while reaching 😊?

\[ 5t + 10(2 - t) + 1 \]
A simple weighted timed game

Question: what is the optimal cost we can ensure while reaching $\smiley$?

$$5t + 10(2 - t) + 1, \ 5t + (2 - t) + 7$$
A simple weighted timed game

Question: what is the optimal cost we can ensure while reaching 😊 ?

\[
\max ( 5t + 10(2 - t) + 1 , 5t + (2 - t) + 7 )
\]
A simple weighted timed game

Question: what is the optimal cost we can ensure while reaching 😊?

\[
\inf_{0 \leq t \leq 2} \max \left( 5t + 10(2 - t) + 1 \right, 5t + (2 - t) + 7 \right) = 14 + \frac{1}{3}
\]
A simple weighted timed game

\[ \ell_0 + 5 \xrightarrow{x \leq 2, c, y := 0} \ell_1 \]  
\[ (y = 0) \xrightarrow{u} \ell_2 +10 \]  
\[ \ell_2 \xrightarrow{x=2, c,} +1 \]  
\[ +7 \]  
\[ \ell_3 +1 \xrightarrow{x=2, c} \]  

**Question:** what is the optimal cost we can ensure while reaching \( \square \)?

\[ \inf_{0 \leq t \leq 2} \max (5t + 10(2 - t) + 1, 5t + (2 - t) + 7) = 14 + \frac{1}{3} \]

\( \leadsto \) **strategy:** wait in \( \ell_0 \), and when \( t = \frac{4}{3} \), go to \( \ell_1 \)
This topic has been fairly hot these last fifteen years...

[LMM02, ABM04, BCFL04, BBR05, BBM06, BLMR06, Rut11, HIM13, BGK+14]
Optimal reachability in weighted timed games (1)

This topic has been fairly hot these last fifteen years...

[LMM02, ABM04, BCFL04, BBR05, BBM06, BLMR06, Rut11, HIM13, BGK+14]

[LMM02]

Tree-like weighted timed games can be solved in 2EXPTIME.
Optimal reachability in weighted timed games (1)

This topic has been fairly hot these last fifteen years...

[LMM02,ABM04,BCFL04,BBR05,BBM06,BLMR06,Rut11,HIM13,BGK+14]

[LMM02]
Tree-like weighted timed games can be solved in 2EXPTIME.

[ABM04,BCFL04]
Depth-$k$ weighted timed games can be solved in EXPTIME. There is a symbolic algorithm to solve weighted timed games with a strongly non-Zeno cost.
In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more.
In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more.

Turn-based optimal timed games are decidable in \textsc{EXPTIME} (resp. \textsc{PTIME}) when automata have a single clock (resp. with two rates). They are \textsc{PTIME}-hard.
Computing the optimal cost: why is that hard?

Given two clocks $x$ and $y$, we can check whether $y = 2x$. 
Computing the optimal cost: why is that hard?

Given two clocks $x$ and $y$, we can check whether $y = 2x$.

The cost is increased by $x_0$.

The cost is increased by $1 - x_0$.
Computing the optimal cost: why is that hard?

Given two clocks $x$ and $y$, we can check whether $y = 2x$. 

\[
\begin{align*}
&x = x_0 \\
y = y_0 \\
z = 0
\end{align*}
\]
Computing the optimal cost: why is that hard?

Given two clocks $x$ and $y$, we can check whether $y = 2x$.

$$\text{In } \{\text{green}, x = x_0, y = y_0 \}, \text{ cost } = 2x_0 + (1 - y_0) + 2$$
Computing the optimal cost: why is that hard?

Given two clocks $x$ and $y$, we can check whether $y = 2x$. 

In blue, cost $= 2x_0 + (1 - y_0) + 2$

In pink, cost $= 2(1 - x_0) + y_0 + 1$
Computing the optimal cost: why is that hard?

Given two clocks $x$ and $y$, we can check whether $y = 2x$.

\[ \text{In } \begin{pmatrix} \varepsilon \end{pmatrix}, \text{ cost } = 2x_0 + (1 - y_0) + 2 \]
\[ \text{In } \begin{pmatrix} 1 \end{pmatrix}, \text{ cost } = 2(1 - x_0) + y_0 + 1 \]

if $y_0 < 2x_0$, player 2 chooses the first branch: cost $> 3$
Computing the optimal cost: why is that hard?

Given two clocks \( x \) and \( y \), we can check whether \( y = 2x \).

- If \( y_0 < 2x_0 \), player 2 chooses the first branch: cost \( > 3 \)
- If \( y_0 > 2x_0 \), player 2 chooses the second branch: cost \( > 3 \)
- If \( y_0 = 2x_0 \), in both branches, cost = 3; player 2 can enforce cost 3 + |\( y_0 - 2x_0 | \)

\[ \text{In } \bigcirc, \text{ cost } = 2x_0 + (1 - y_0) + 2 \]
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In the image, the diagram illustrates the process with two paths:

- For $x = x_0$, cost = $2x_0 + (1 - y_0) + 2$
- For $y = y_0$, cost = $2(1 - x_0) + y_0 + 1$

- If $y_0 < 2x_0$, player 2 chooses the first branch: cost > 3
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$\leadsto$ player 2 can enforce cost $3 + |y_0 - 2x_0|$
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  \( \leadsto \) player 2 can enforce cost \( 3 + |y_0 - 2x_0| \)

- Player 1 has a winning strategy with cost \( \leq 3 \) iff \( y_0 = 2x_0 \)
Computing the optimal cost: why is that hard?

Player 1 will simulate a two-counter machine:
- each instruction is encoded as a module;
- the counter values $c_1$ and $c_2$ are encoded by two clocks:

$$x = \frac{1}{2^{c_1}} \quad \text{and} \quad y = \frac{1}{3^{c_2}}$$
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The two-counter machine has a halting computation iff player 1 has a winning strategy to ensure a cost no more than 3.
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Globally, \((x \leq 1, y \leq 1, u \leq 1)\)

\[
\begin{align*}
x=1, x:=0 & \quad \lor \quad y=1, y:=0 & \quad x=1, x:=0 & \quad \lor \quad y=1, y:=0 \\
\text{Test}_y(x=2z) & \quad (u=0)
\end{align*}
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Are we done?
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12/25
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$$\text{cost}(\sigma) = \sup \{ \text{cost}(\rho) \mid \rho \text{ outcome of } \sigma \text{ up to the target} \}$$
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**Two problems of interest**
- The **value problem** asks, given $\mathcal{G}$ and a threshold $\triangleright c$, whether $\text{optcost}_\mathcal{G} \triangleright c$?
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- The **value problem** asks, given $\mathcal{G}$ and a threshold $\bowtie c$, whether $\text{optcost}_\mathcal{G} \bowtie c$?
- The **existence problem** asks, given $\mathcal{G}$ and a threshold $\bowtie c$, whether there exists a winning strategy in $\mathcal{G}$ such that $\text{cost}(\sigma) \bowtie c$?
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- The existence problem asks, given $\mathcal{G}$ and a threshold $\blacktriangleleft c$, whether there exists a winning strategy in $\mathcal{G}$ such that $\text{cost}(\sigma) \blacktriangleleft c$?

Note: These problems are distinct...
The value of the game is 3, but no strategy has cost 3.
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Weighted timed automata

In weighted timed automata, the optimal cost is an integer, and can be computed in PSPACE.
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In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more. The existence problem is undecidable in weighted timed games.
Our recent developments

The value problem is undecidable in weighted timed games
  ∼ Intellectually satisfactory to not have this discrepancy in the set of results
  ∼ A first proof based on a diagonal construction (originally proposed in the context of quantitative temporal logics [BMM14] – see Nicolas Markey’s talk)
  ∼ A second direct proof

Our recent developments

1. The value problem is undecidable in weighted timed games
   - Intellectually satisfactory to not have this discrepancy in the set of results
   - A first proof based on a diagonal construction (originally proposed in the context of quantitative temporal logics [BMM14] – see Nicolas Markey’s talk)
   - A second direct proof

2. An approximation algorithm for a large class of weighted timed games (that comprises the class of games used for proving the above undecidability)
   - Almost-optimality in practice should be sufficient
   - Even when we know how to compute the value, we are only able to synthesize almost-optimal strategies...

A snapshot on the undecidability proof

\[ M \text{ does not halt iff the value of } G_M \text{ is } 3 + \frac{1}{2n} \quad (n: \text{length of the path}) \]
A snapshot on the undecidability proof
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Leave with cost $3 + 1/2^n$ ($n$: length of the path)
A snapshot on the undecidability proof

$M$ does not halt iff the value of $G_M$ is 3

Leave with cost $3 + \frac{1}{2^n}$ ($n$: length of the path)
Optimal cost is computable...

... when cost is strongly non-zeno.  \[\text{[AM04,BCFL04]}\]
That is, there exists $\kappa > 0$ such that for every region cycle $C$, for every real run $\varrho$ read on $C$,
\[\text{cost}(\varrho) \geq \kappa\]

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\textbf{Note:} In both cases, we can assume $\kappa = 1$. 
Optimal cost is computable... [AM04,BCFL04]

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Optimal cost is not computable... but is approximable! [BJM15]

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Approximation of the optimal cost

**Theorem**

Let $G$ be a weighted timed game, in which the cost is almost-strongly non-zeno. For every $\epsilon > 0$, one can compute:

- two values $v_\epsilon^-$ and $v_\epsilon^+$ such that

\[|v_\epsilon^+ - v_\epsilon^-| < \epsilon \quad \text{and} \quad v_\epsilon^- \leq \text{optcost}_G \leq v_\epsilon^+\]
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- one strategy $\sigma_\epsilon$ such that
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It is an $\epsilon$-optimal winning strategy.

- Standard technics: unfold the game to get more precision, and compute two adjacency sequences
- This is not possible here
  - There might be runs with prefixes of arbitrary length and cost 0 (e.g. the game of the undecidability proof)
Idea for approximation

Idea

Only partially unfold the game:

- Keep components with cost 0 untouched – we call it the kernel
- Unfold the rest of the game
Semi-unfolding

Hypothesis: cost > 0 implies cost ≥ κ

Conclusion: we can stop unfolding the game after N steps (e.g. N = (M + 2) · |R(A)|, where M is a pre-computed bound on optcost(G)).
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Approximation scheme
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Approximation scheme

Exact computation
Approximation scheme

Exact computation

Approximation
First step: Tree-like parts

\[ O(\ell, v) = \inf_{t'} |v + t'| = g' \]

\[ (\alpha) = t'_c + c' + O(\ell', v') \]

\[ (\beta) = \sup_{t'' \leq t'} |v + t''| = g'' \]

\[ t''_c + c'' + O(\ell'', v'') \]

\[ v' = [Y' ← 0](v + t') \]

\[ v'' = [Y'' ← 0](v + t'') \]

\[ \rightsquigarrow \text{Goes back to [LMM02]} \]
First step: Tree-like parts

\[ O(\ell, v) = \inf t' \mid v + t' = g' \]
\[ O(\ell', v') = \max (\alpha, \beta) \]
\[ (\alpha) = t' + c' + O(\ell', v') \]
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[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata (TCS@02).
First step: Tree-like parts

\[ O(\ell, v) = \]

\[ c', Y' \]
\[ g', Y' \]
\[ O(\ell', v') \]
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Output cost functions $f$
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Second step: Kernels

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2. Under- and over-approximate by piecewise constant functions $f_{\epsilon}^-$ and $f_{\epsilon}^+$
Second step: Kernels

4. Refine/split the kernel along the new small regions and fix $f_\epsilon^-$ or $f_\epsilon^+$, write $f_\epsilon$

$f_\epsilon$: constant    $f_\epsilon$: constant
Second step: Kernels

3. Refine/split the kernel along the new small regions and fix $f^-_\epsilon$ or $f^+_\epsilon$, write $f_\epsilon$

4. Since cost is 0 everywhere, the resulting game is nothing more than a reachability timed game with an order on target (output) edges (given by $f_\epsilon$)

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3. Those can be solved using standard technics based on attractors: small regions are sufficient, and the local optimal cost (for output $f_\epsilon$) is constant within a small region

\[ \sim \] We have computed $\epsilon$-approximations of the optimal cost, which are constant within small regions. Corresponding strategies can be inferred
Conclusion

Summary of the talk

- Very quick overview of results concerning the optimal reachability problem in weighted timed games
- Some new insight into the value problem for this model:
  - Undecidability of this problem
  - Approximability of the optimal cost
    (under some conditions)

Future work

- Improve the approximation scheme
- Extend to the whole class of weighted timed games, or understand why it is not possible
- Assume stochastic uncertainty?
Conclusion

Summary of the talk

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