Approximation of the value in a weighted timed game

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Joint work with Samy Jaziri and Nicolas Markey

An example: The task graph scheduling problem

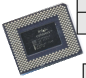
Compute $D\times(C\times(A+B))+(A+B)+(C\times D)$ using two processors:

P_1 (fast):

time	
$+$	2 picoseconds
$\times$	3 picoseconds

energy	
idle	10 Watt
in use	90 Watts

P_2 (slow):



time	
$+$	5 picoseconds
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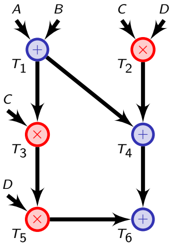
energy	
idle	20 Watts
in use	30 Watts



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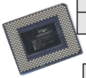

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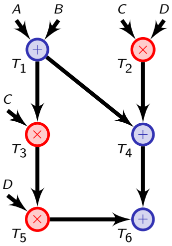

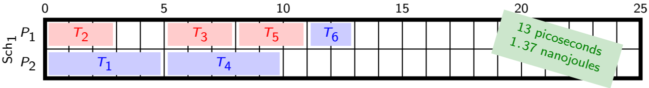

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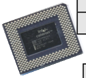

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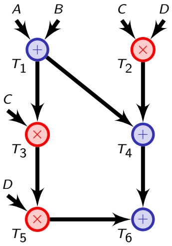

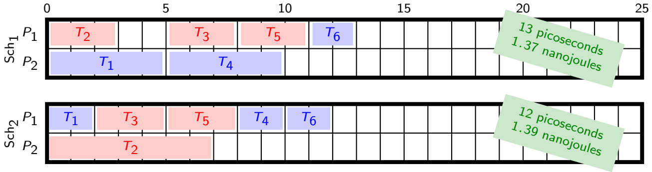

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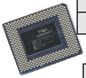

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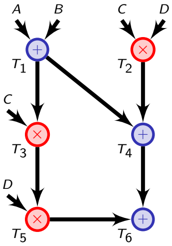
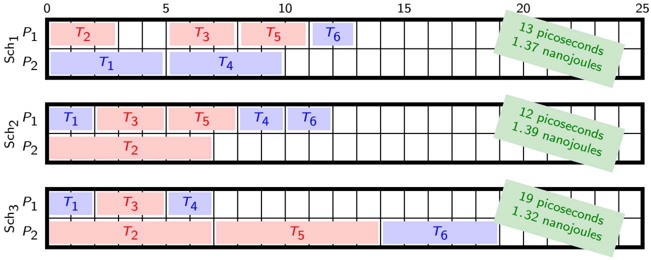
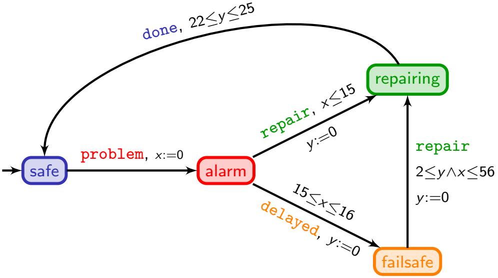




The model of timed automata



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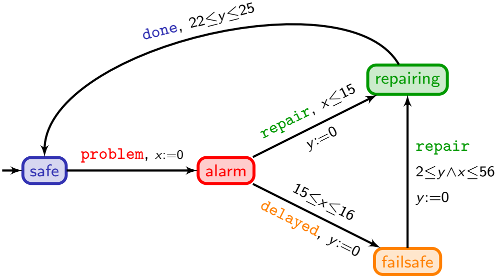


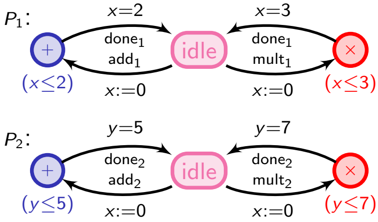
	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe	
x	0		23		0		15.6		15.6	$\cdots$
y	0		23		23		38.6		0	

	failsafe	$\xrightarrow{2.3}$	failsafe	$\xrightarrow{\texttt{repair}}$	repairing	$\xrightarrow{22.1}$	repairing	$\xrightarrow{\texttt{done}}$	safe
$\cdots$	15.6		17.9		17.9		40		40
	0		2.3		0		22.1		22.1

Modelling the task graph scheduling problem

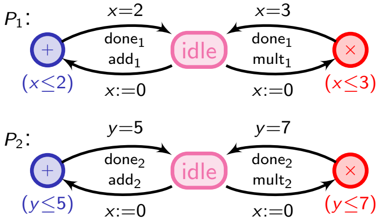
Modelling the task graph scheduling problem

- Processors

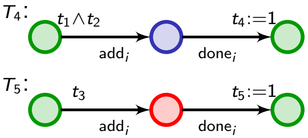


Modelling the task graph scheduling problem

- Processors



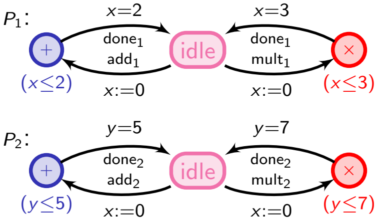
- Tasks



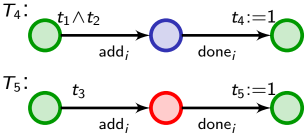
A schedule is a path in the product automaton

Modelling the task graph scheduling problem

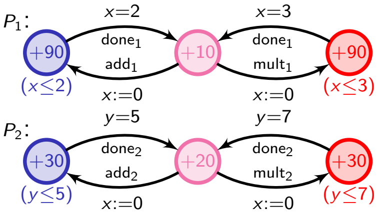
- Processors



- Tasks



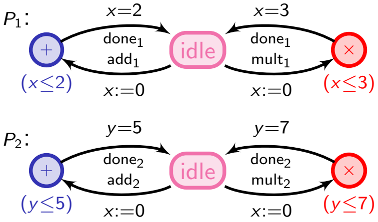
- Modelling energy



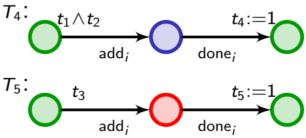
A good schedule is a path in the product automaton with a low cost

Modelling the task graph scheduling problem

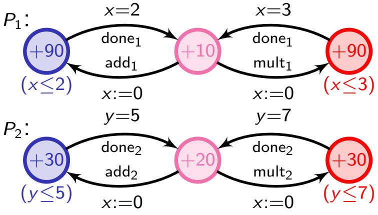
- Processors



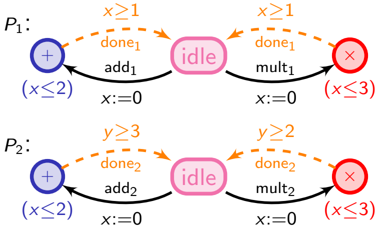
- Tasks



- Modelling energy



- Modelling uncertainty



Modelling the task graph scheduling problem

- Processors

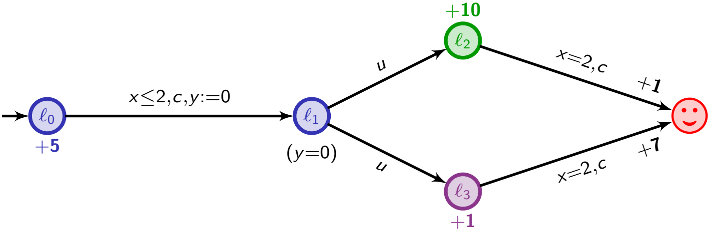
- Tasks

> A (good) schedule is a strategy in the product game (with a low cost)

- Modelling energy

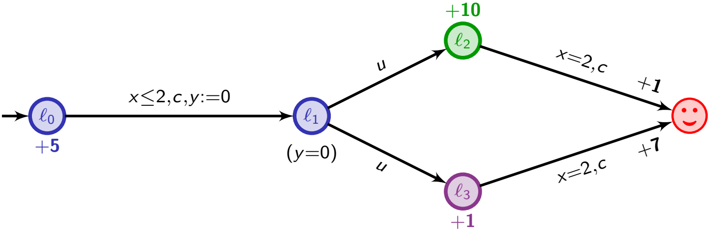
- Modelling uncertainty

Weighted/priced timed automata [ALP01,BFH+01]



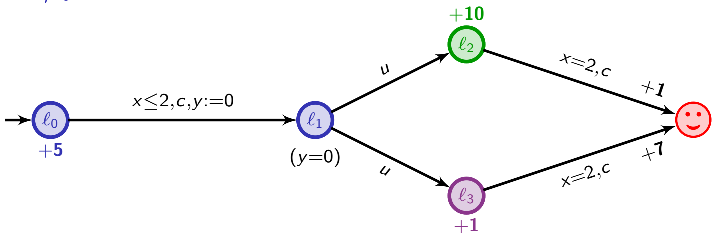
[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.

Weighted/priced timed automata



	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

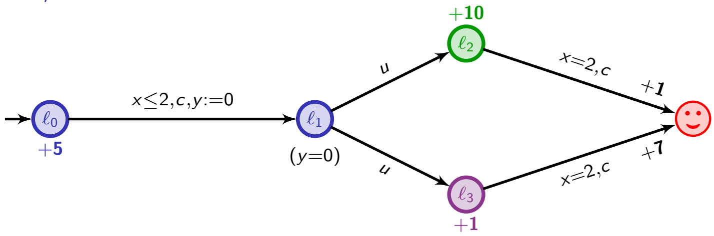
Weighted/priced timed automata



$$\begin{array}{lccccccccccc}
 & \ell_0 & \xrightarrow{1.3} & \ell_0 & \xrightarrow{c} & \ell_1 & \xrightarrow{u} & \ell_3 & \xrightarrow{0.7} & \ell_3 & \xrightarrow{c} & ☺ \\
x & 0 & & 1.3 & & 1.3 & & 1.3 & & 2 & & \\
y & 0 & & 1.3 & & 0 & & 0 & & 0.7 & &
\end{array}$$

cost :

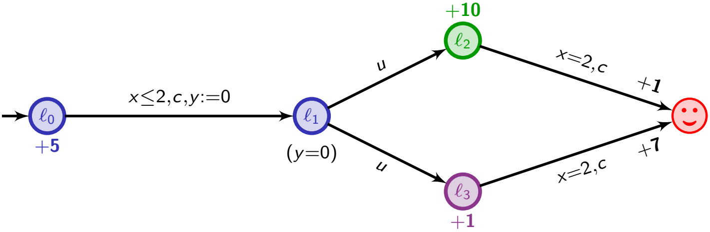
Weighted/priced timed automata



	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5

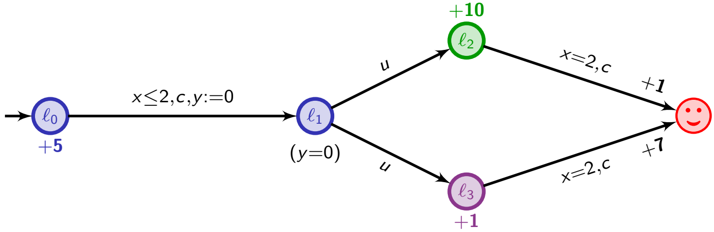
Weighted/priced timed automata



$$
\begin{array}{ccccccccccccc}
 & \ell_0 & \xrightarrow{1.3} & \ell_0 & \xrightarrow{c} & \ell_1 & \xrightarrow{u} & \ell_3 & \xrightarrow{0.7} & \ell_3 & \xrightarrow{c} & ☺ \\
x & 0 & & 1.3 & & 1.3 & & 1.3 & & 2 & & \\
y & 0 & & 1.3 & & 0 & & 0 & & 0.7 & &
\end{array}
$$

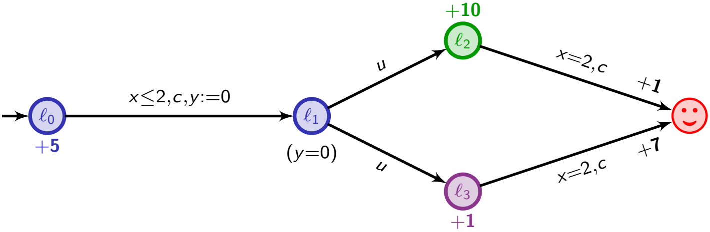
cost : 6.5 $+$ 0

Weighted/priced timed automata



	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		
cost :		6.5	+	0	+	0					

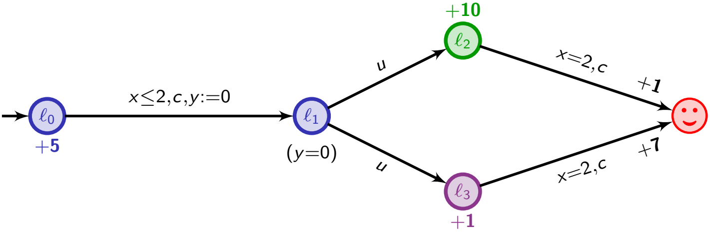
Weighted/priced timed automata



	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

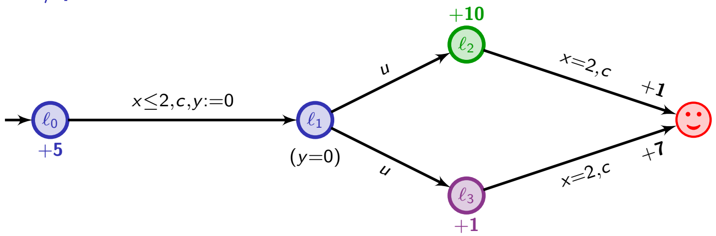
cost : 6.5 + 0 + 0 + 0.7

Weighted/priced timed automata



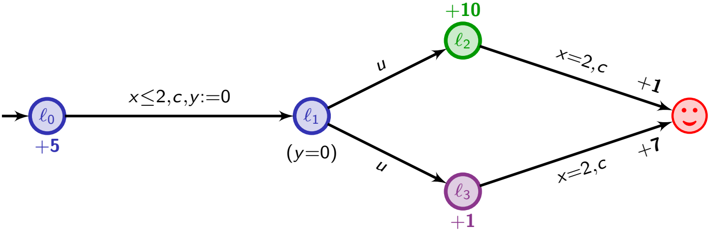
	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		
cost :		6.5	+	0	+	0	+	0.7	+	7	

Weighted/priced timed automata



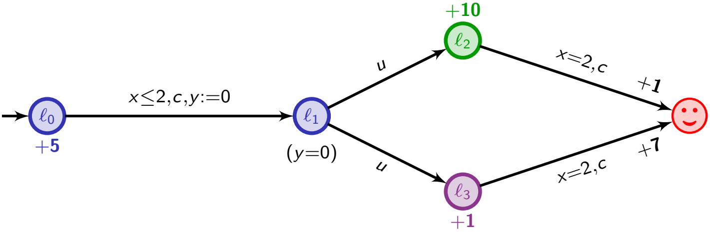
	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺	
x	0		1.3		1.3		1.3		2			
y	0		1.3		0		0		0.7			
cost :		6.5	+	0	+	0	+	0.7	+	7	=	14.2

Weighted/priced timed automata



Question: what is the optimal cost for reaching ☺ ?

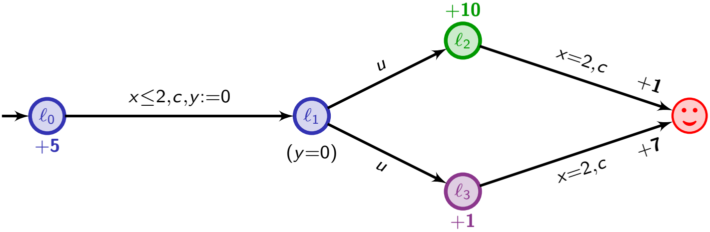
Weighted/priced timed automata



Question: what is the optimal cost for reaching  ?

$$5t + 10(2 - t) + 1$$

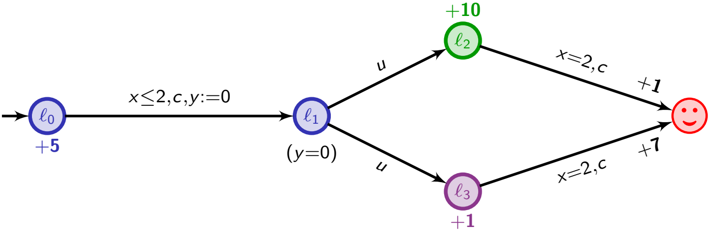
Weighted/priced timed automata



Question: what is the optimal cost for reaching ☺?

$$5t + 10(2-t) + 1 \ , \ 5t + (2-t) + 7$$

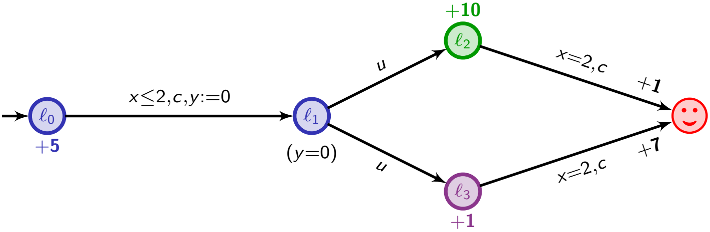
Weighted/priced timed automata



Question: what is the optimal cost for reaching ☺ ?

$$\min\ (\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ )$$

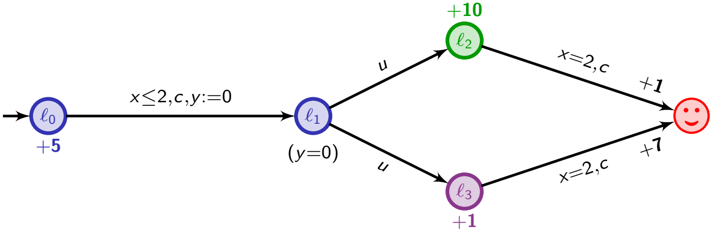
Weighted/priced timed automata



Question: what is the optimal cost for reaching ?

$$\inf_{0\leq t\leq 2} \ \min\ (\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ ) = 9$$

Weighted/priced timed automata

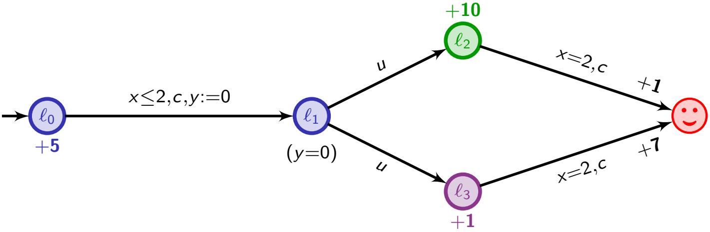


Question: what is the optimal cost for reaching ☺ ?

$$\inf_{0\leq t\leq 2} \min\left(\, 5t + 10(2-t) + 1 \,,\; 5t + (2-t) + 7 \,\right) = 9$$

$\rightsquigarrow$ *strategy:* leave immediately ℓ_0, go to ℓ_3, and wait there 2 t.u.

Weighted/priced timed automata



Question: what is the optimal cost for reaching ☺ ?

$$\inf_{0\leq t\leq 2} \min\left(5t+10(2-t)+1 \,,\, 5t+(2-t)+7 \right) = 9$$

$\rightsquigarrow$ *strategy:* leave immediately ℓ_0, go to ℓ_3, and wait there 2 t.u.

That can be generalized!

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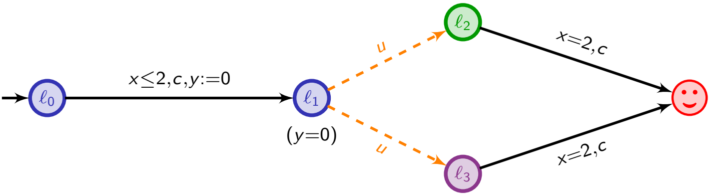
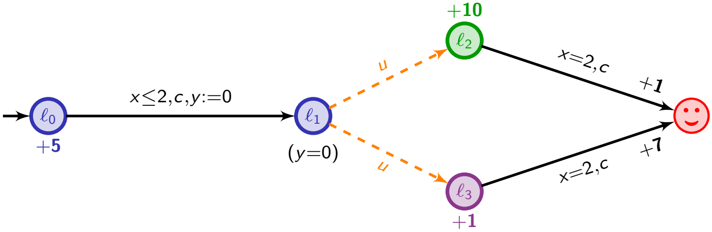
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.
[BBBR07] Bouyer, Brihaye, Bruyère, Raskin. On the optimal reachability problem *(Formal Methods in System Design)*.

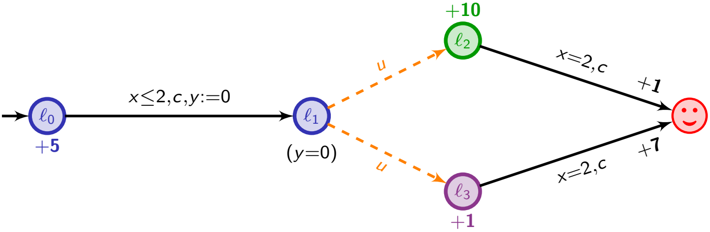
A simple timed game



A simple weighted timed game

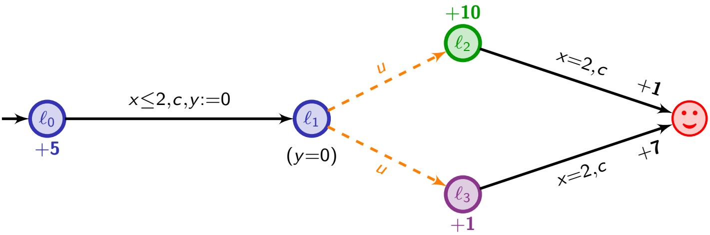


A simple weighted timed game



Question: what is the optimal cost we can ensure while reaching 🙂?

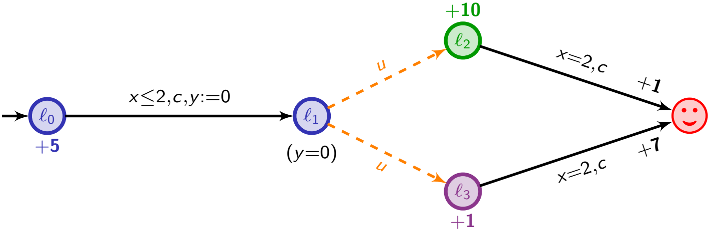
A simple weighted timed game



Question: what is the optimal cost we can ensure while reaching 🙂?

$$5t + 10(2 - t) + 1$$

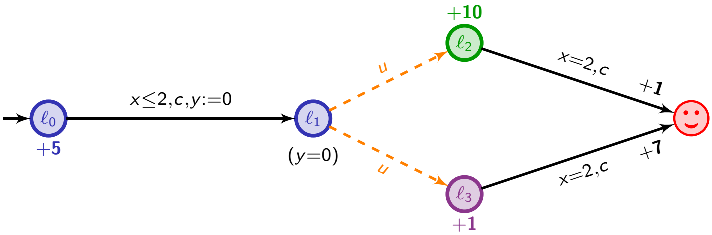
A simple weighted timed game



Question: what is the optimal cost we can ensure while reaching ?

$$5t + 10(2-t) + 1 \ , \ 5t + (2-t) + 7$$

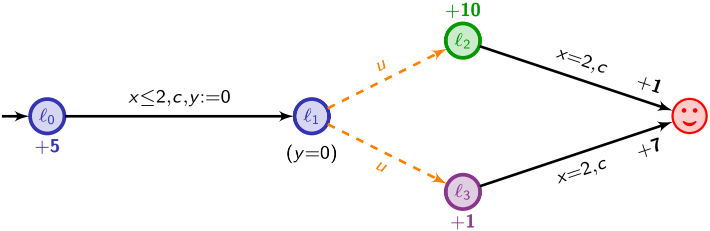
A simple weighted timed game



Question: what is the optimal cost we can ensure while reaching 🙂 ?

$$\max\left(\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ \right)$$

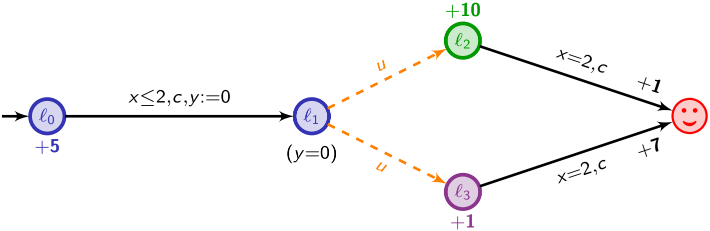
A simple weighted timed game



Question: what is the optimal cost we can ensure while reaching ?

$$\inf_{0 \leq t \leq 2} \max\left(5t + 10(2-t) + 1 \,,\; 5t + (2-t) + 7 \right) = 14 + \frac{1}{3}$$

A simple weighted timed game



Question: what is the optimal cost we can ensure while reaching ?

$$\inf_{0\leq t\leq 2} \max\big(\, 5t + 10(2-t) + 1 \,,\, 5t + (2-t) + 7 \,\big) = 14 + \frac{1}{3}$$

$\rightsquigarrow$ *strategy:* wait in ℓ_0, and when $t = \frac{4}{3}$, go to ℓ_1

Optimal reachability in weighted timed games (1)

This topic has been fairly hot these last fifteen years...

[LMM02,ABM04,BCFL04,BBR05,BBM06,BLMR06,Rut11,HIM13,BGK+14]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.
[ABM04] Alur, Bernardsky, Madhusudan. Optimal reachability in weighted timed games *(ICALP'04)*.
[BCFL04] Bouyer, Cassez, Fleury, Larsen. Optimal strategies in priced timed game automata *(FSTTCS'04)*.
[BBR05] Brihaye, Bruyère, Raskin. On optimal timed strategies *(FORMATS'05)*.
[BBM06] Bouyer, Brihaye, Markey. Improved undecidability results on weighted timed automata *(Information Processing Letters)*.
[BLMR06] Bouyer, Larsen, Markey, Rasmussen. Almost-optimal strategies in one-clock priced timed automata *(FSTTCS'06)*.
[Rut11] Rutkowski. Two-player reachability-price games on single-clock timed automata *(QAPL'11)*.
[HIM13] Hansen, Ibsen-Jensen, Miltersen. A faster algorithm for solving one-clock priced timed games *(CONCUR'13)*.
[BGK+14] Brihaye, Geeraerts, Krishna, Manasa, Monmege, Trivedi. Adding Negative Prices to Priced Timed Games *(CONCUR'14)*.

Optimal reachability in weighted timed games (1)

This topic has been fairly hot these last fifteen years...

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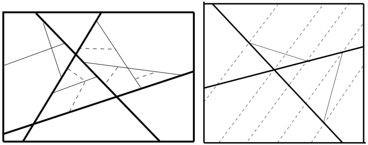
Depth-k weighted timed games can be solved in EXPTIME. There is a symbolic algorithm to solve weighted timed games **with a strongly non-Zeno cost**.



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In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more.

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In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more.

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Turn-based optimal timed games are decidable in EXPTIME (resp. PTIME) when automata have a single clock (resp. with two rates). They are PTIME-hard.

Computing the optimal cost: why is that hard?

Given two clocks x and y, we can check whether $y = 2x$.

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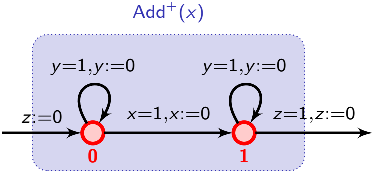
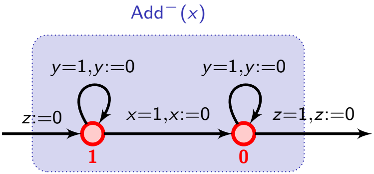


The cost is increased by x_0



The cost is increased by $1-x_0$

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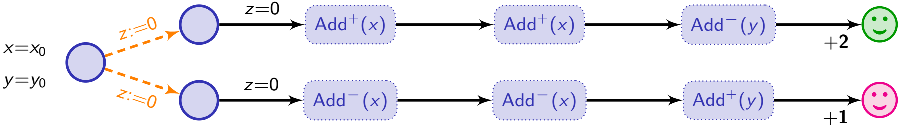

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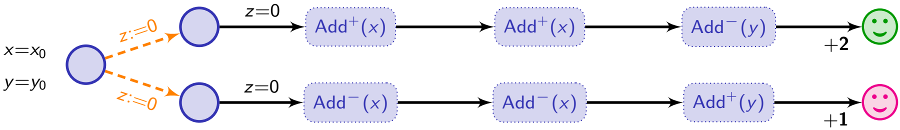

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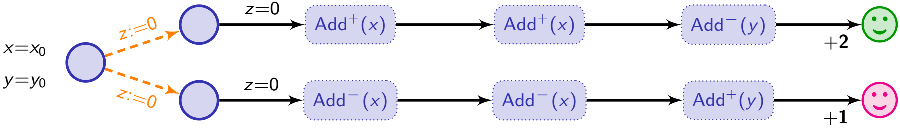

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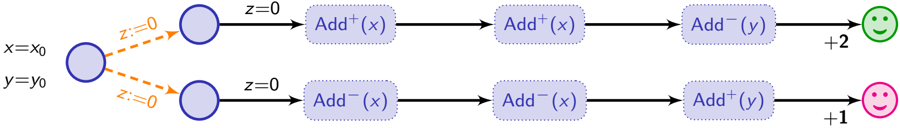

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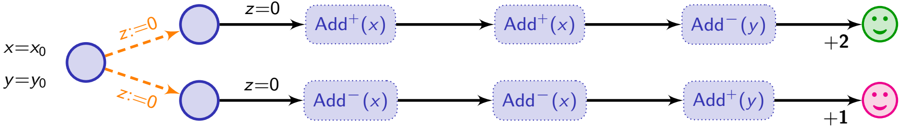

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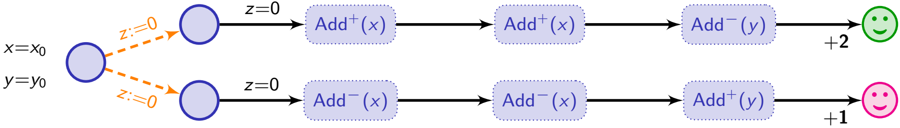

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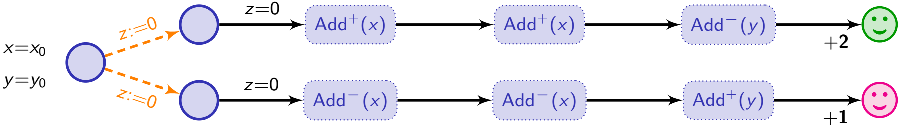

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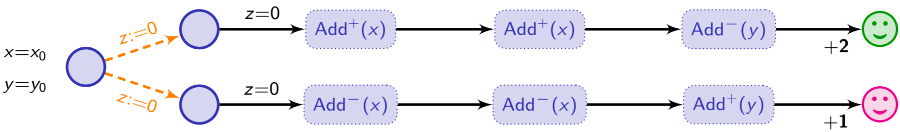

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  $\rightsquigarrow$ player 2 can enforce cost $3 + |y_0 - 2x_0|$
- Player 1 has a winning strategy with cost ≤ 3 iff $y_0 = 2x_0$

Computing the optimal cost: why is that hard?

Player 1 will simulate a two-counter machine:

- each instruction is encoded as a module;
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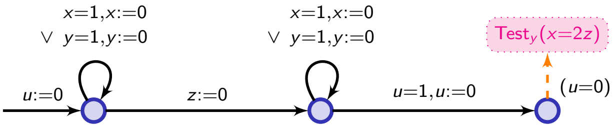
Globally, $(x \leq 1, y \leq 1, u \leq 1)$



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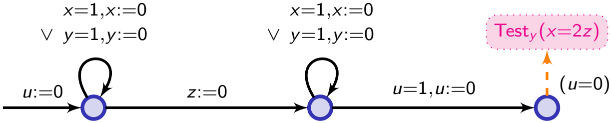

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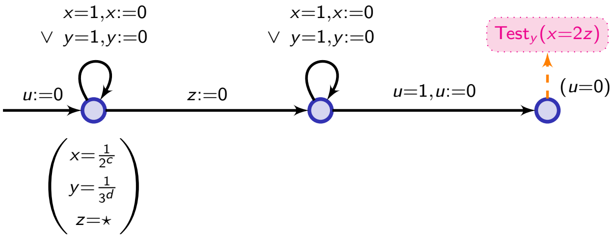

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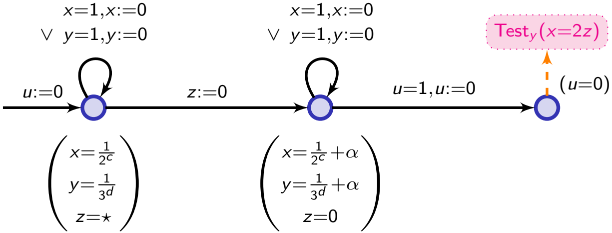

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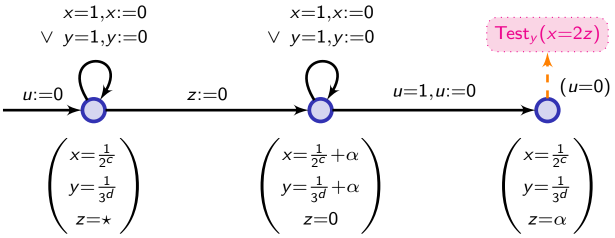

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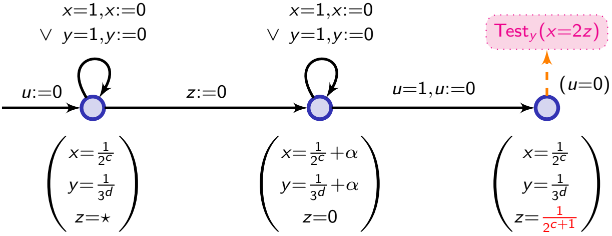

Shape of the reduction

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Note: These problems are distinct...

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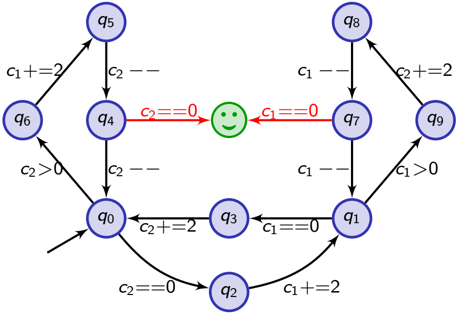

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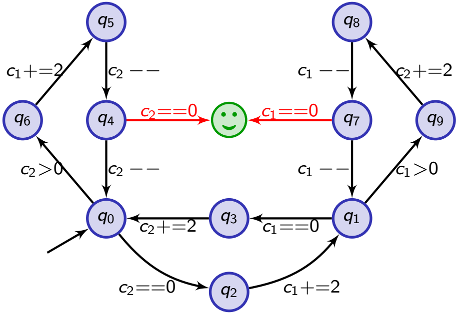

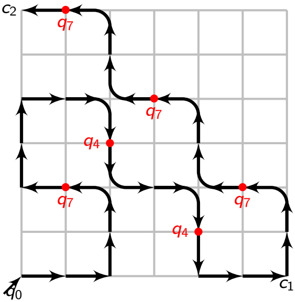

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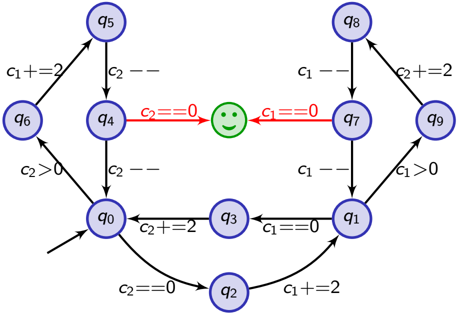

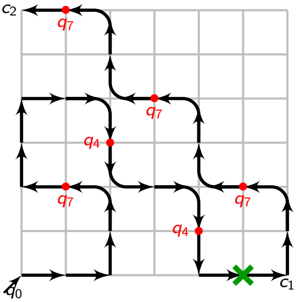

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1. The value problem is undecidable in weighted timed games
   - $\rightsquigarrow$ Intellectually satisfactory to not have this discrepancy in the set of results
   - $\rightsquigarrow$ Proof based on a diagonal construction (originally proposed in the context of quantitative temporal logics [BMM14])

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2. An approximation algorithm for a large class of weighted timed games (that comprises the class of games used for proving the above undecidability)
   - Almost-optimality in practice should be sufficient
   - Even when we know how to compute the value, we are only able to synthesize almost-optimal strategies...

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Optimal cost is computable...

... when cost is strongly non-zeno. [AM04,BCFL04]

That is, there exists $\kappa > 0$ such that for every region cycle C, for every real run ϱ read on C,

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[BJM15] Bouyer, Jaziri, Markey. On the value problem in weighted timed games.

Approximation of the optimal cost

Theorem

Let $\mathcal{G}$ be a weighted timed game, in which the cost is almost-strongly non-zeno. For every $\epsilon > 0$, one can compute:

- two values v_ϵ^- and v_ϵ^+ such that

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- one strategy σ_ϵ such that

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  It is an ϵ-optimal winning strategy.

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$$|v_\epsilon^+ - v_\epsilon^-| < \epsilon \quad \text{and} \quad v_\epsilon^- \leq \mathsf{optcost}_{\mathcal{G}} \leq v_\epsilon^+$$

- one strategy σ_ϵ such that

$$\mathsf{optcost}_{\mathcal{G}} \leq \mathsf{cost}(\sigma_\epsilon) \leq \mathsf{optcost}_{\mathcal{G}} + \epsilon$$

  It is an ϵ-optimal winning strategy.

- Standard technics: unfold the game to get more precision, and compute two adjacency sequences

Approximation of the optimal cost

Theorem

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  It is an ϵ-optimal winning strategy.

- Standard technics: unfold the game to get more precision, and compute two adjacency sequences

$\rightsquigarrow$ This is not possible here
There might be runs with prefixes of arbitrary length and cost 0 (e.g. the game of the undecidability proof)

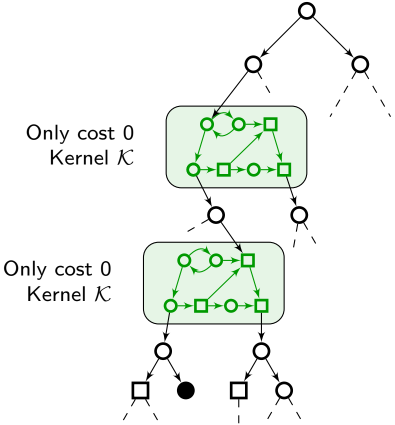
Idea for approximation

Idea

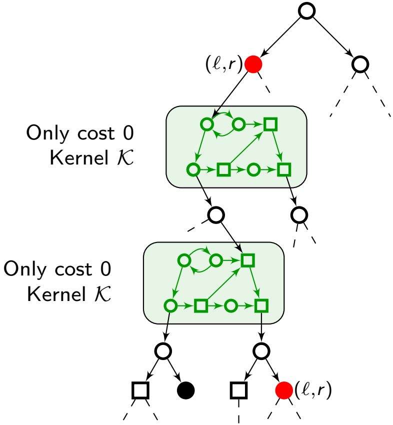
Only partially unfold the game:

- Keep components with cost 0 untouched – we call it the kernel
- Unfold the rest of the game

Semi-unfolding



Semi-unfolding



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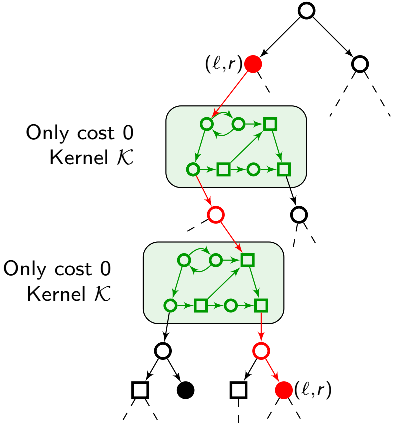
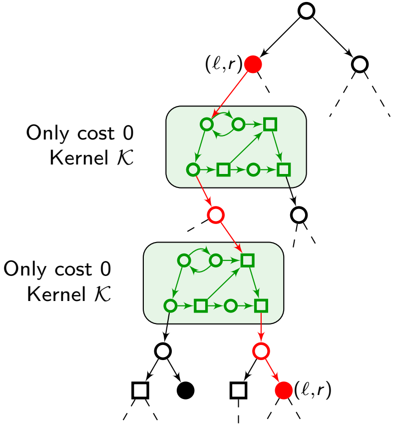
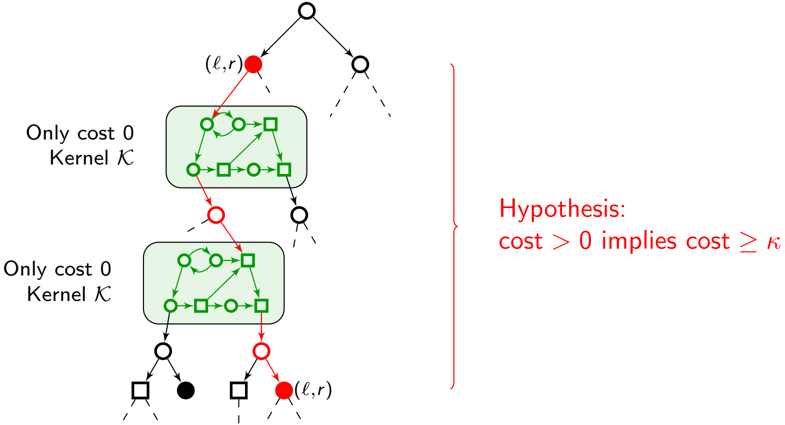


Semi-unfolding



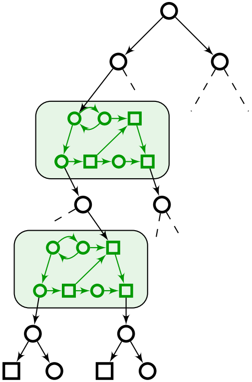
Hypothesis:
cost > 0 implies cost $\geq \kappa$

Semi-unfolding

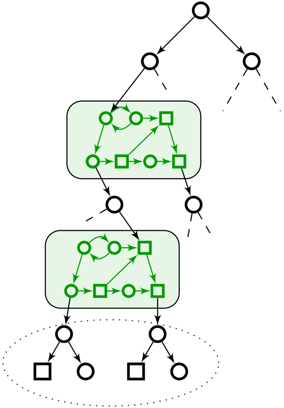


Conclusion: we can stop unfolding the game after N steps
(e.g. $N = (M+2) \cdot |\mathcal{R}(\mathcal{A})|$, where M is a pre-computed bound on $\mathrm{optcost}_{\mathcal{G}}$)

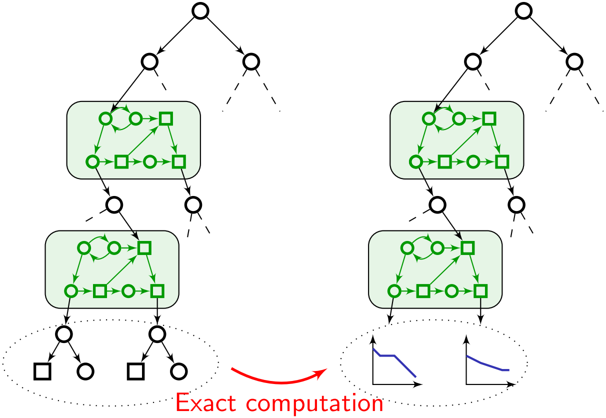
Approximation scheme



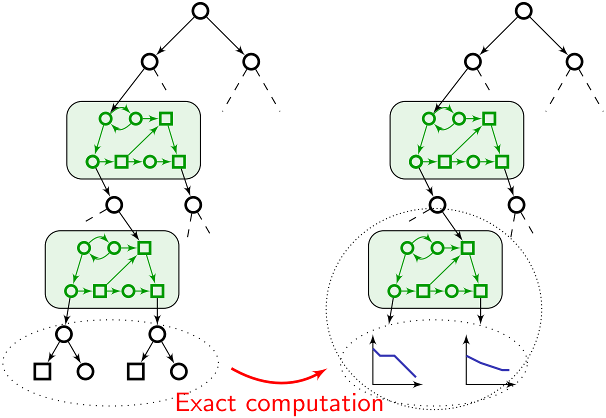
Approximation scheme



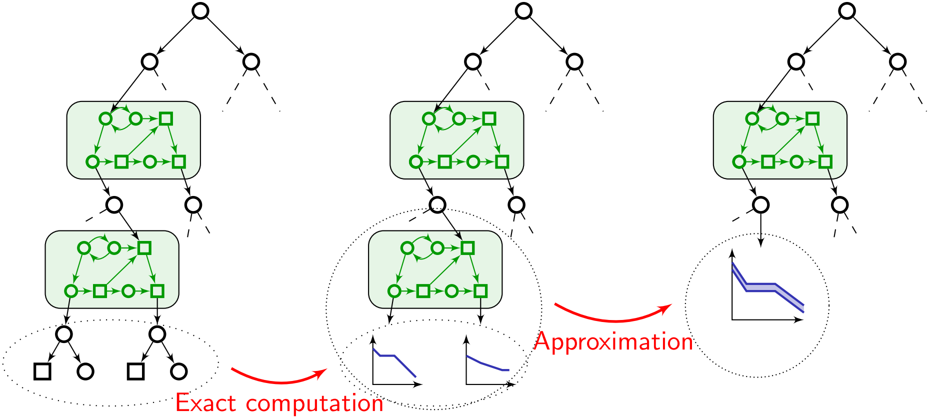
Approximation scheme



Approximation scheme



Approximation scheme



First step: Tree-like parts

$\rightsquigarrow$ Goes back to [LMM02]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

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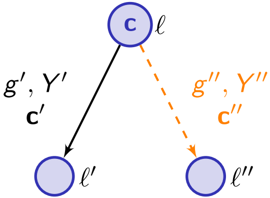

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

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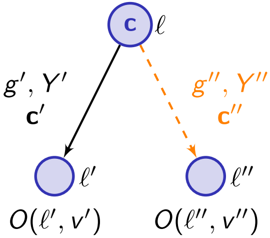

$O(\ell, v) =$

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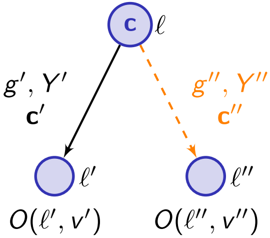

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'}$$

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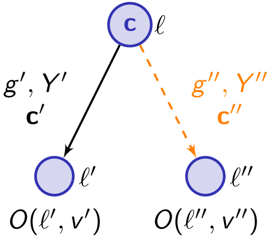

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max(\quad , \quad)$$

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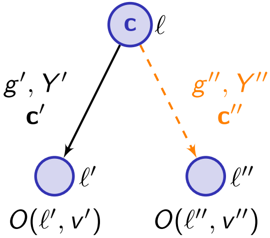

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max\big((\alpha), \qquad \big)$$

$$(\alpha) = t'\mathbf{c} + \mathbf{c}' + O(\ell', v')$$

$v' = [Y' \leftarrow 0](v+t')$

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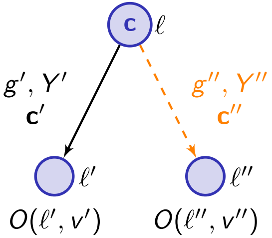

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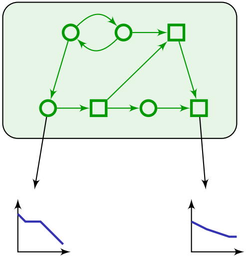
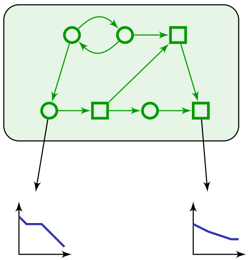
[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

Second step: Kernels



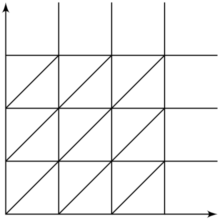
Output cost functions f

Second step: Kernels



Output cost functions f

1. Refine the regions such that f differs of at most ϵ within a small region



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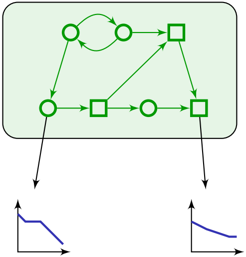

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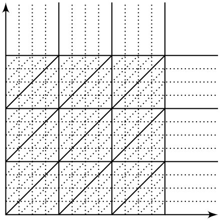

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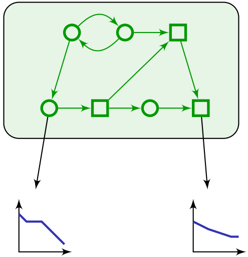

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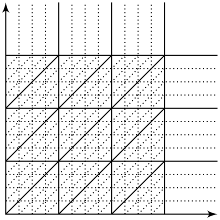

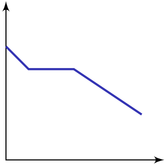

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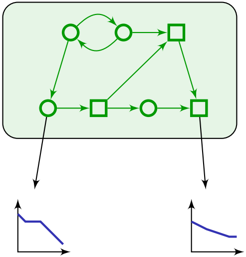

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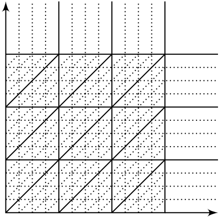
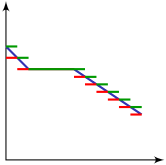
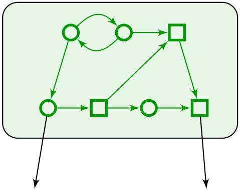


2. Under- and over-approximate by piecewise constant functions f_ϵ^- and f_ϵ^+



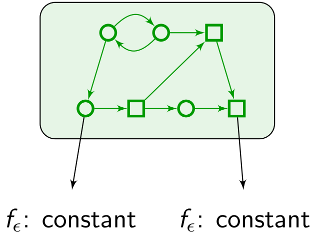
Second step: Kernels

3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ



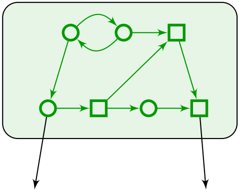
f_ϵ: constant $\qquad$ f_ϵ: constant

Second step: Kernels



3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ
4. Since cost is 0 everywhere, the resulting game is nothing more than a reachability timed game with an order on target (output) edges (given by f_ϵ)

Second step: Kernels



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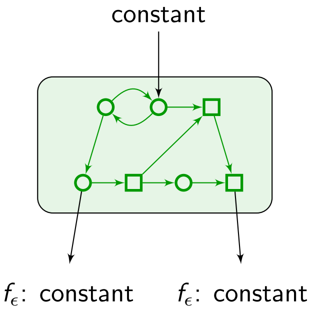
Second step: Kernels



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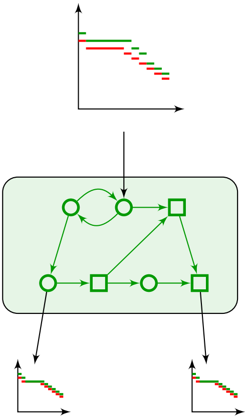

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$\leadsto$ We have computed ϵ-approximations of the optimal cost, which are constant within small regions. Corresponding strategies can be inferred

Conclusion

Summary of the talk

- Very quick overview of results concerning the optimal reachability problem in weighted timed games
- Some new insight into the value problem for this model:
  - Undecidability of this problem
  - Approximability of the optimal cost
    (under some conditions)

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Future work

- Improve the approximation scheme $\left(2\mathsf{EXP}(|\mathcal{G}|) \cdot \left(1/\epsilon\right)^{|X|}\right)$
- Extend to the whole class of weighted timed games, or understand why it is not possible
- Assume stochastic uncertainty?
- Multiplayer setting?