Averaging in LTL

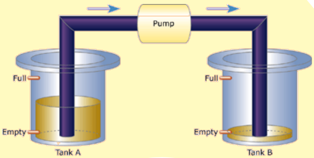
Patricia Bouyer Nicolas Markey Raj Mohan Matteplackel

LSV, CNRS & ENS Cachan, France

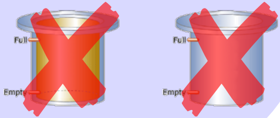
Outline

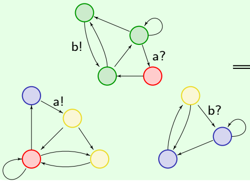
Model-checking

system:



property:





model-checking algorithm

$\mathbf{AG}(\neg\texttt{B.overfull} \wedge \neg\texttt{B.dried\_up})$

yes/no

Boolean verification

Standard model-checking

Strict dichotomy between correct and incorrect systems

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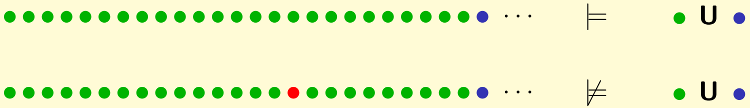

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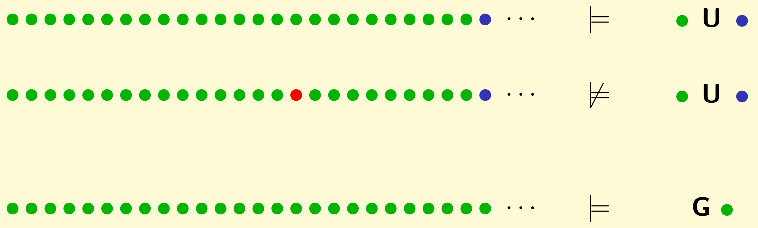

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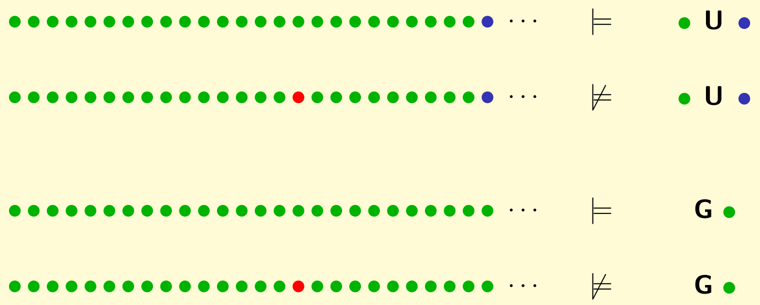

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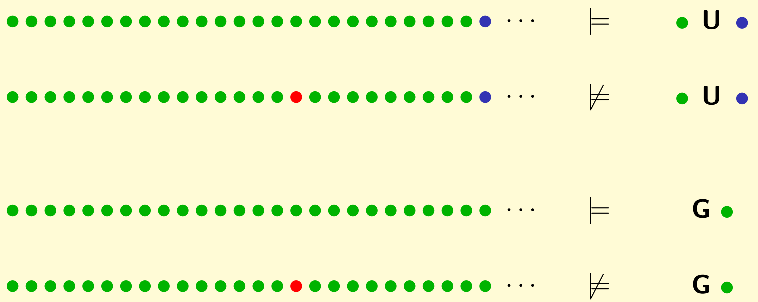

$\rightsquigarrow$ this Boolean approach might be too crude

Towards quantitative verification

Quantitative model-checking

Measure the accuracy of a system w.r.t. a property.

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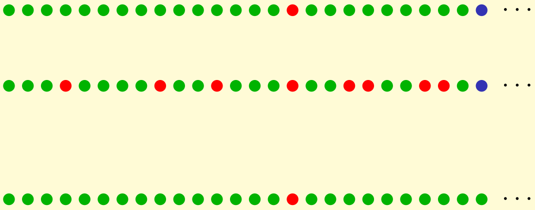

high value to • **U** •

smaller value to • **U** •

high value to **G** •

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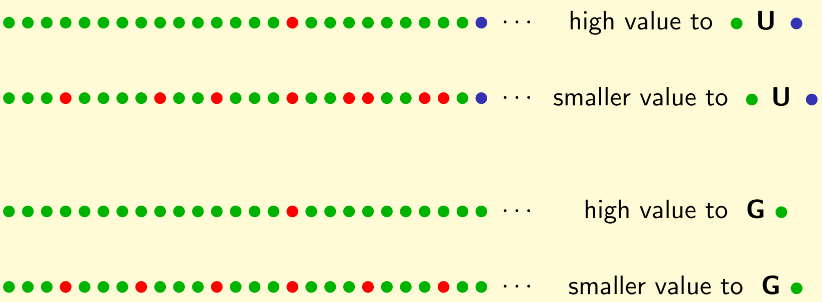

Related work

Quantitative verification

- value of executions given by weighted automata [DCH10,...]
- accuracy of a model given by a distance to another model or to a specification (e.g. simulation distance [CHR12], model measuring [HO13])
- quantitative specification languages/logics: standard in probabilistic model-checking (e.g. CSL and PCTL logics)
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Related work – cont'd

Quantitative logics based on LTL

- Logics yielding finitely many values
  - min/max extension of LTL to quant. Kripke structures [FLS08]
  - extension of LTL with functions [ABK13]
  - $\rightsquigarrow$ the model-checking is decidable

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- LTL over weighted Kripke structures using numerical assertions over weight variables [BCHK11,THHY12,BGM14]

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- Frequency LTL [BDL12]
  - boolean interpretation with quantitative constraints
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  $\rightsquigarrow$ the model-checking is undecidable
- Discounted LTL [ABK14]
  - the until modality discounts over the future

  $\rightsquigarrow$ the (threshold) model-checking is decidable, undecidable with an extra local average operator

Outline

1. Introduction
2. Average-LTL
3. Results
4. Conclusion

The logic Average-LTL

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi \mid \mathbf{G}\,\varphi \mid \varphi\,\widetilde{\mathrm{U}}\,\varphi \mid \widetilde{\mathrm{G}}\,\varphi.$$

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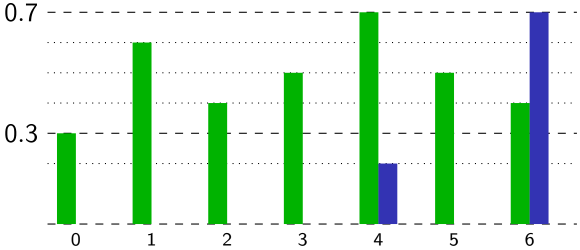
Continue



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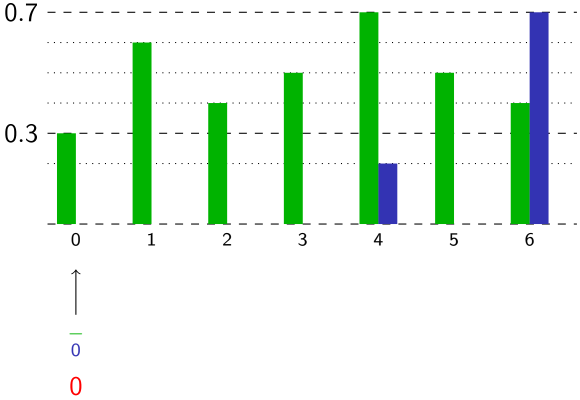

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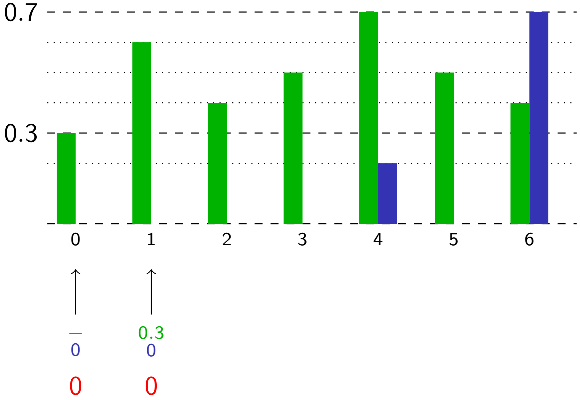

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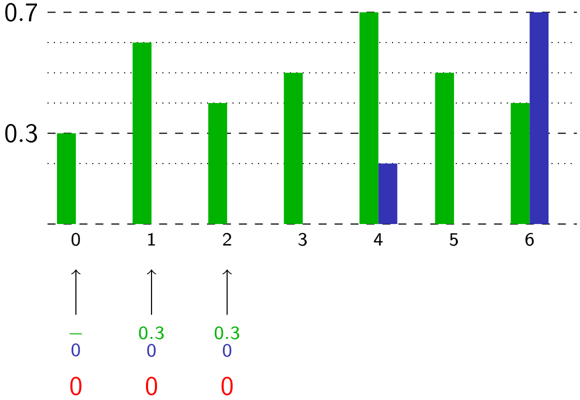

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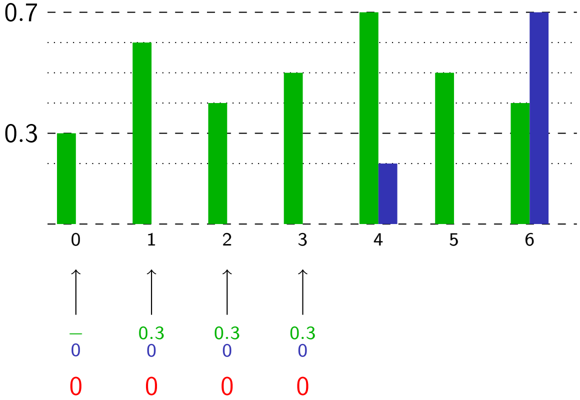

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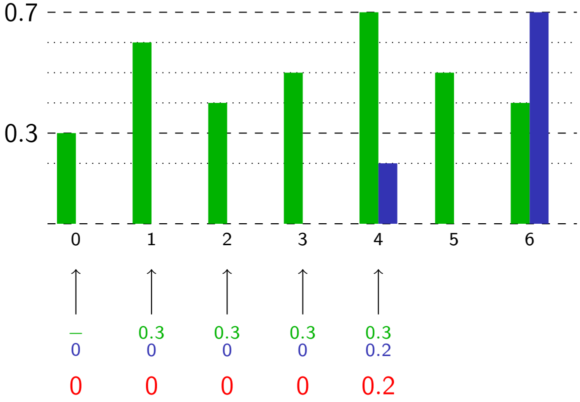

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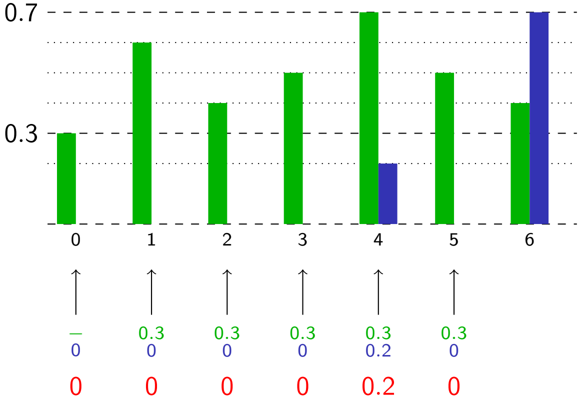

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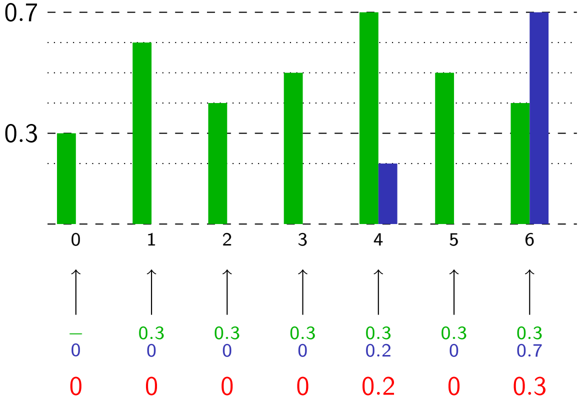

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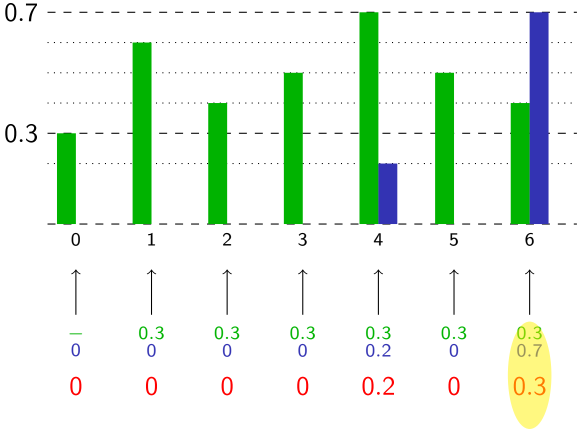

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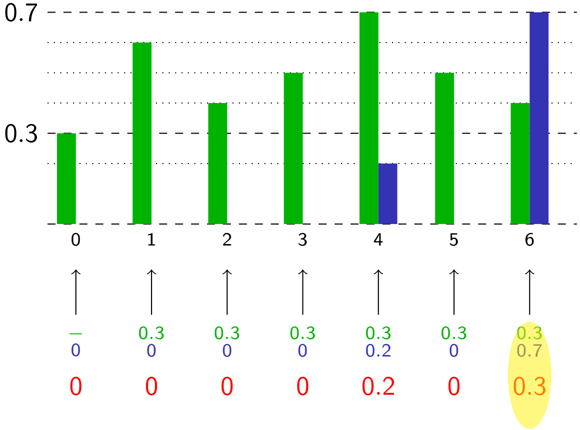

Back

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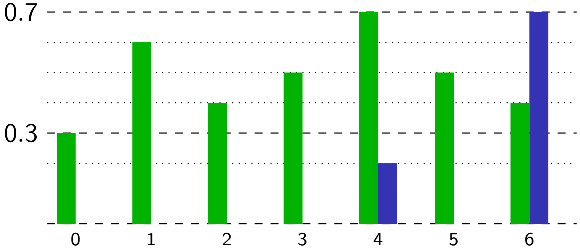

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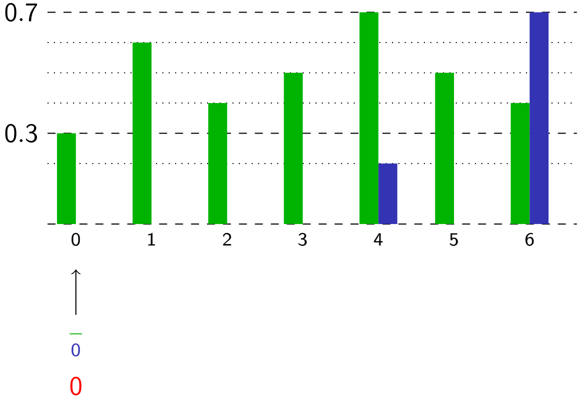

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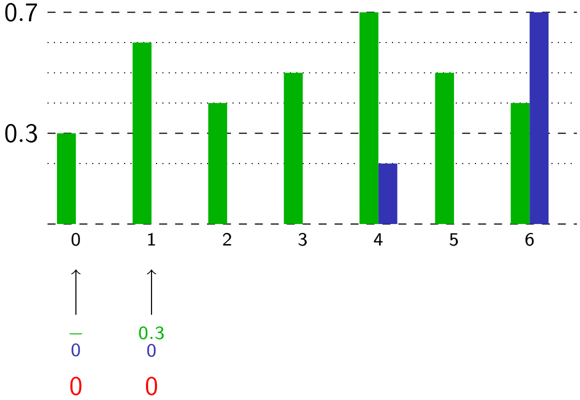

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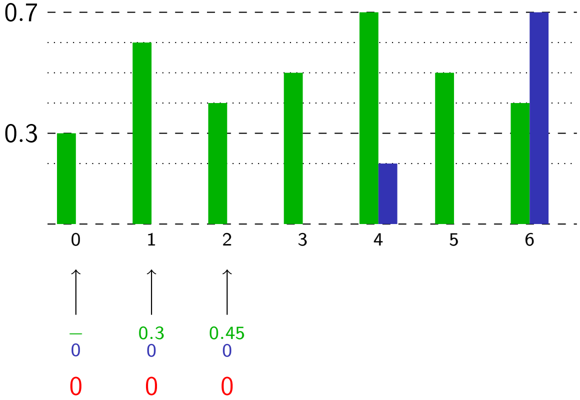

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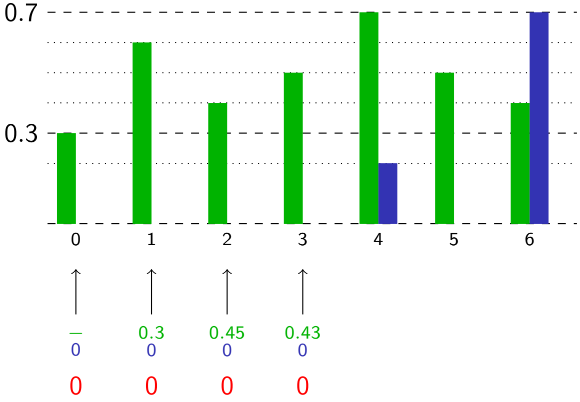

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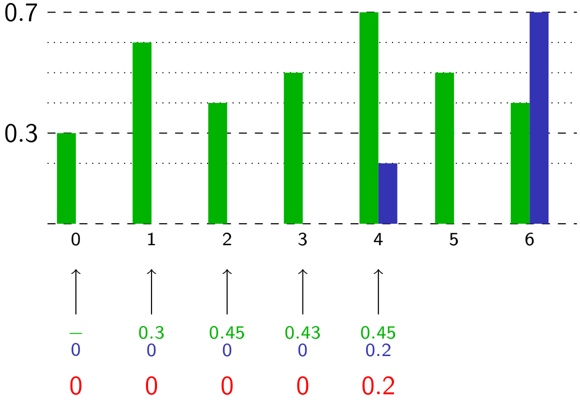

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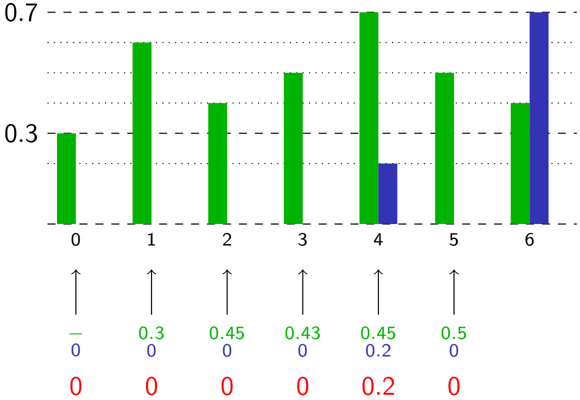

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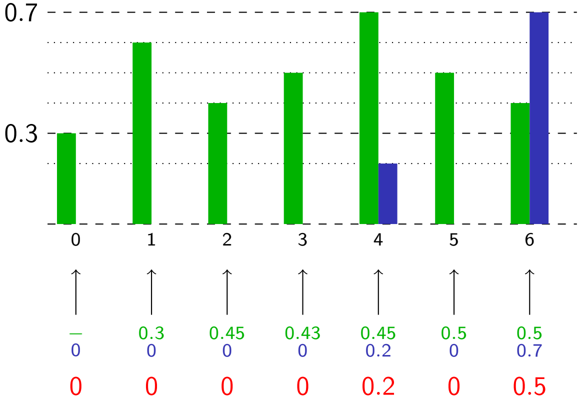

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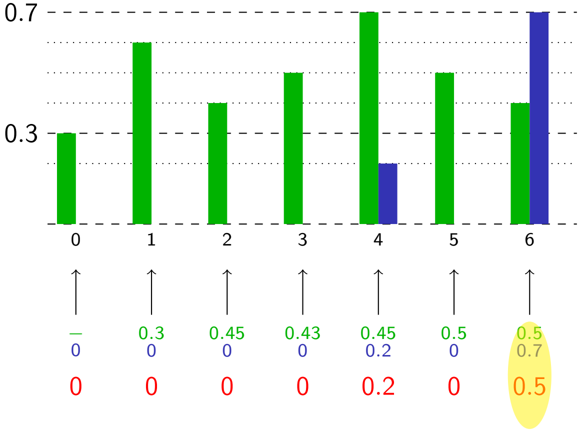

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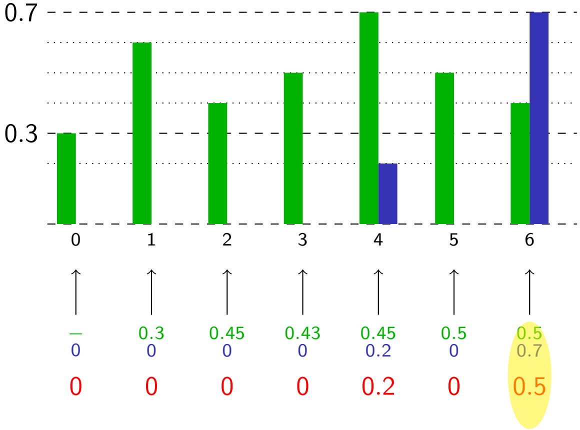

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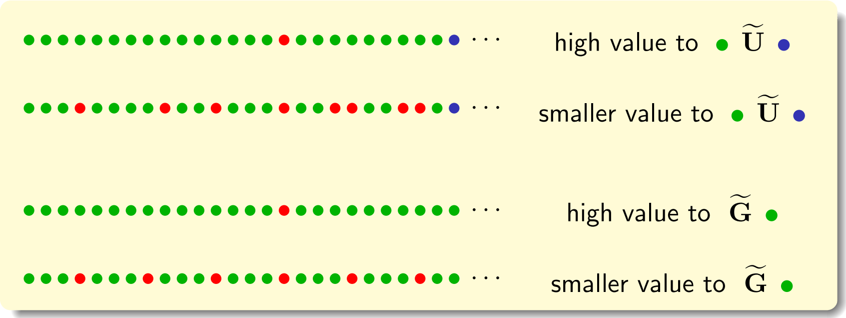

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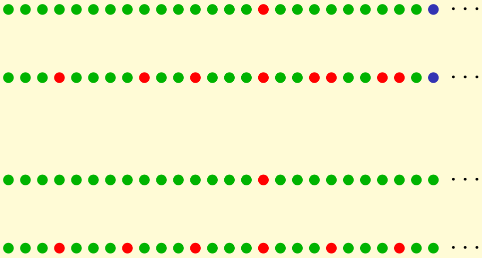

gives value 0.96 to ● $\widetilde{\mathbf{U}}$ ●

smaller value to ● $\widetilde{\mathbf{U}}$ ●

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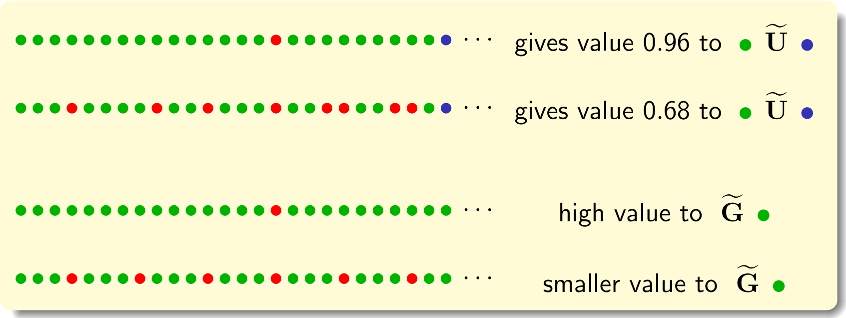
Back to the introductory examples



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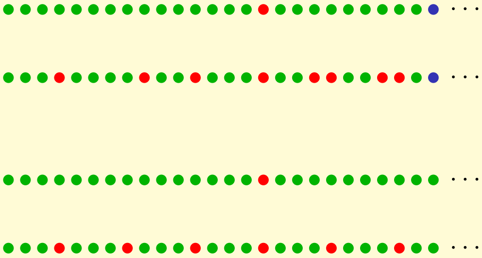


gives value 0.96 to $\bullet\ \widetilde{\mathbf{U}}\ \bullet$

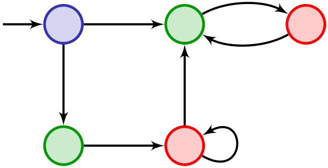
gives value 0.68 to $\bullet\ \widetilde{\mathbf{U}}\ \bullet$

gives value 1 to $\widetilde{\mathbf{G}}\ \bullet$

smaller value to $\widetilde{\mathbf{G}}\ \bullet$

Back to the introductory examples

Examples of computations in Kripke structure $\mathcal{K}$



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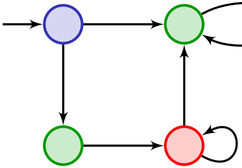

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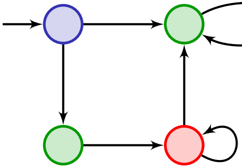

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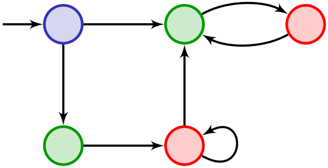

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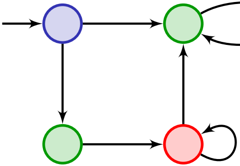

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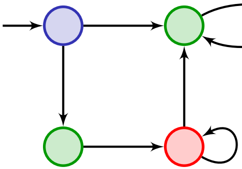

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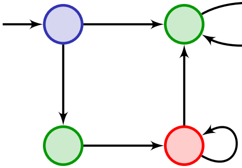

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Hence $[\![\mathcal{K}, \textcolor{red}{\bullet}\,\widetilde{\mathbf{U}}\,\textcolor{green}{\bullet}]\!] = 1$, even though no run witnesses that value.

Outline

1. Introduction
2. Average-LTL
3. Results
4. Conclusion

Our results

For any threshold $\bowtie c$:

- Existence problem: given $\mathcal{K}$, φ, is there π in $\mathcal{K}$ s.t. $[\![\pi, \varphi]\!] \bowtie c$?
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  I.e.

$$[\![q \bullet^{n_0} \bullet^{n_1} q' \bullet^{n_0'} \bullet^{n_1'} q'', \psi_\alpha \,\widetilde{\mathbf{U}}\, Q]\!] = \frac{1}{2}$$

  where $\psi_\alpha = q' \vee \bullet \vee \bullet$ when $\alpha = 1 + n_0 + n_1'$

Undecidability of the existence problem (cont'd)

Global reduction

- Formula to be checked:

$$\mathtt{halt}_{\mathcal{M}} = \mathbf{F}\, q_{halt} \wedge \mathbf{G}\, \mathtt{consec}_{\mathcal{M}}$$

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$\rightsquigarrow$ This yields the undecidability of the existence problem for threshold $= 1/2$

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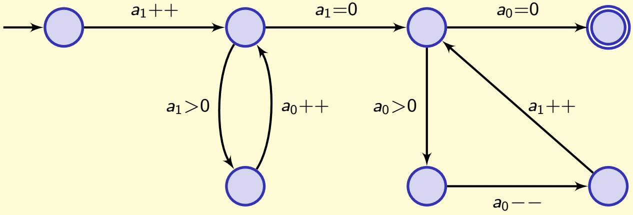

☞ We need to be careful with machines having such a converging phenomenon

Arguments for the undecidability proof

Technical lemmas

- For a finite invalid run, if the first invalid consecution assigns a value smaller than $1/2 - 1/n$ to $\texttt{consec}_{\mathcal{M}}$, then the value of formula $\mathbf{G}\,\texttt{consec}_{\mathcal{M}}$ along the run is smaller than $1/2 - 1/n$.

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- If $[\![\mathcal{K}_{\mathcal{M}}, \mathtt{accept}_{\mathcal{M}}]\!] = 1/2$ but no run gives value $1/2$ to formula $\mathtt{accept}_{\mathcal{M}}$, then the unique valid run is infinite.

The diagonal argument

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Towards a contradiction, assume it is computable by det. Turing machine $\mathcal{T}_{\mathcal{H}}$,

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Therefore, $\mathcal{H}$ is not computable.

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