On the optimal reachability problem in weighted timed games

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Based on former works with Thomas Brihaye, Kim G. Larsen, Nicolas Markey, etc...
And on recent work with Samy Jaziri and Nicolas Markey

Outline

Time-dependent systems

- We are interested in timed systems

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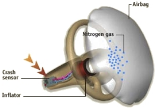

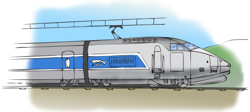

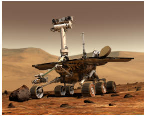

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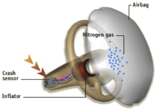
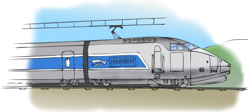
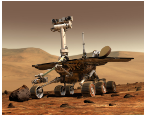




- ... and in their analysis and control

An example: The task graph scheduling problem

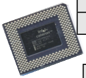
Compute $D\times(C\times(A+B))+(A+B)+(C\times D)$ using two processors:

P_1 (fast):

time	
$+$	2 picoseconds
$\times$	3 picoseconds

energy	
idle	10 Watt
in use	90 Watts

P_2 (slow):



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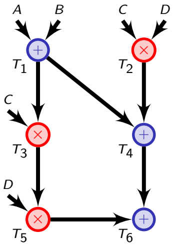

[BFLM10] Bouyer, Fahrenberg, Larsen, Markey. Quantitative Analysis of Real-Time Systems using Priced Timed Automata.

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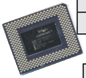

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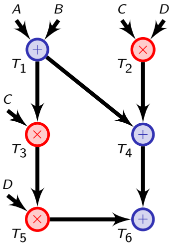

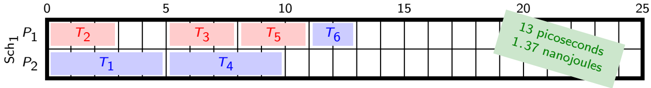

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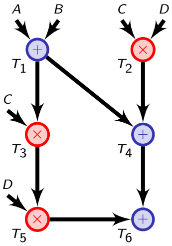

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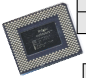

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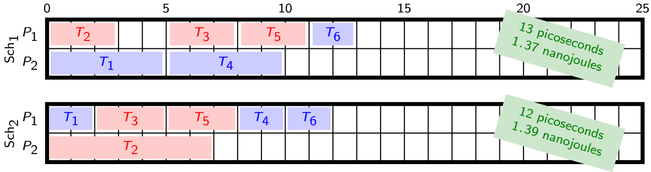

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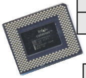

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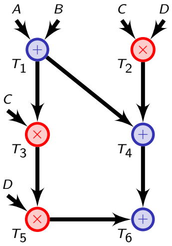

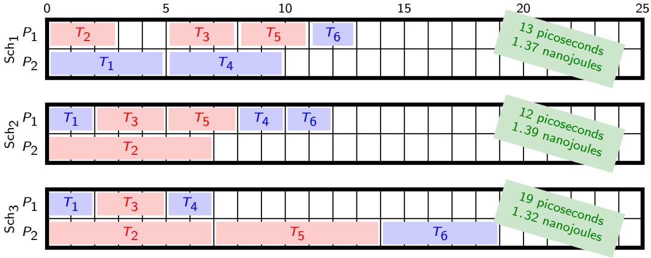

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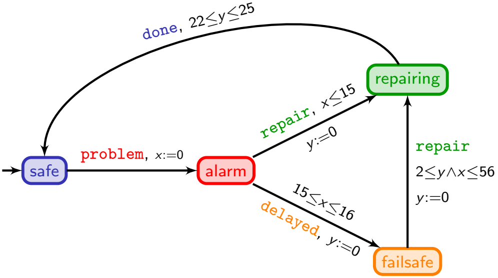
The model of timed automata



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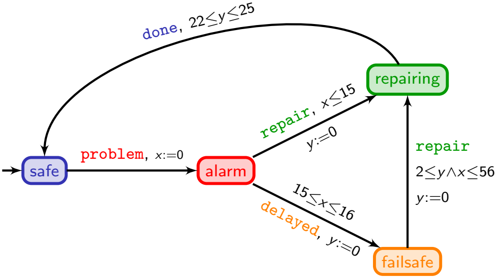

	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe	
x	0		23		0		15.6		15.6	⋯
y	0		23		23		38.6		0	

	failsafe	$\xrightarrow{2.3}$	failsafe	$\xrightarrow{\texttt{repair}}$	repairing	$\xrightarrow{22.1}$	repairing	$\xrightarrow{\texttt{done}}$	safe	
⋯	15.6		17.9		17.9		40		40	
	0		2.3		0		22.1		22.1	

Modelling the task graph scheduling problem

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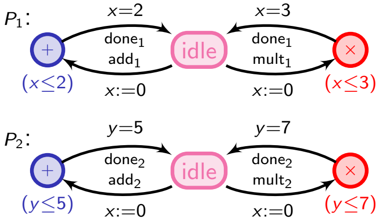
- Processors



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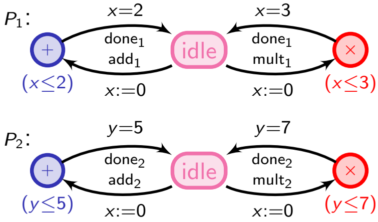
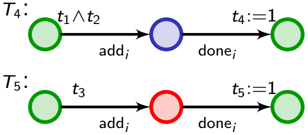
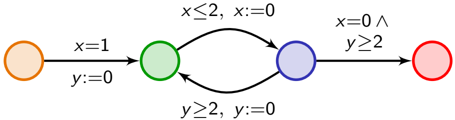


- Tasks



A schedule is a path in the product automaton

Analyzing timed automata



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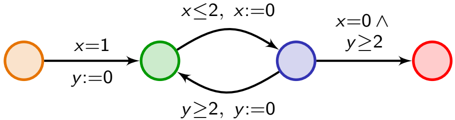

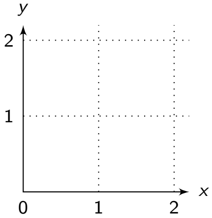

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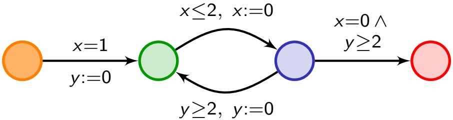

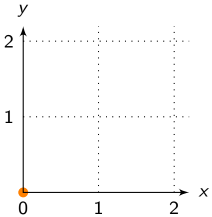

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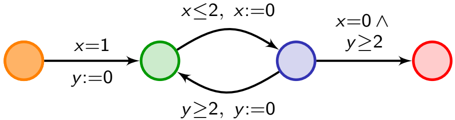

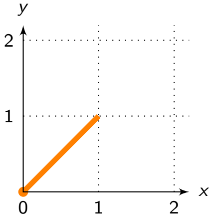

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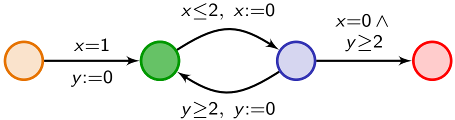

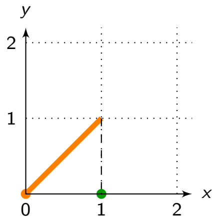

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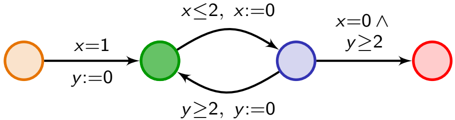

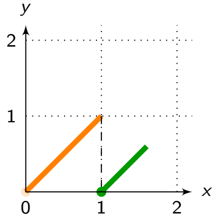

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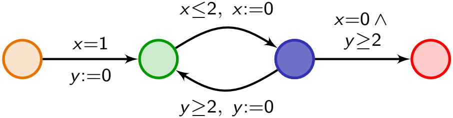

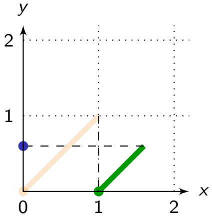

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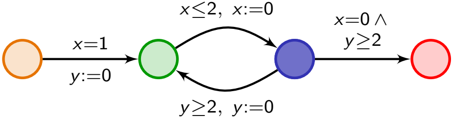

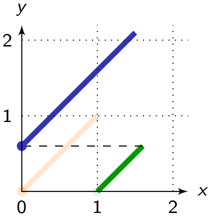

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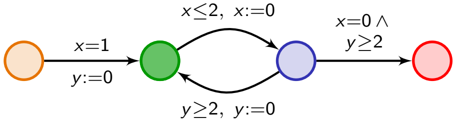

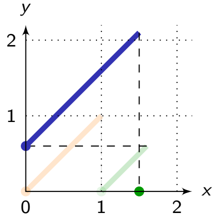

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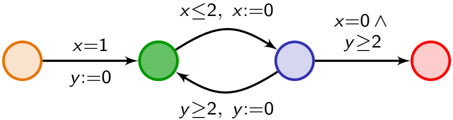

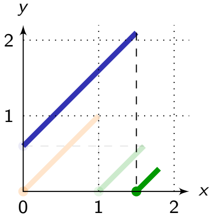

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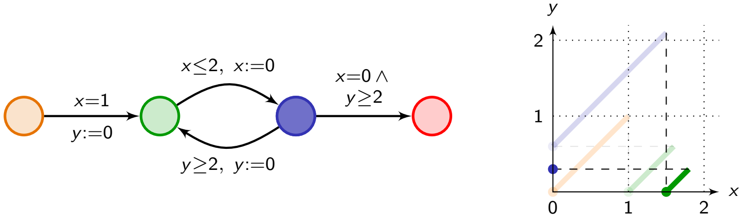

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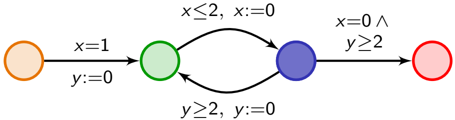

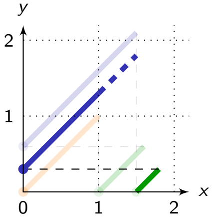

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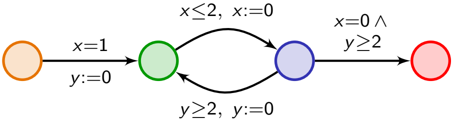

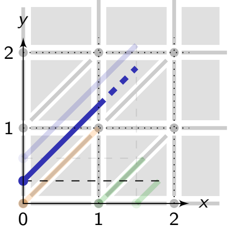

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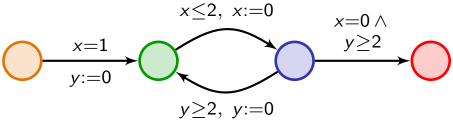
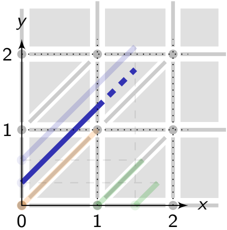


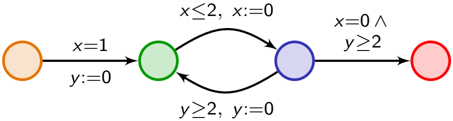


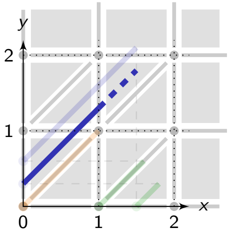
Theorem [AD94]

Reachability in timed automata is decidable (as well as many other important properties). It is PSPACE-complete.

- Technical tool: region abstraction

Analyzing timed automata





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- Efficient symbolic technics based on zones, implemented in tools

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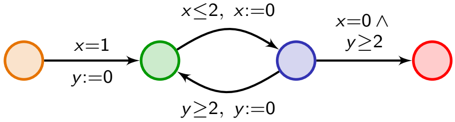

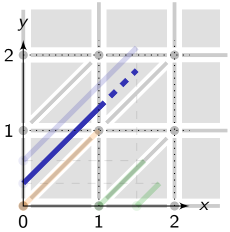

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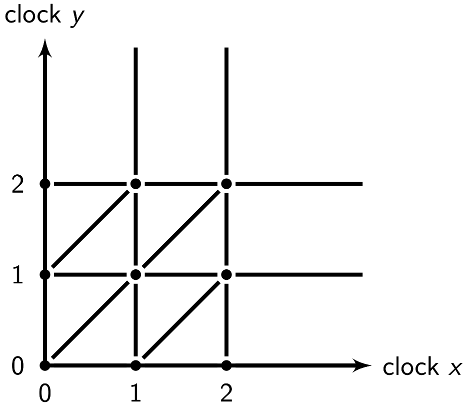
Skip regions

Technical tool: Region abstraction



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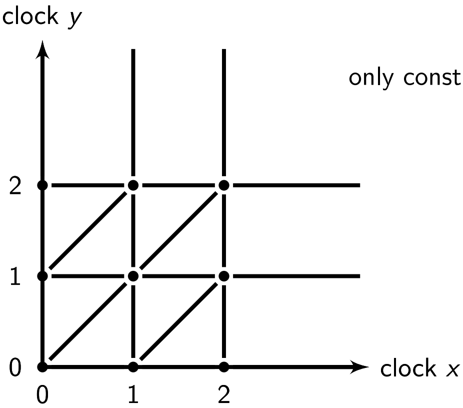
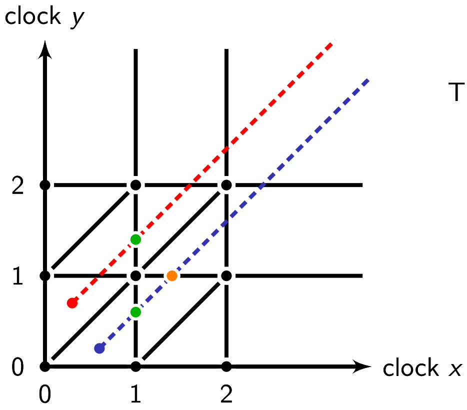
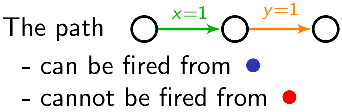


only constraints: $x \sim c$ with $c \in \{0, 1, 2\}$
$y \sim c$ with $c \in \{0, 1, 2\}$

- "compatibility" between regions and constraints

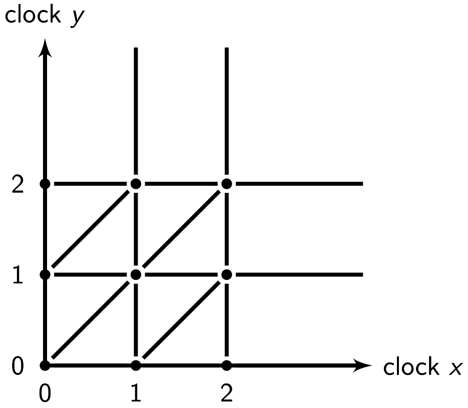
Technical tool: Region abstraction





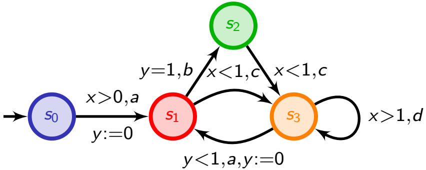
- "compatibility" between regions and constraints
- "compatibility" between regions and time elapsing

Technical tool: Region abstraction



$\rightsquigarrow$ This is a finite time-abstract bisimulation!

Technical tool: Region abstraction – An example [AD94]



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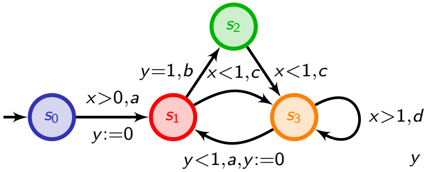

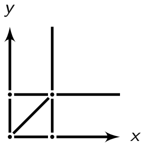

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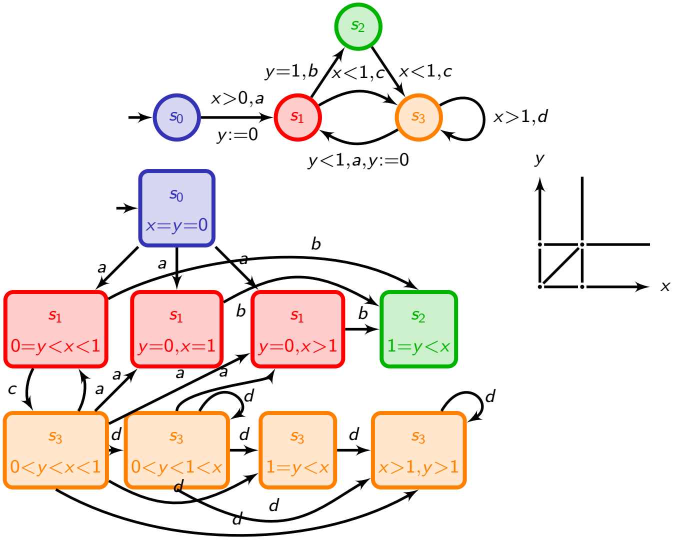

Outline

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1. Introduction
2. Overview of “old” results
   - Weighted timed automata
   - Timed games
   - Weighted timed games
3. Some recent developments
   - Undecidability of the value problem
   - Approximation of the optimal cost
   - Back to the undecidability
4. Conclusion

Modelling resources in timed systems

- System resources might be relevant and even crucial information

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  - memory usage,
  - ...
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- A possible solution: use hybrid automata

  a discrete control (the mode of the system)
  $+$ continuous evolution of the variables within a mode

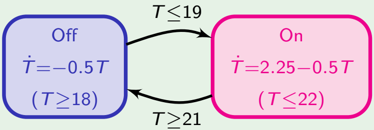
Modelling resources in timed systems

- System resources might be relevant and even crucial information
  - energy consumption,
  - memory usage,
  - ...
  - price to pay,
  - bandwidth,

$\rightsquigarrow$ timed automata are not powerful enough!

- A possible solution: use hybrid automata

The thermostat example



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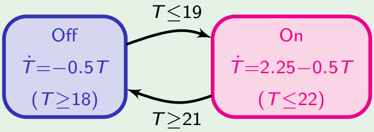
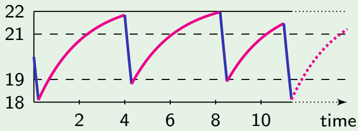
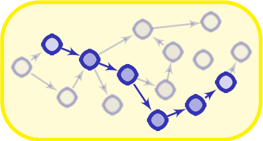




Ok...



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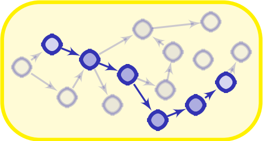

Easy...

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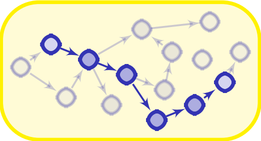

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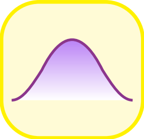

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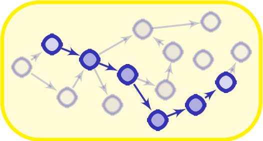

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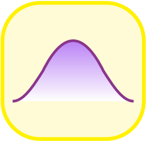
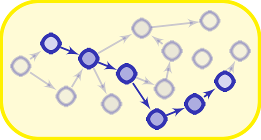
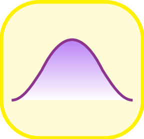


Easy...

Ok... but?



Easy...



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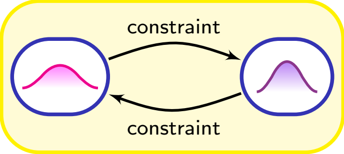

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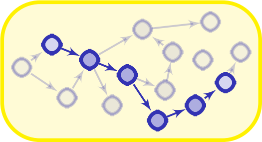

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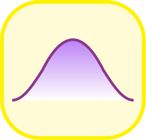

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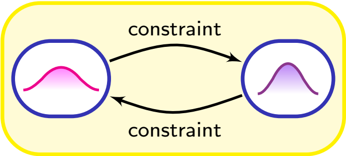

Hard!

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Theorem [HKPV95]

The reachability problem is undecidable in hybrid automata. Even for the simplest, the so-called stopwatch automata (clocks can be stopped).

[HKPV95] Henzinger, Kopke, Puri, Varaiya. What's decidable wbout hybrid automata? *(SToC'95)*.

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Theorem [HKPV95]

The reachability problem is undecidable in hybrid automata. Even for the simplest, the so-called stopwatch automata (clocks can be stopped).

- An alternative: weighted/priced timed automata [ALP01,BFH+01]
  $\rightsquigarrow$ hybrid variables do not constrain the system
  hybrid variables are observer variables

[HKPV95] Henzinger, Kopke, Puri, Varaiya. What's decidable wbout hybrid automata? *(SToC'95)*.
[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
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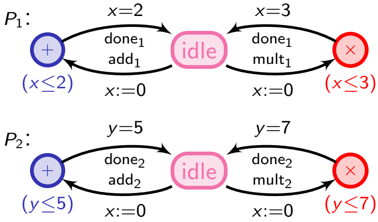
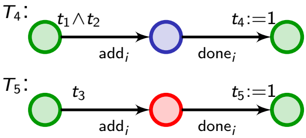
Modelling the task graph scheduling problem

- Processors



- Tasks



Modelling the task graph scheduling problem

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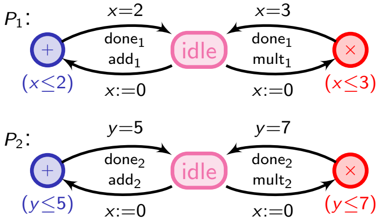

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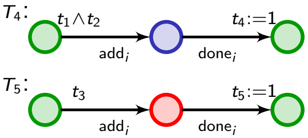
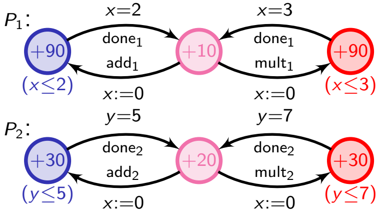
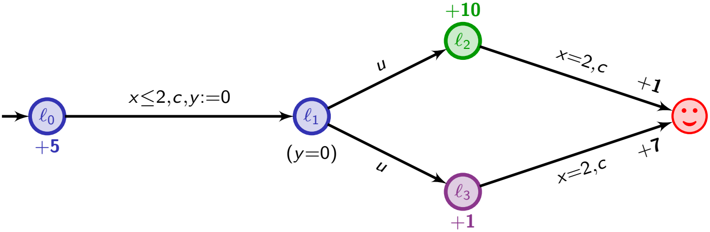


- Modelling energy



A good schedule is a path in the product automaton with a low cost

Weighted/priced timed automata [ALP01,BFH+01]



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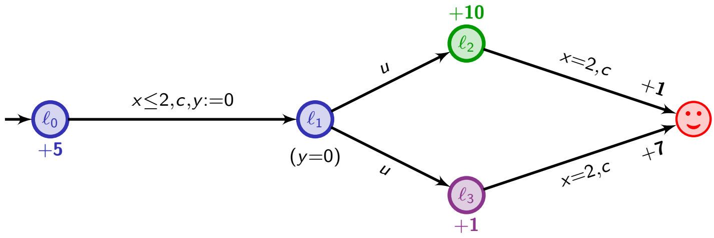

$$\begin{array}{cccccccccccc} & \ell_0 & \xrightarrow{1.3} & \ell_0 & \xrightarrow{c} & \ell_1 & \xrightarrow{u} & \ell_3 & \xrightarrow{0.7} & \ell_3 & \xrightarrow{c} & \text{☺} \\ x & 0 & & 1.3 & & 1.3 & & 1.3 & & 2 & & \\ y & 0 & & 1.3 & & 0 & & 0 & & 0.7 & & \end{array}$$

[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
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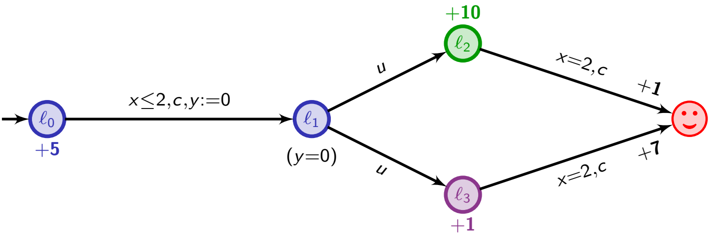

$$
\begin{array}{ccccccccccccc}
 & \ell_0 & \xrightarrow{1.3} & \ell_0 & \xrightarrow{c} & \ell_1 & \xrightarrow{u} & \ell_3 & \xrightarrow{0.7} & \ell_3 & \xrightarrow{c} & ☺ \\
x & 0 & & 1.3 & & 1.3 & & 1.3 & & 2 & & \\
y & 0 & & 1.3 & & 0 & & 0 & & 0.7 & &
\end{array}
$$

cost :

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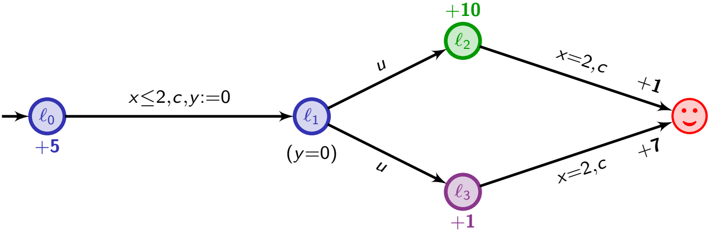

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5

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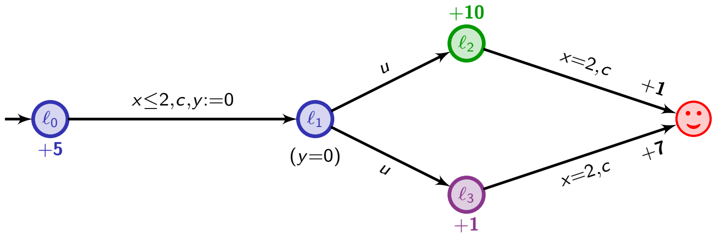

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x & 0 & & 1.3 & & 1.3 & & 1.3 & & 2 & & \\
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\end{array}$$

cost : 6.5 + 0

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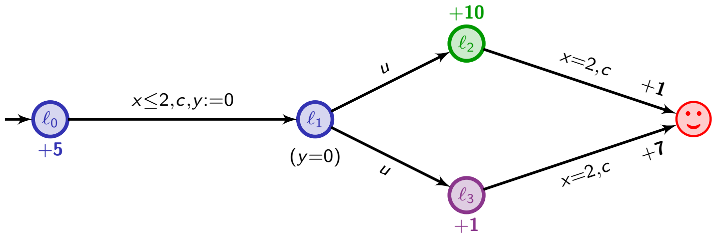

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x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0

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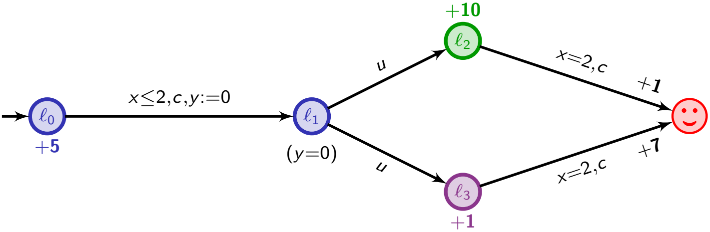

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x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		
cost :		6.5	+	0	+	0	+	0.7			

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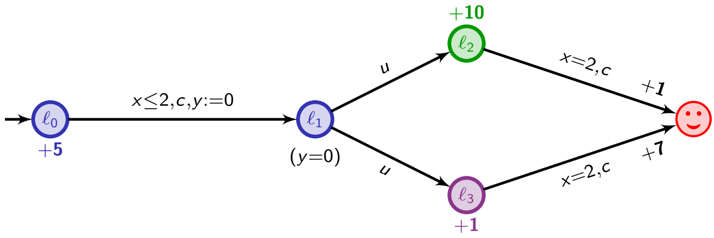

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0 + 0.7 + 7

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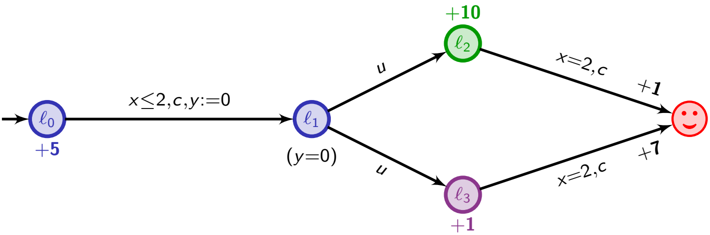

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0 + 0.7 + 7 = 14.2

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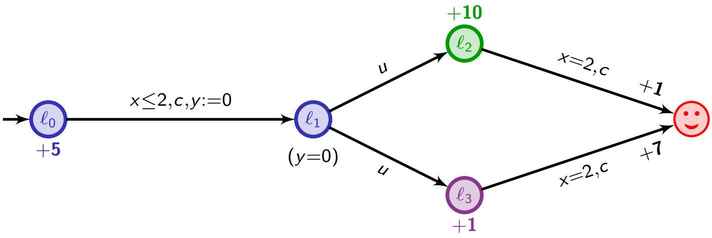
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching 🙂 ?

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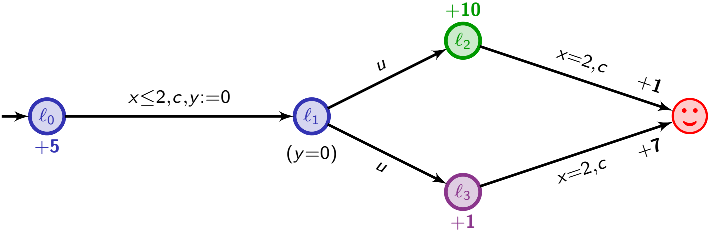

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$$5t + 10(2 - t) + 1$$

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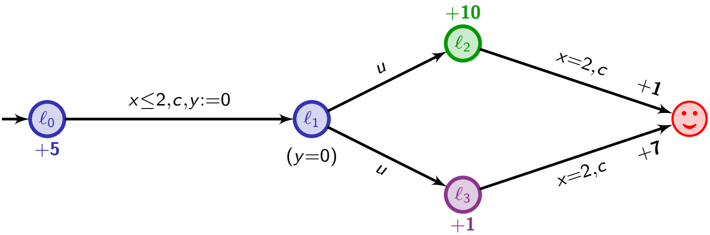

Question: what is the optimal cost for reaching ☺ ?

$$5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7$$

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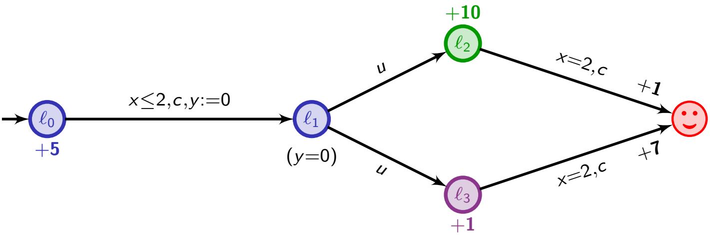
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching ☺ ?

$$\min\ (\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ )$$

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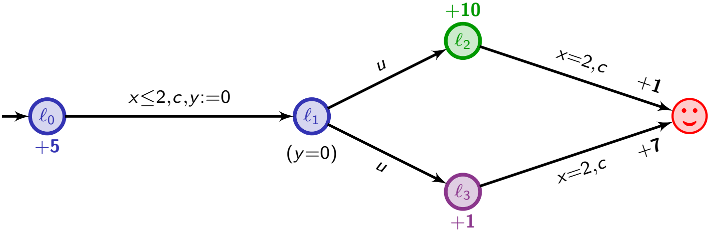
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching 🙂?

$$\inf_{0 \leq t \leq 2} \min\left(5t + 10(2-t) + 1 \ , \ 5t + (2-t) + 7 \right) = 9$$

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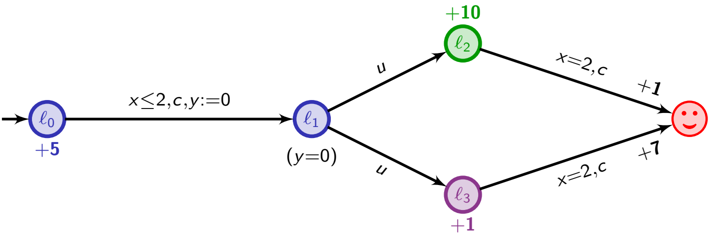
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching ☺?

$$\inf_{0 \leq t \leq 2} \min\ (\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ ) = 9$$

$\rightsquigarrow$ *strategy:* leave immediately ℓ_0, go to ℓ_3, and wait there 2 t.u.

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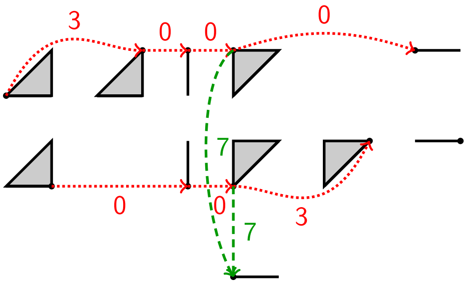
Optimal-cost reachability

Theorem [ALP01,BFH+01,BBBR07]

In weighted timed automata, the optimal cost is an integer and can be computed in PSPACE.

- Technical tool: a refinement of the regions, the corner-point abstraction



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[BBBR07] Bouyer, Brihaye, Bruyère, Raskin. On the optimal reachability problem *(Formal Methods in System Design)*.

From timed to discrete behaviours

Optimal reachability as a linear programming problem

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow{t_1} \circ \xrightarrow{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \left\{ \begin{array}{l} t_1+t_2\leq 2 \\ \end{array} \right.$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \begin{cases} t_1+t_2\leq 2 \\ t_2+t_3+t_4\geq 5 \end{cases}$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\begin{array}{cccccc} & T_1 & T_2 & T_3 & T_4 & T_5 \end{array}$$

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \left\{ \begin{array}{ll} t_1+t_2\leq 2 & T_2\leq 2 \\ t_2+t_3+t_4\geq 5 & T_4-T_1\geq 5 \end{array} \right.$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5$

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \begin{cases} t_1+t_2\leq 2 & T_2\leq 2 \\ t_2+t_3+t_4\geq 5 & T_4-T_1\geq 5 \end{cases}$$

Lemma

Let Z be a bounded zone and f be a function

$$f : (T_1, \ldots, T_n) \mapsto \sum_{i=1}^{n} c_i T_i + c$$

well-defined on $\overline{Z}$. Then $inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\begin{array}{cccccccccccc} & T_1 & & T_2 & & T_3 & & T_4 & & T_5 & & \\ \circ & \xrightarrow[y:=0]{t_1} & \circ & \xrightarrow[x\leq 2]{t_2} & \circ & \xrightarrow{t_3} & \circ & \xrightarrow[y\geq 5]{t_4} & \circ & \xrightarrow{t_5} & \circ & \cdots \end{array} \quad \begin{cases} t_1+t_2\leq 2 & T_2\leq 2 \\ t_2+t_3+t_4\geq 5 & T_4-T_1\geq 5 \end{cases}$$

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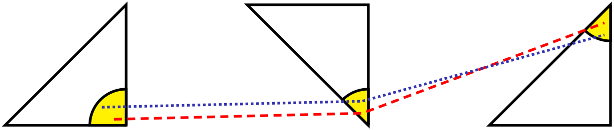
$\rightsquigarrow$ for every finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{\mathsf{cp}}$ such that

$$\mathsf{cost}(\Pi) \leq \mathsf{cost}(\pi)$$

[Π is a "corner-point projection" of π]

From discrete to timed behaviours

Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$,

From discrete to timed behaviours

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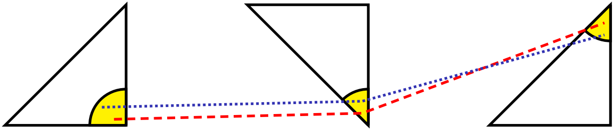

For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$,

From discrete to timed behaviours

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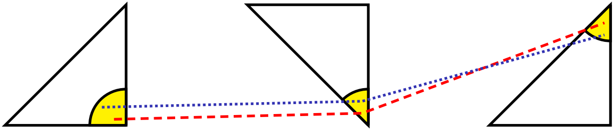

For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

From discrete to timed behaviours

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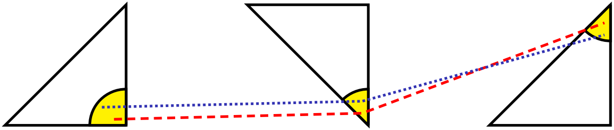

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$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon \Rightarrow |\mathsf{cost}(\Pi) - \mathsf{cost}(\pi_\varepsilon)| < \eta$$

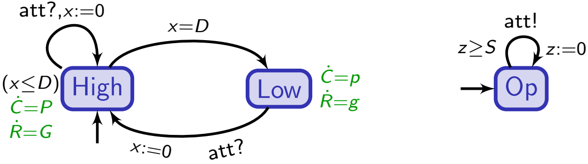
Note on the corner-point abstraction

It is a very interesting abstraction, that can be used in several other contexts:

- for mean-cost optimization [BBL04,BBL08]
- for discounted-cost optimization [FL08]
- for all concavely-priced timed automata [JT08]
- for deciding frequency objectives [BBBS11,Sta12]
- ...

[BBL04] Bouyer, Brinksma, Larsen. Staying Alive As Cheaply As Possible *(HSCC'04)*.
[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.
[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.
[JT08] Judziński, Trivedi. Concavely-priced timed automata *(FORMATS'08)*.
[BBBS11] Bertrand, Bouyer, Brihaye, Stainer. Emptiness and universality problems in timed automata with positive frequency *(ICALP'11)*.
[Sta12] Stainer. Frequencies in forgetful timed automata *(FORMATS'12)*.

Going further 1: mean-cost optimization



[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.

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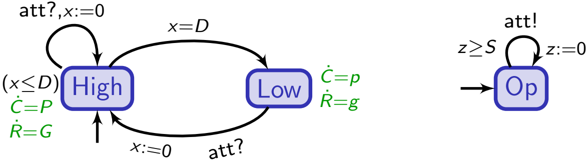

$\rightsquigarrow$ compute optimal infinite schedules that minimize

$$\text{mean-cost}(\pi) = \limsup_{n \to +\infty} \frac{\text{cost}(\pi_n)}{\text{reward}(\pi_n)}$$

[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.

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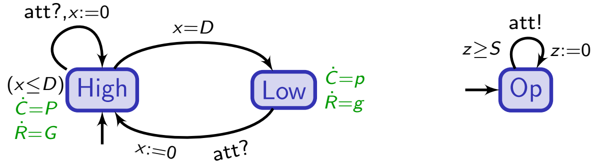

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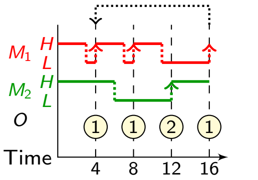
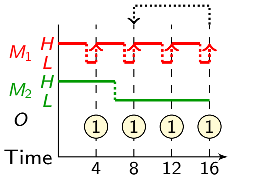


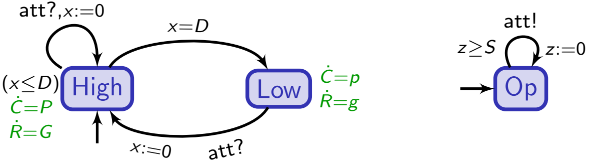
Schedule with ratio ≈1.455



Schedule with ratio ≈1.478

[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.

Going further 1: mean-cost optimization



$\rightsquigarrow$ compute optimal infinite schedules that minimize

$$\text{mean-cost}(\pi) = \limsup_{n \to +\infty} \frac{\text{cost}(\pi_n)}{\text{reward}(\pi_n)}$$

Theorem [BBL08]

In weighted timed automata, the optimal mean-cost can be compute in PSPACE.

$\rightsquigarrow$ the corner-point abstraction can be used

[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.

From timed to discrete behaviours

- **Finite behaviours:** based on the following property

Lemma

Let Z be a bounded zone and f be a function

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well-defined on $\overline{Z}$. Then $inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

$\leadsto$ for every finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{\text{cp}}$ s.t.
mean-cost(Π) $\leq$ mean-cost(π)

From timed to discrete behaviours

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Let Z be a bounded zone and f be a function

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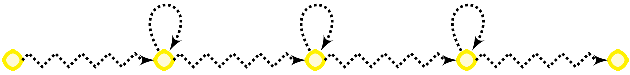
$$\text{mean-cost}(\Pi) \leq \text{mean-cost}(\pi)$$

- **Infinite behaviours:** decompose each sufficiently long projection into cycles:



The (acyclic) linear part will be negligible!

From timed to discrete behaviours

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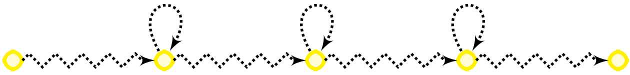
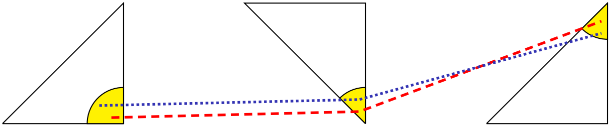


The (acyclic) linear part will be negligible!

$\rightsquigarrow$ the optimal cycle of $\mathcal{A}_{\mathsf{cp}}$ is better than any infinite path of $\mathcal{A}$!

From discrete to timed behaviours

Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathrm{cp}}$,

From discrete to timed behaviours

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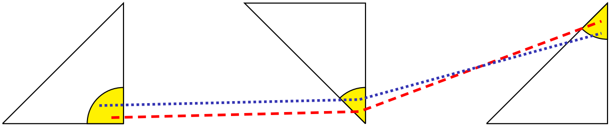
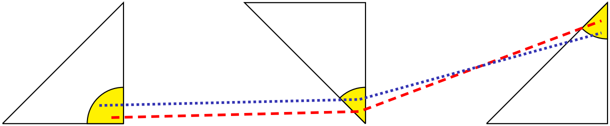


For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$,

From discrete to timed behaviours

Approximation of abstract paths:



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From discrete to timed behaviours

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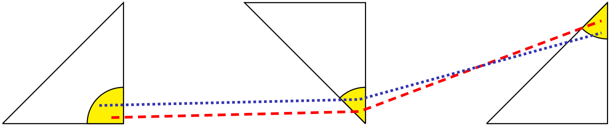

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Going further 2: concavely-priced cost functions

$\rightsquigarrow$ A general abstract framework for quantitative timed systems

Theorem [JT08]

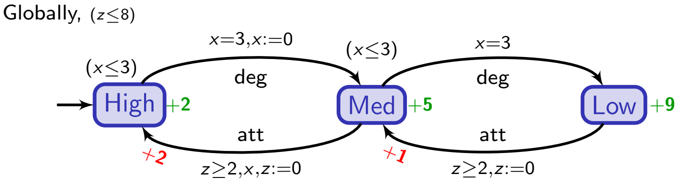
In concavely-priced timed automata, optimal cost is computable, if we restrict to quasi-concave cost functions. For the following cost functions, the (decision) problem is even PSPACE-complete:

- optimal-time and optimal-cost reachability;
- optimal discrete discounted cost;
- optimal mean-cost.

$\rightsquigarrow$ the corner-point abstraction can be used

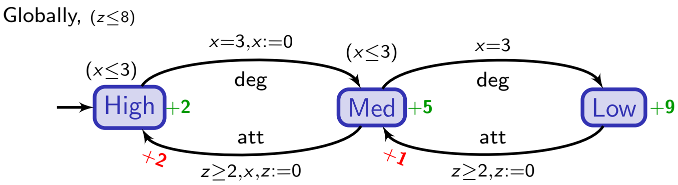
[JT08] Judziński, Trivedi. Concavely-priced timed automata *(FORMATS'08)*.

Going further 3: discounted-time cost optimization



[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.

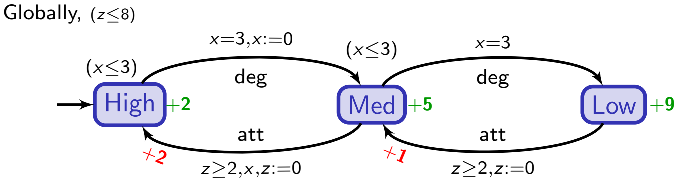
Going further 3: discounted-time cost optimization



$\rightsquigarrow$ compute optimal infinite schedules that minimize
discounted cost over time

[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.

Going further 3: discounted-time cost optimization



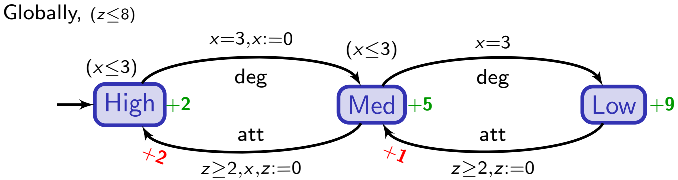
$\rightsquigarrow$ compute optimal infinite schedules that minimize

$$\text{discounted-cost}_{\lambda}(\pi) = \sum_{n\geq 0} \lambda^{T_n} \int_{t=0}^{\tau_{n+1}} \lambda^{t} \text{cost}(\ell_n)\, \mathrm{d}t + \lambda^{T_{n+1}} \text{cost}(\ell_n \xrightarrow{a_{n+1}} \ell_{n+1})$$

if $\pi = (\ell_0, v_0) \xrightarrow{\tau_1, a_1} (\ell_1, v_1) \xrightarrow{\tau_2, a_2} \cdots$ and $T_n = \sum_{i\leq n} \tau_i$

[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.

Going further 3: discounted-time cost optimization



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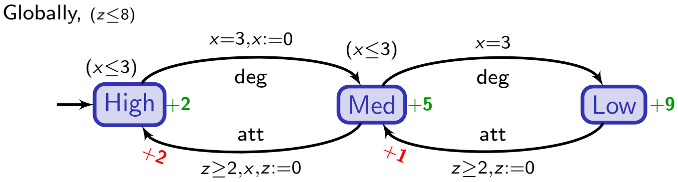

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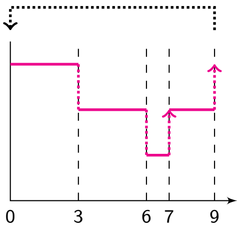

if $\lambda = e^{-1}$, the discounted cost of that infinite schedule is ≈ 2.16

[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.

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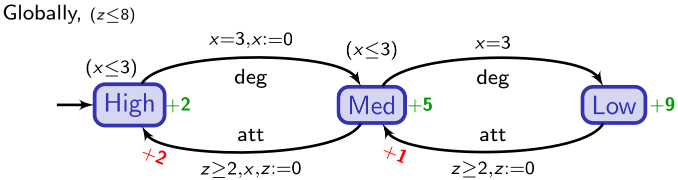

$\rightsquigarrow$ compute optimal infinite schedules that minimize
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Theorem [FL08]

In weighted timed automata. the optimal discounted cost is computable in EXPTIME.

$\rightsquigarrow$ the corner-point abstraction can be used

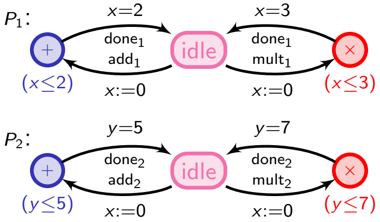
[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.

Outline

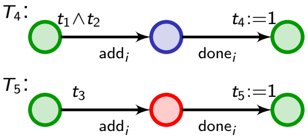
1. Introduction
2. Overview of “old” results
   - Weighted timed automata
   - Timed games
   - Weighted timed games
3. Some recent developments
   - Undecidability of the value problem
   - Approximation of the optimal cost
   - Back to the undecidability
4. Conclusion

Modelling the task graph scheduling problem

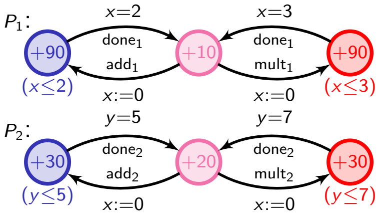
- Processors



- Tasks



- Modelling energy



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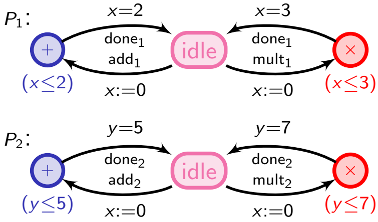

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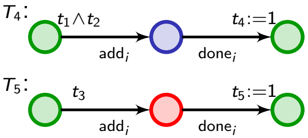

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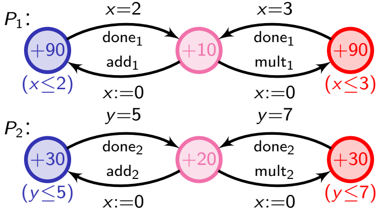
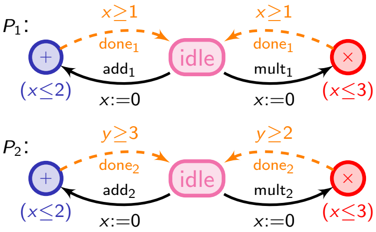


- Modelling uncertainty



Modelling the task graph scheduling problem

- Processors

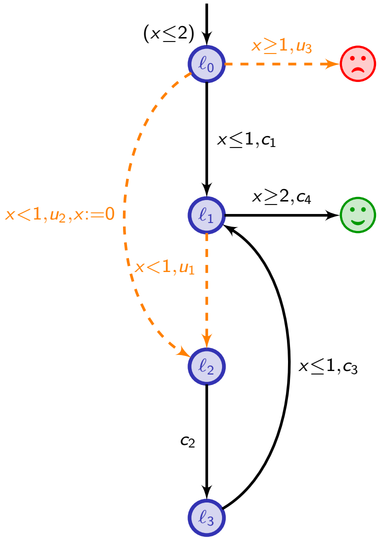
- Tasks

A (good) schedule is a strategy in the product game (with a low cost)

- Modelling energy

- Modelling uncertainty

An example of a timed game



Rule of the game

- Aim: avoid ☹ and reach ☺

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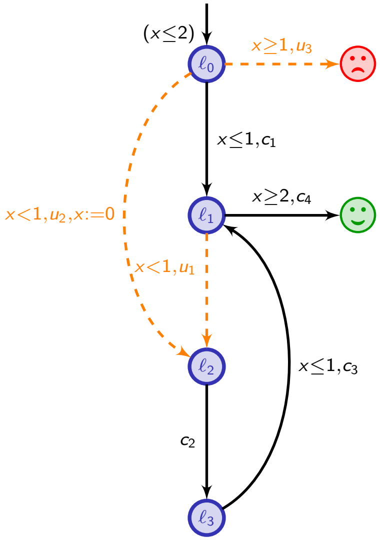

Rule of the game

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- How do we play? According to a strategy:

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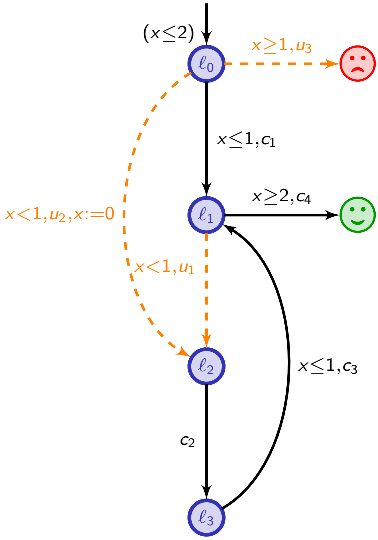

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$$f : \text{history} \mapsto (\text{delay}, \text{cont. transition})$$

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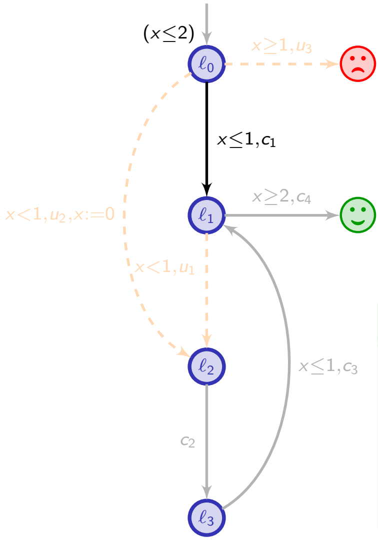

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A (memoryless) winning strategy

- from $(\ell_0, 0)$, play $(0.5, c_1)$

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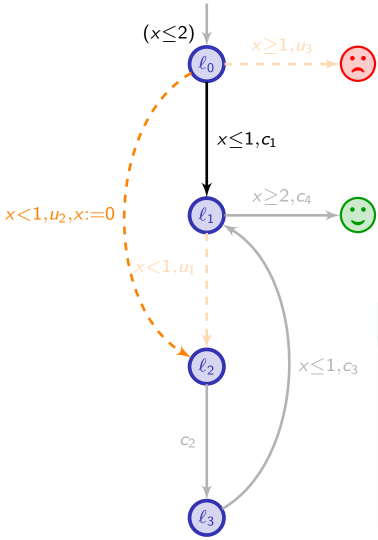

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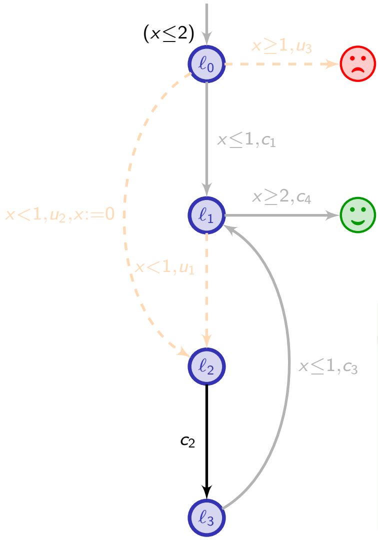

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- from $(\ell_2, \star)$, play $(1 - \star, c_2)$

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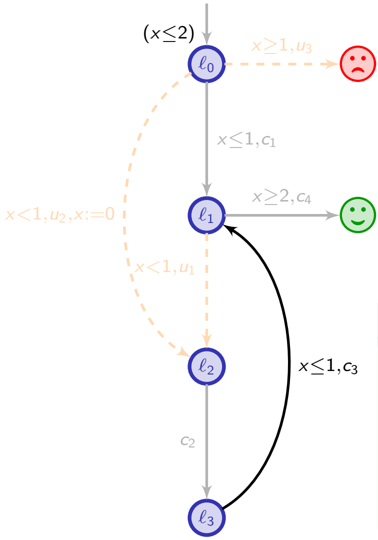

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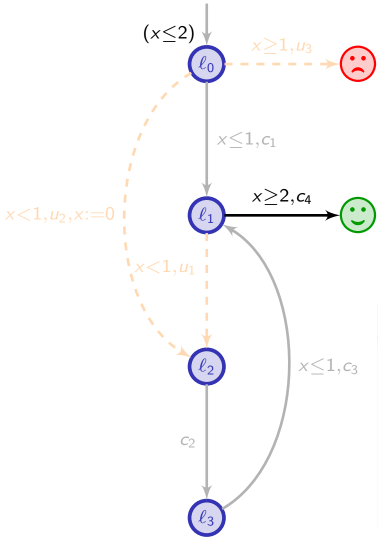
- from $(\ell_2, \star)$, play $(1-\star, c_2)$
- from $(\ell_3, 1)$, play $(0, c_3)$

An example of a timed game



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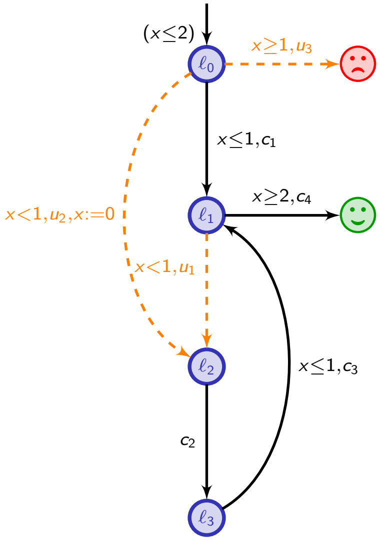
- How do we play? According to a strategy:

$$f : \text{history} \mapsto (\text{delay}, \text{cont. transition})$$

A (memoryless) winning strategy

- from $(\ell_0, 0)$, play $(0.5, c_1)$
  $\rightsquigarrow$ can be preempted by u_2
- from $(\ell_2, \star)$, play $(1 - \star, c_2)$
- from $(\ell_3, 1)$, play $(0, c_3)$
- from $(\ell_1, 1)$, play $(1, c_4)$

An example of a timed game



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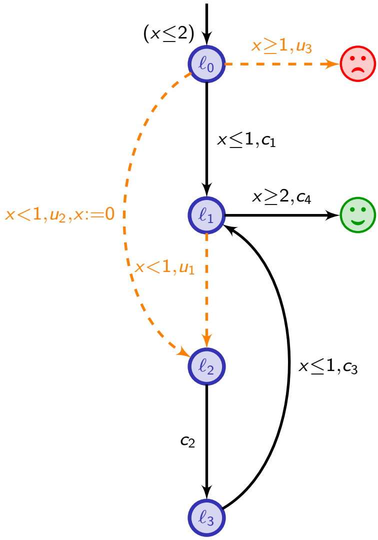

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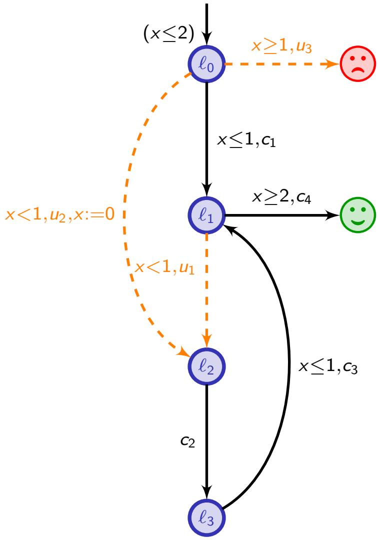

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Problems to be considered

- Does there exist a winning strategy?
- If yes, compute one (as simple as possible).

Decidability of timed games

Theorem [AMPS98,HK99]

Reachability and safety timed games are decidable and EXPTIME-complete. Furthermore memoryless and "region-based" strategies are sufficient.

[AMPS98] Asarin, Maler, Pnueli, Sifakis. Controller synthesis for timed automata *(SSC'98)*.
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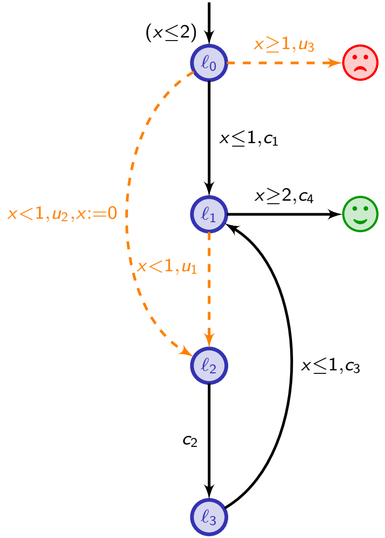
Theorem [AM99,BHPR07,JT07]

Optimal-time reachability timed games are decidable and EXPTIME-complete.

[AM99] Asarin, Maler. As soon as possible: time optimal control for timed automata *(HSCC'99)*.
[BHPR07] Brihaye, Henzinger, Prabhu, Raskin. Minimum-time reachability in timed games *(ICALP'07)*.
[JT07] Jurdziński, Trivedi. Reachability-time games on timed automata *(ICALP'07)*.

Back to the example: computing winning states



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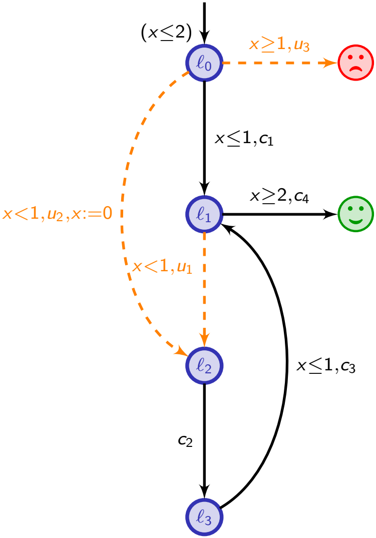

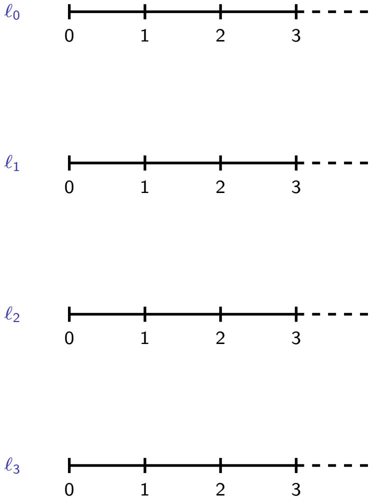

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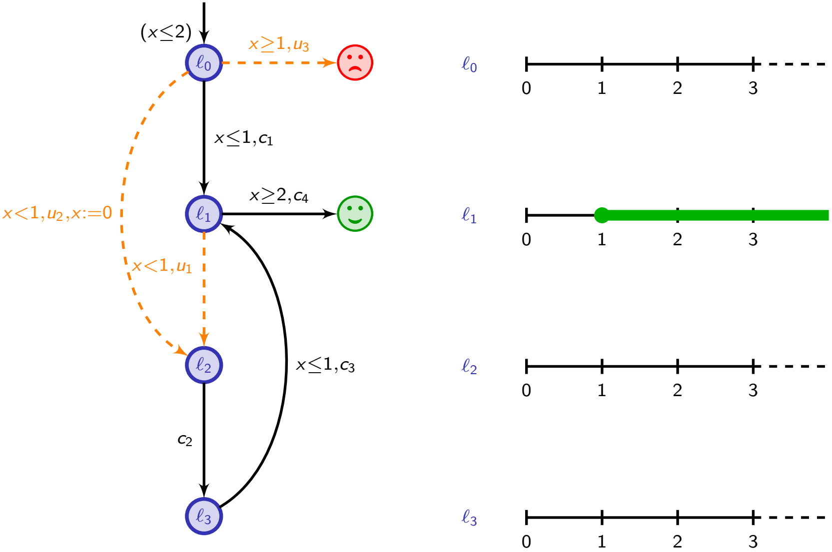

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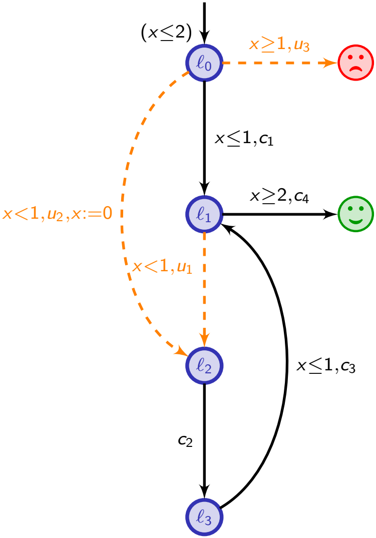

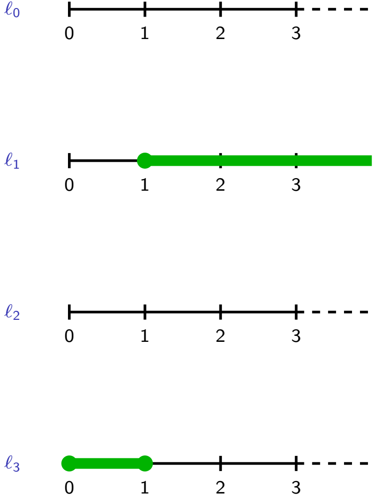

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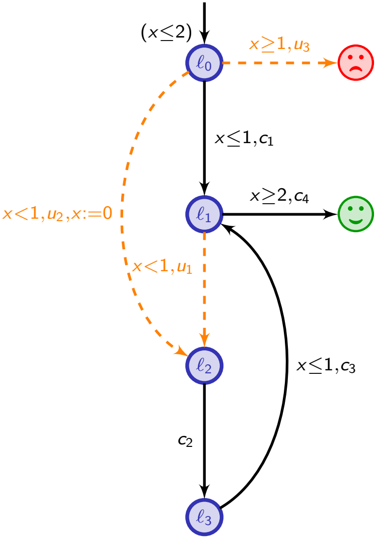

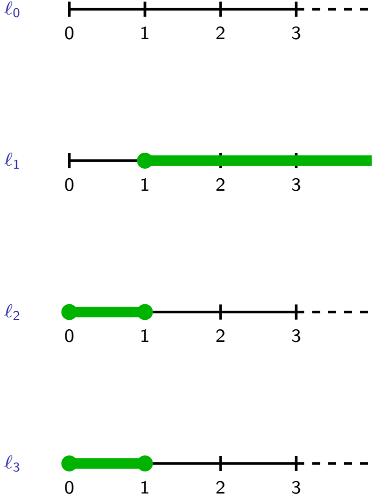

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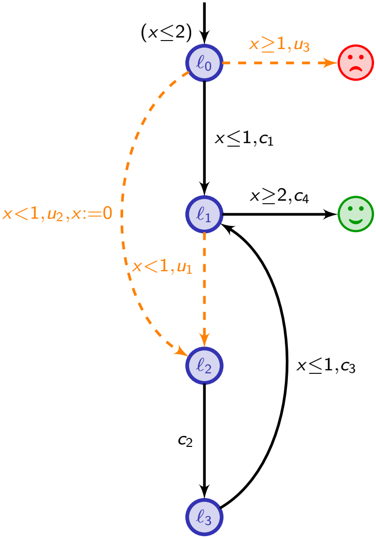

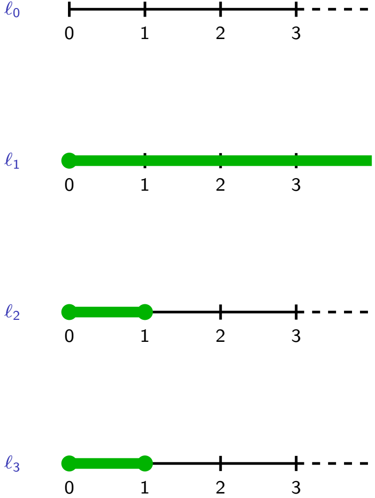

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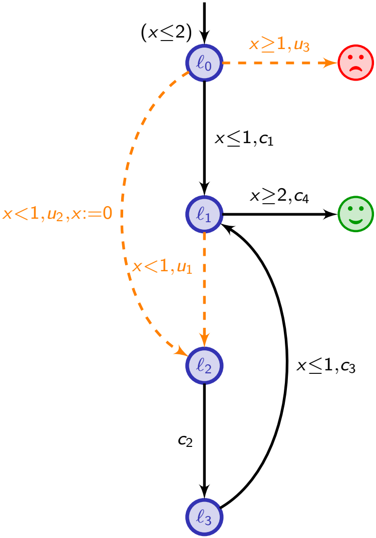

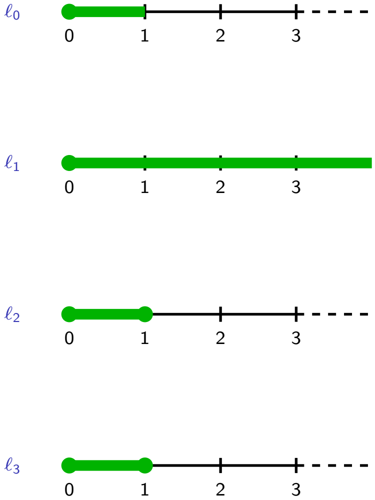

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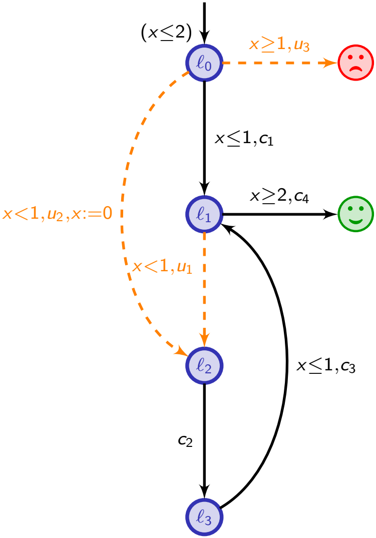

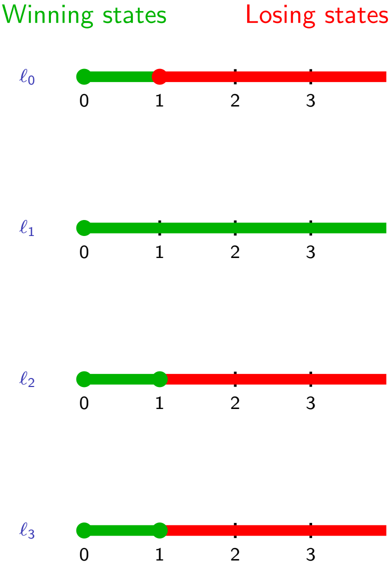

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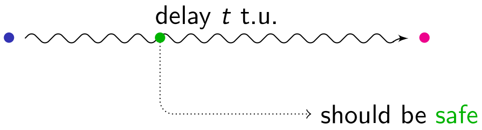

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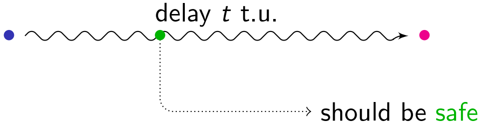

$$\mathsf{Pred}_\delta(X, \mathsf{Safe}) = \{\bullet \mid \exists t \geq 0,\ \bullet \xrightarrow{\delta(t)} \bullet$$
$$\text{and } \forall 0 \leq t' \leq t,\ \bullet \xrightarrow{\delta(t')} \bullet \in \mathsf{Safe}\}$$

Timed games with a reachability objective

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$$\pi(X) = X \cup \mathsf{Pred}_\delta(\mathsf{cPred}(X), \neg\mathsf{uPred}(\neg X))$$

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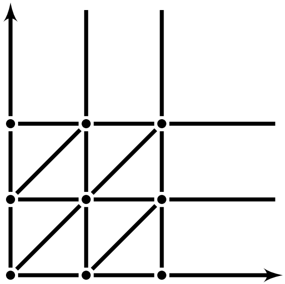

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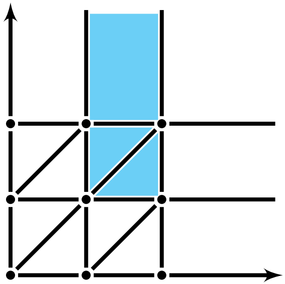

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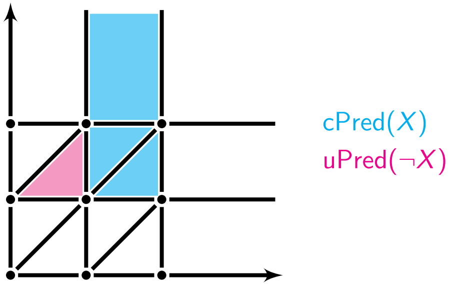

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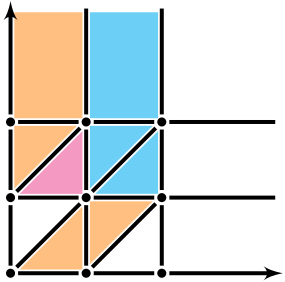

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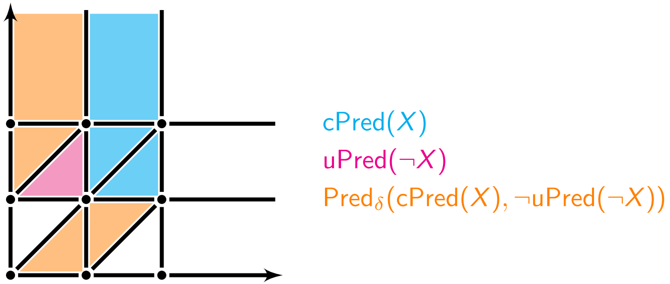

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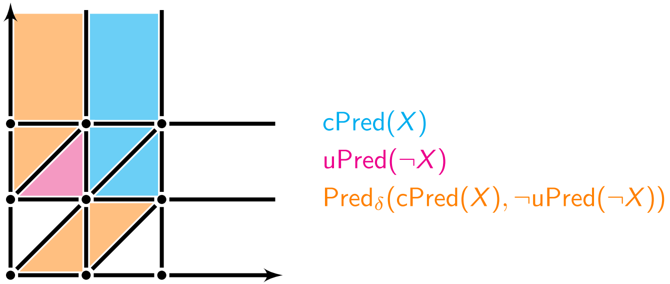

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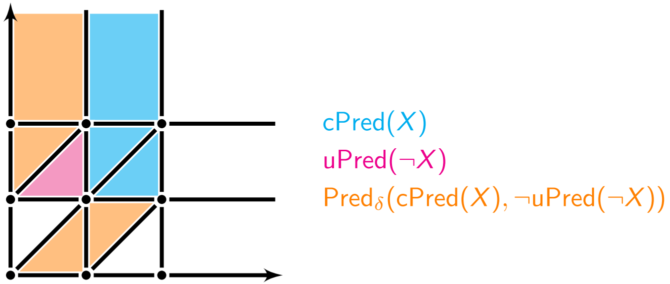

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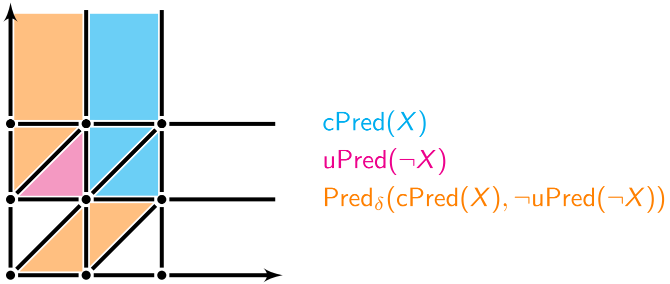

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- We can use operator $\widetilde{\pi}$ defined by

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Outline

1. Introduction
2. Overview of “old” results
   - Weighted timed automata
   - Timed games
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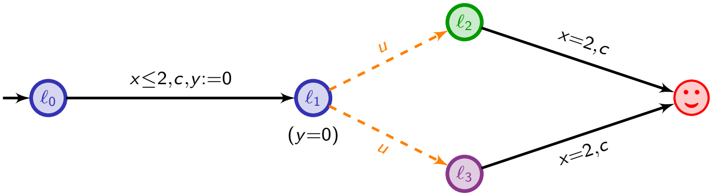
3. Some recent developments
   - Undecidability of the value problem
   - Approximation of the optimal cost
   - Back to the undecidability
4. Conclusion

A simple timed game



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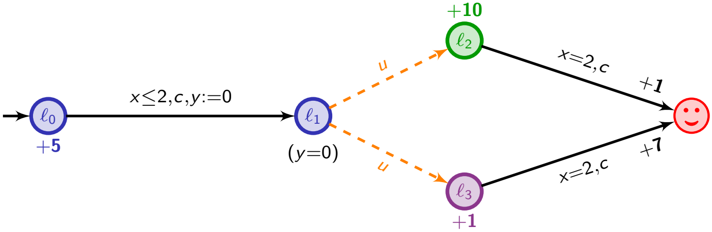

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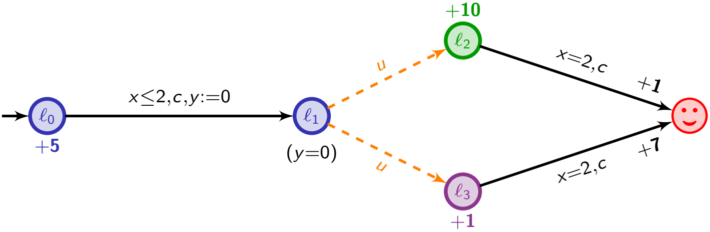

Question: what is the optimal cost we can ensure while reaching 🙂?

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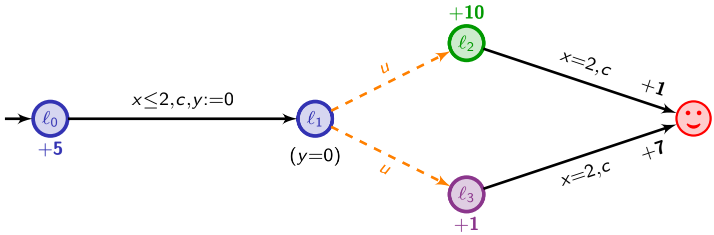

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$$5t + 10(2 - t) + 1$$

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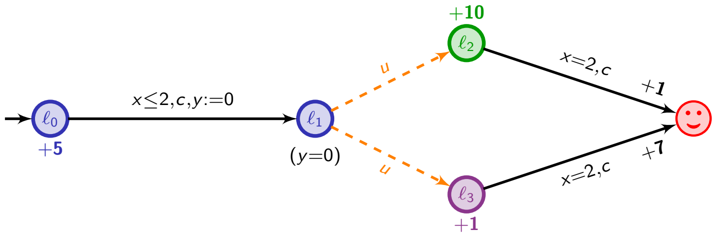

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$$5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7$$

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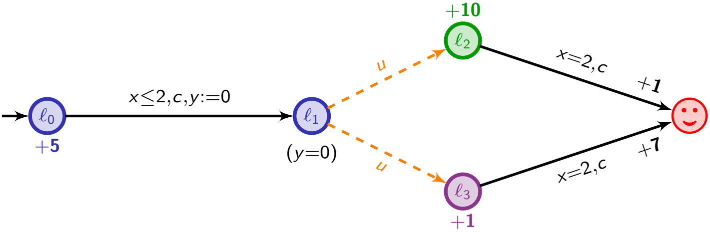

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$$\max\left(\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ \right)$$

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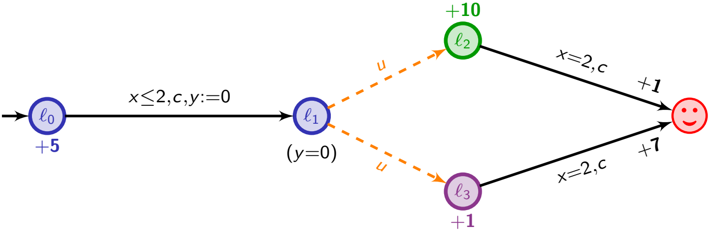

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$$\inf_{0\leq t\leq 2} \max\left(5t+10(2-t)+1 \;,\; 5t+(2-t)+7 \right) = 14+\frac{1}{3}$$

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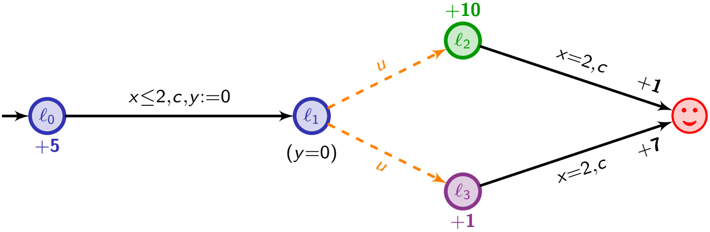

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$\rightsquigarrow$ *strategy:* wait in ℓ_0, and when $t=\frac{4}{3}$, go to ℓ_1

Optimal reachability in weighted timed games (1)

This topic has been fairly hot these last fifteen years...

[LMM02,ABM04,BCFL04,BBR05,BBM06,BLMR06,Rut11,HIM13,BGK+14]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.
[ABM04] Alur, Bernardsky, Madhusudan. Optimal reachability in weighted timed games *(ICALP'04)*.
[BCFL04] Bouyer, Cassez, Fleury, Larsen. Optimal strategies in priced timed game automata *(FSTTCS'04)*.
[BBR05] Brihaye, Bruyère, Raskin. On optimal timed strategies *(FORMATS'05)*.
[BBM06] Bouyer, Brihaye, Markey. Improved undecidability results on weighted timed automata *(Information Processing Letters)*.
[BLMR06] Bouyer, Larsen, Markey, Rasmussen. Almost-optimal strategies in one-clock priced timed automata *(FSTTCS'06)*.
[Rut11] Rutkowski. Two-player reachability-price games on single-clock timed automata *(QAPL'11)*.
[HIM13] Hansen, Ibsen-Jensen, Miltersen. A faster algorithm for solving one-clock priced timed games *(CONCUR'13)*.
[BGK+14] Brihaye, Geeraerts, Krishna, Manasa, Monmege, Trivedi. Adding Negative Prices to Priced Timed Games *(CONCUR'14)*.

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Tree-like weighted timed games can be solved in 2EXPTIME.

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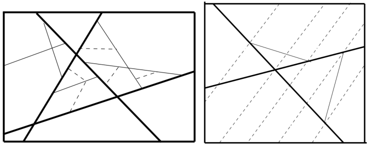
[LMM02]

Tree-like weighted timed games can be solved in 2EXPTIME.

[ABM04,BCFL04]

Depth-k weighted timed games can be solved in EXPTIME. There is a symbolic algorithm to solve weighted timed games **with a strongly non-Zeno cost**.



Optimal reachability in weighted timed games (2)

[BBR05,BBM06]

In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more.

Optimal reachability in weighted timed games (2)

[BBR05,BBM06]

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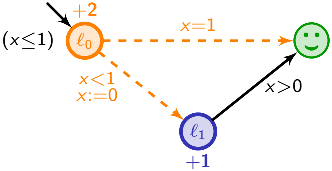
[BLMR06,Rut11,HIM13,BGK+14]

Turn-based optimal timed games are decidable in EXPTIME (resp. PTIME) when automata have a single clock (resp. with two rates). They are PTIME-hard.

What is easier with a single clock?

- Memoryless strategies can be non-optimal...



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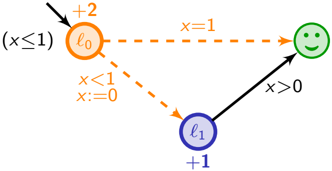

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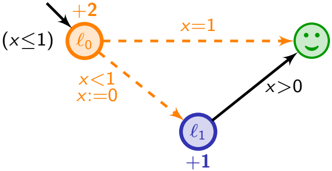

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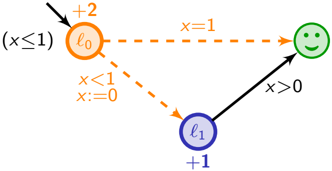

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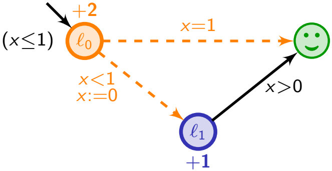

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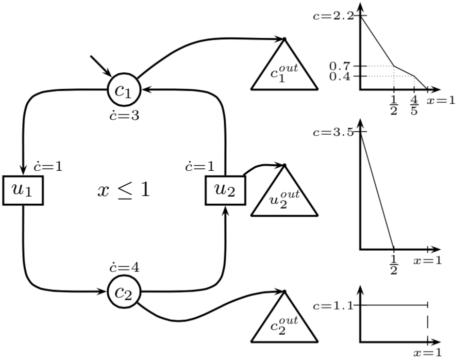
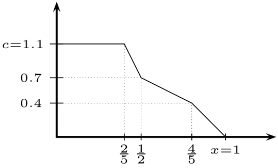
- Rather involved proofs of correctness





$$\sigma(c_2, x) = \begin{cases} c_2^{out} & \text{if } 0 \leq x < 2/5 \\ c_2 & \text{if } 2/5 \leq x < 1/2 \\ u_2 & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

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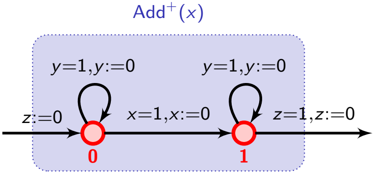
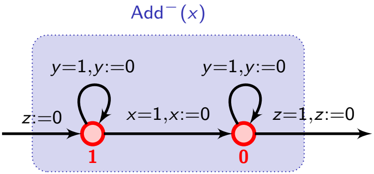


The cost is increased by x_0



The cost is increased by $1-x_0$

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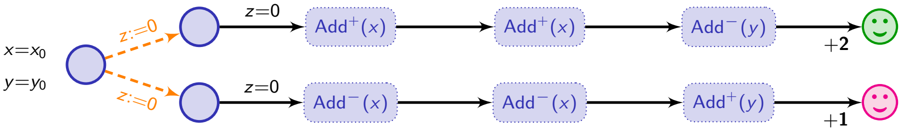

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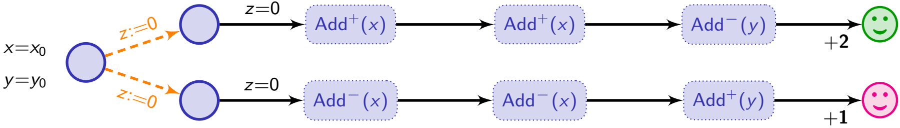

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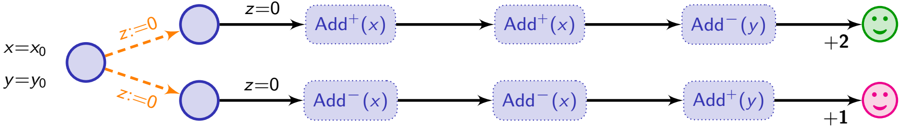

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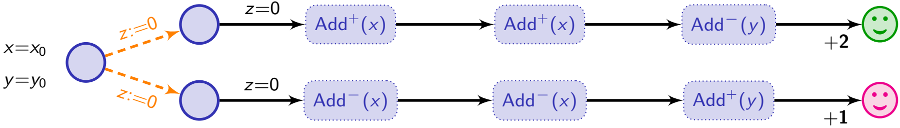

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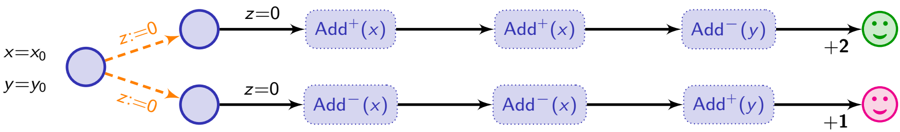

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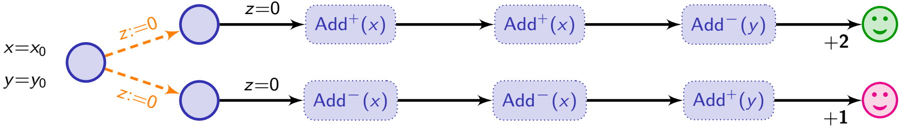

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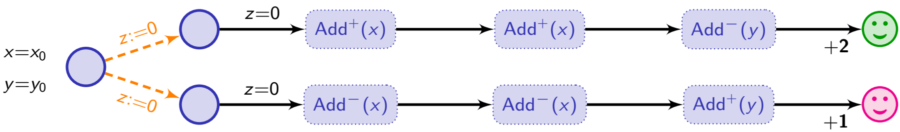

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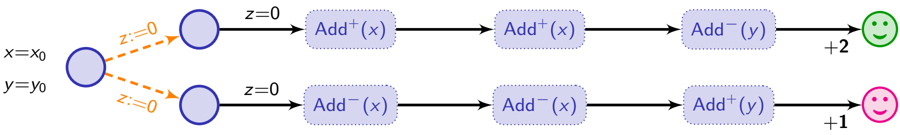

- In 🙂 (green), cost $= 2x_0 + (1 - y_0) + 2$
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  $\rightsquigarrow$ player 2 can enforce cost $3 + |y_0 - 2x_0|$
- Player 1 has a winning strategy with cost ≤ 3 iff $y_0 = 2x_0$

Computing the optimal cost: why is that hard?

Player 1 will simulate a two-counter machine:

- each instruction is encoded as a module;
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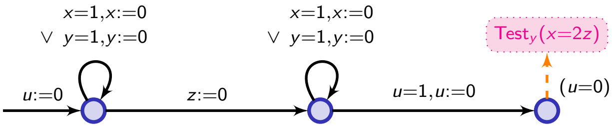

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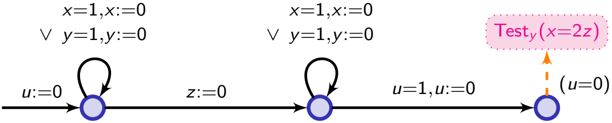

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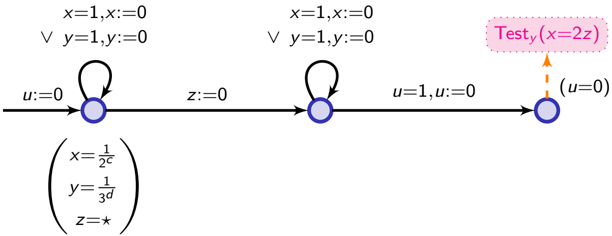

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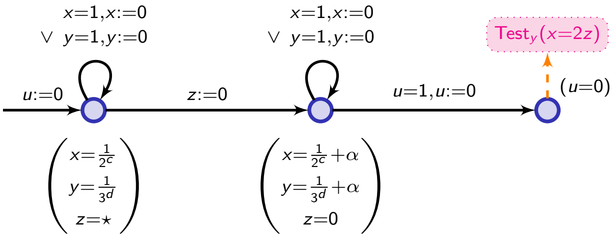

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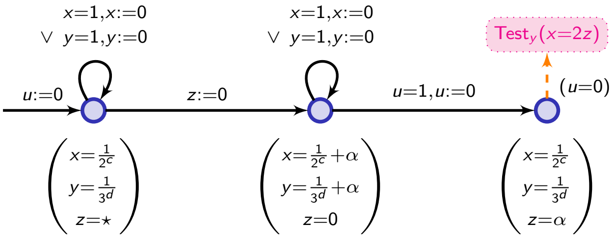

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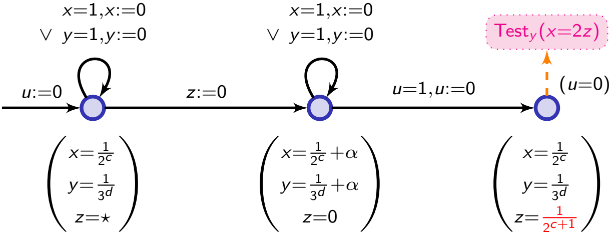

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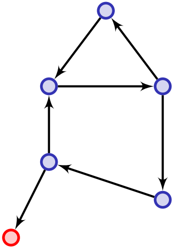

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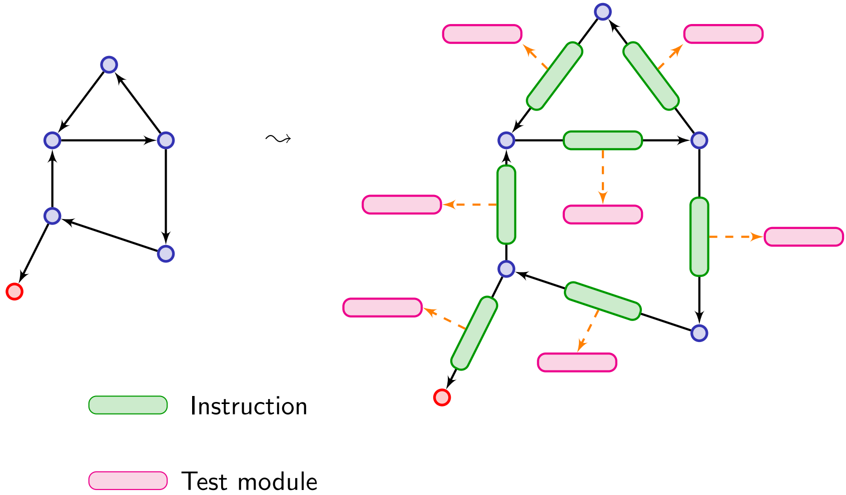

Are we done?

Outline

1. Introduction
2. Overview of "old" results
   - Weighted timed automata
   - Timed games
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3. Some recent developments
   - Undecidability of the value problem
   - Approximation of the optimal cost
   - Back to the undecidability
4. Conclusion

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Note: These problems are distinct...

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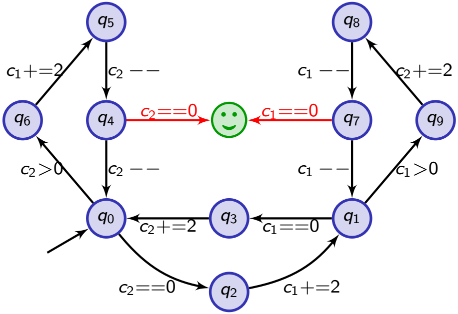

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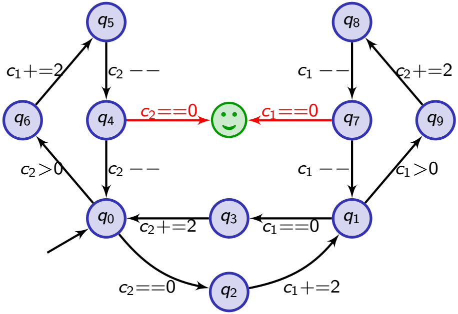

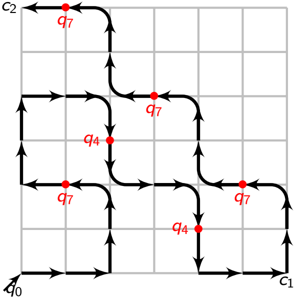

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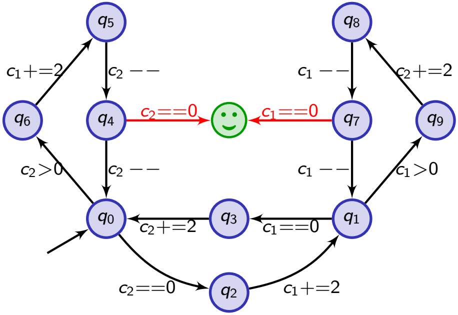

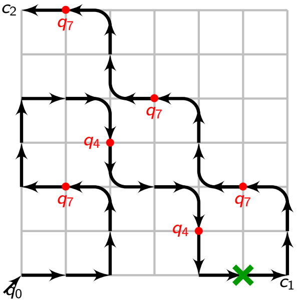

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The existence problem is undecidable in weighted timed games.

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[BMM14] Bouyer, Markey, Matteplackel. Averaging in LTL *(CONCUR'14)*.

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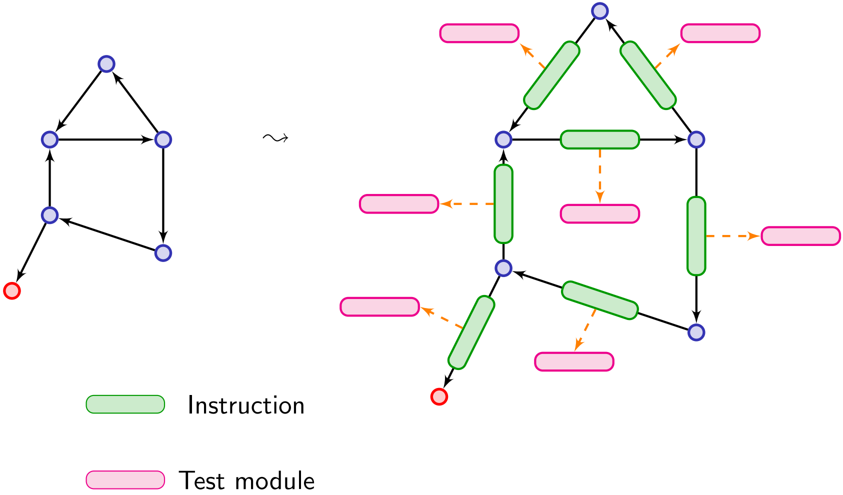
   - $\rightsquigarrow$ A second direct proof

2. Propose an approximation algorithm for a large class of weighted timed games (that comprises the class of games used for proving the above undecidability)
   - Almost-optimality in practice should be sufficient
   - Even when we know how to compute the value, we are only able to synthesize almost-optimal strategies...

Outline

A snapshot on the undecidability proof



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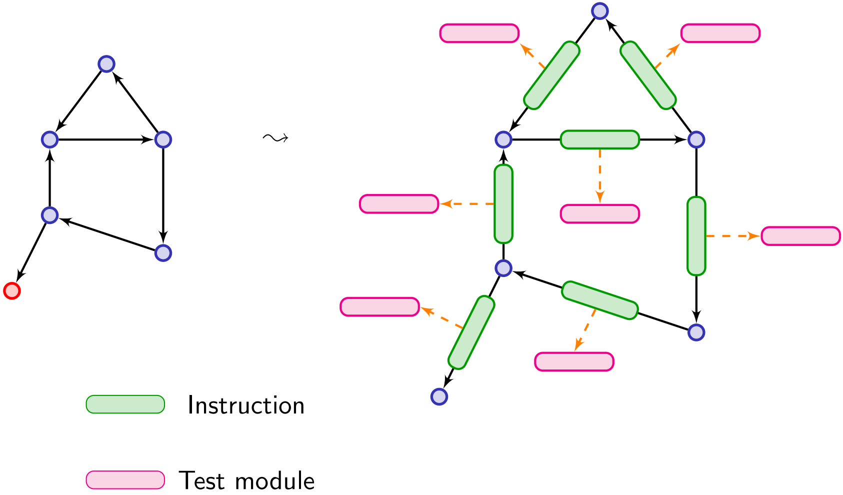

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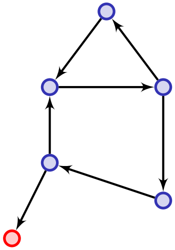
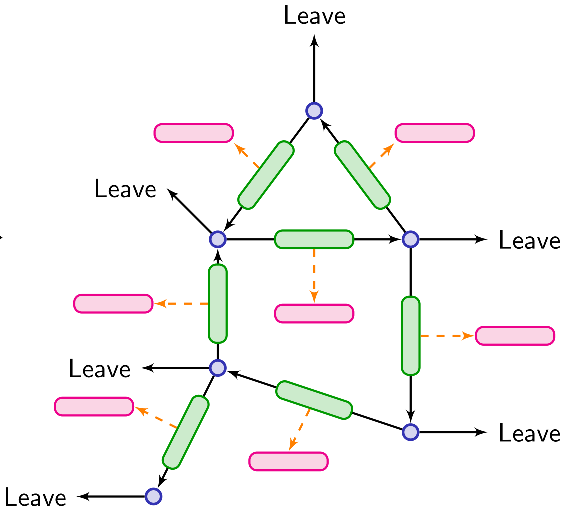


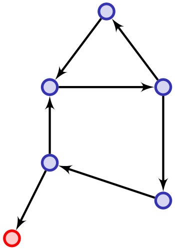
$\rightsquigarrow$



Leave with cost $3 + 1/2^n$ (n: length of the path)

A snapshot on the undecidability proof

$\mathcal{M}$ does not halt iff the value of $\mathcal{G}_{\mathcal{M}}$ is 3



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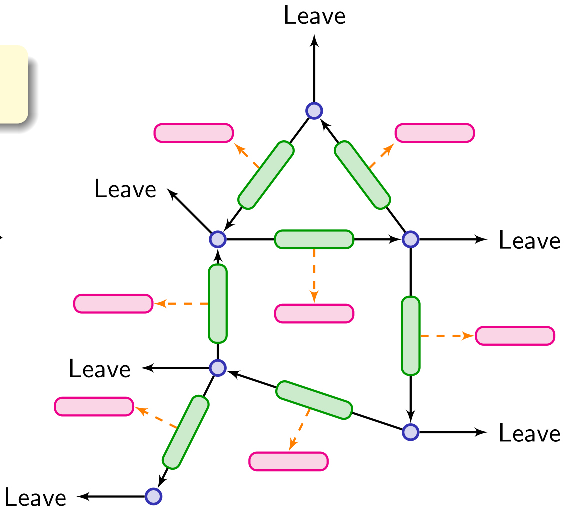

Leave with cost $3 + 1/2^n$ (n: length of the path)

Theorem [BJM15]

The value problem is undecidable in weighted timed games (with four clocks or more).

- Remark on the reduction:
  - Cost 0 within the core of the game
  - The rest of the game is acyclic

[BJM15] Bouyer, Jaziri, Markey. On the value problem in weighted timed games.

Outline

Optimal cost is computable...

... when cost is strongly non-zeno. [AM04,BCFL04]

That is, there exists $\kappa > 0$ such that for every region cycle C, for every real run ϱ read on C,

$$\mathsf{cost}(\varrho) \geq \kappa$$

Optimal cost is not computable...

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Optimal cost is not computable... but is approximable!

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Approximation of the optimal cost

Theorem

Let $\mathcal{G}$ be a weighted timed game, in which the cost is almost-strongly non-zeno. For every $\epsilon > 0$, one can compute:

- two values v_ϵ^- and v_ϵ^+ such that

$$|v_\epsilon^+ - v_\epsilon^-| < \epsilon \quad \text{and} \quad v_\epsilon^- \leq \mathsf{optcost}_{\mathcal{G}} \leq v_\epsilon^+$$

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- one strategy σ_ϵ such that

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[it is an ϵ-optimal winning strategy]

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- Standard technics: unfold the game to get more precision, and compute two adjacency sequences

$\rightsquigarrow$ This is not possible here
There might be runs with prefixes of arbitrary length and cost 0 (e.g. the game of the undecidability proof)

Idea for approximation

Idea

Only partially unfold the game:

- Keep components with cost 0 untouched – we call it the kernel
- Unfold the rest of the game

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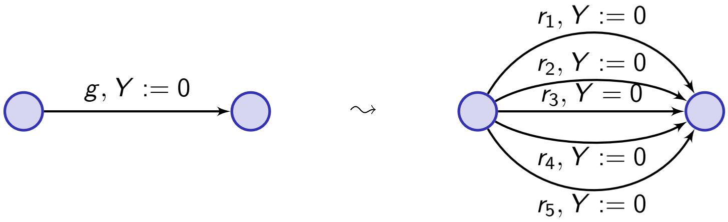
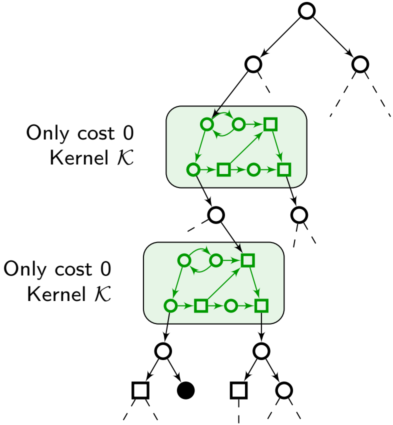
First: split the game along regions!



Semi-unfolding



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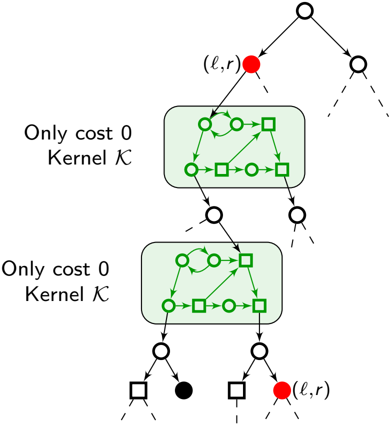

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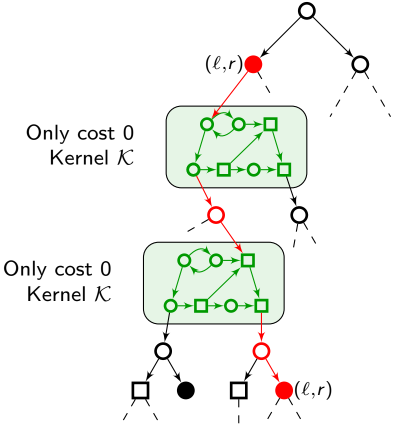

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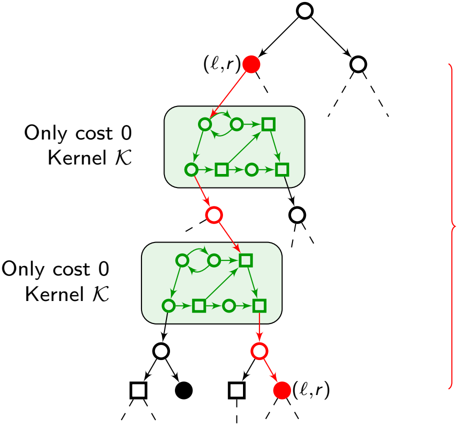
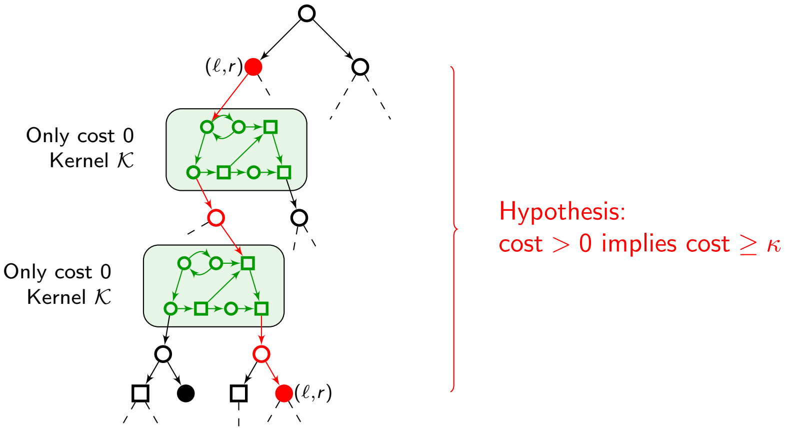


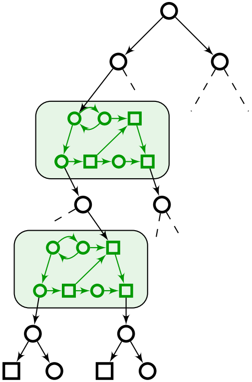
Hypothesis:
cost > 0 implies cost $\geq \kappa$

Semi-unfolding



Conclusion: we can stop unfolding the game after N steps
(e.g. $N = (M+2) \cdot |\mathcal{R}(\mathcal{A})|$, where M is a pre-computed bound on $\text{optcost}_{\mathcal{G}}$)

Approximation scheme



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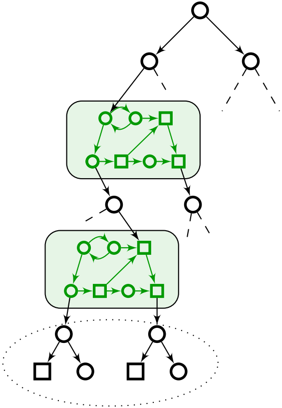

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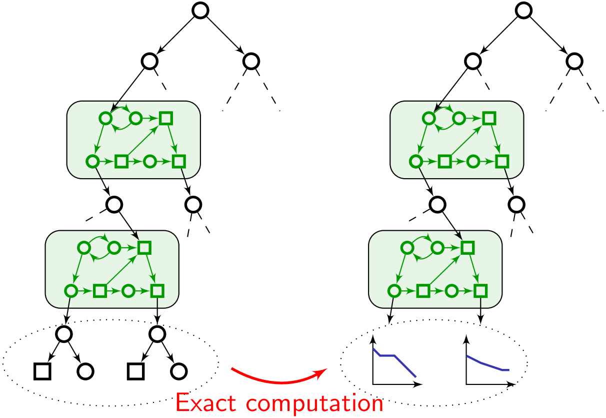

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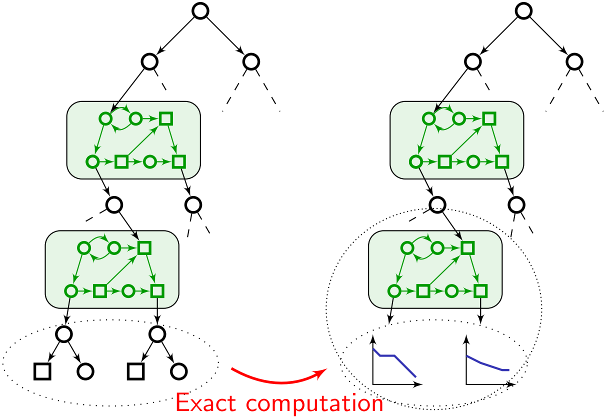

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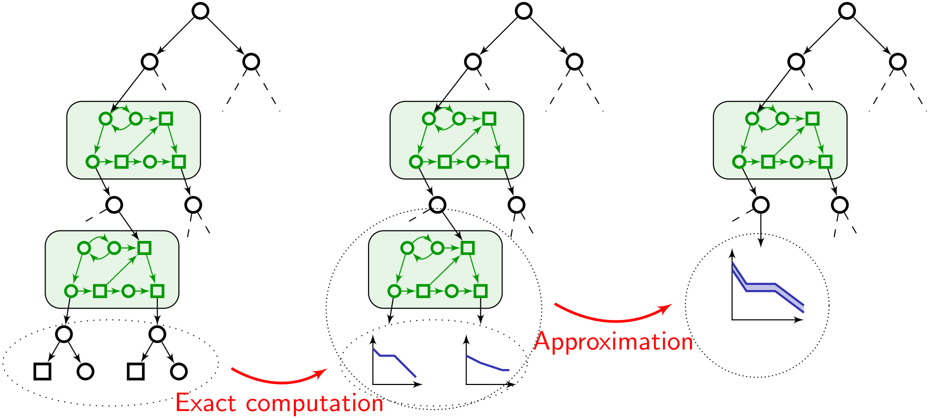

First step: Tree-like parts

$\rightsquigarrow$ Goes back to [LMM02]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

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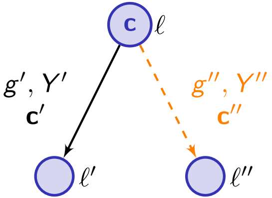

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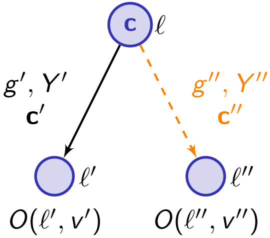

$O(\ell, v) =$

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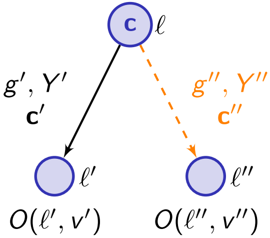

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'}$$

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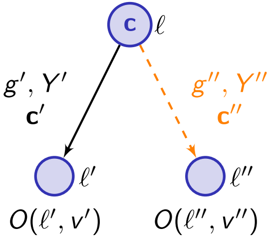

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max(\quad , \quad)$$

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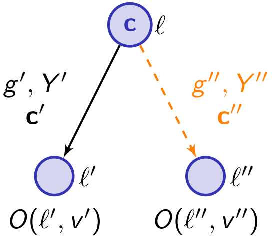

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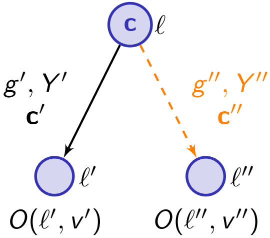

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$$(\alpha) = t'\mathbf{c} + \mathbf{c}' + O(\ell', v')$$

$$(\beta) = \sup_{t'' \leq t' \mid v+t'' \models g''} t''\mathbf{c} + \mathbf{c}'' + O(\ell'', v'')$$

$$v' = [Y' \leftarrow 0](v+t')$$
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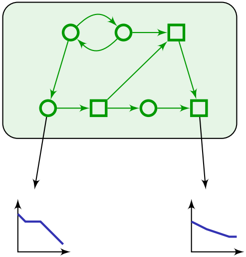
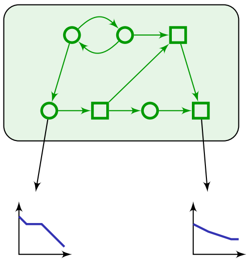
[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

Second step: Kernels



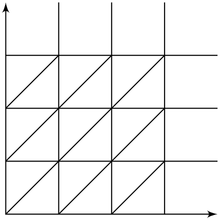
Output cost functions f

Second step: Kernels



Output cost functions f

1. Refine the regions such that f differs of at most ϵ within a small region



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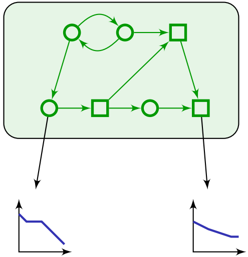

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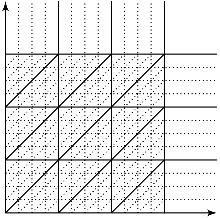

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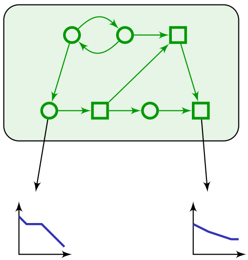

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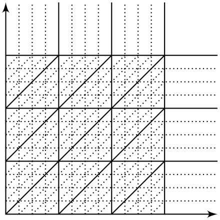

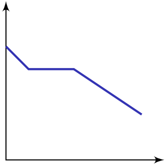

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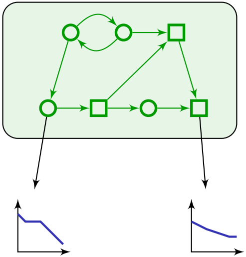

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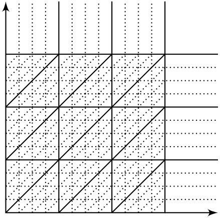
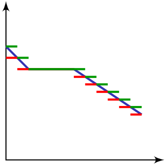
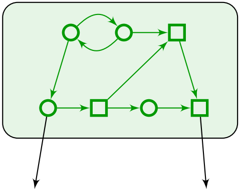


2. Under- and over-approximate by piecewise constant functions f_ϵ^- and f_ϵ^+



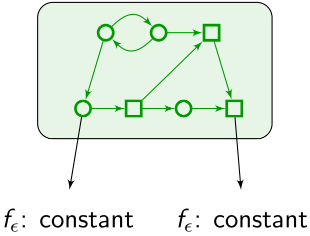
Second step: Kernels

3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ



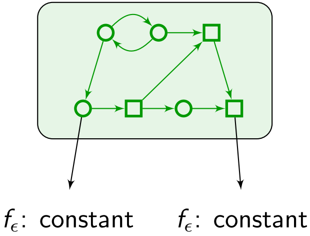
f_ϵ: constant f_ϵ: constant

Second step: Kernels



3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ
4. Since cost is 0 everywhere, the resulting game is nothing more than a reachability timed game with an order on target (output) edges (given by f_ϵ)

Second step: Kernels



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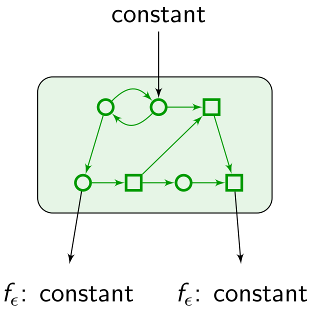

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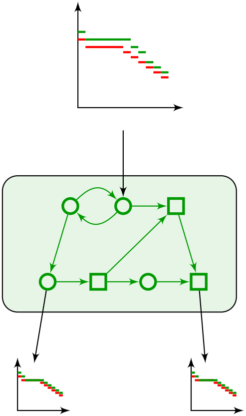

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$\rightsquigarrow$ We have computed ϵ-approximations of the optimal cost, which are constant within small regions. Corresponding strategies can be inferred

Outline

1. Introduction

2. Overview of "old" results
   - Weighted timed automata
   - Timed games
   - Weighted timed games

3. Some recent developments
   - Undecidability of the value problem
   - Approximation of the optimal cost
   - Back to the undecidability

4. Conclusion

Consequence of the approximation algorithm

Theorem

The value problem is co-recursively enumerable (for almost-strongly non-zeno weighted timed games), but not recursively enumerable.

Outline

Conclusion

Summary of the talk

- Quick overview of results concerning the optimal reachability problem in weighted timed games
- New insight into the value problem for this model:
  - Undecidability of this problem
  - Approximability of the optimal cost (under some conditions)

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Future work

- Improve the approximation scheme $(2\mathsf{EXP}(|\mathcal{G}|) \cdot \left(1/\epsilon\right)^{|X|})$
- Extend to the whole class of weighted timed games? understand why it is not possible
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- Is the value of any game a rational number?
- Understand the multiplayer setting (see next slides)

Nash equilibria in weighted timed games

The setting

- One weight function per player, one target state
- Payoff$_i$: weight$_i$ of the outcome if the target is reached; $+\infty$ otherwise (note: the smaller, the better)
- Nash equilibrium: a strategy profile such that the payoff of each player cannot be improved by unilateral deviation by that player

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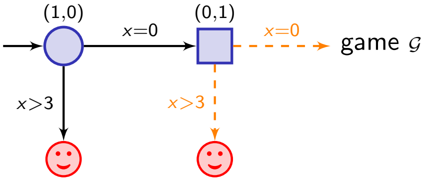
Theorem

In a two-player (non-zero-sum) weighted timed game as given above, we cannot decide whether there is a Nash equilibrium.

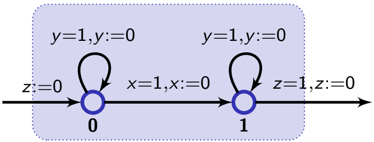
$\rightsquigarrow$ inspired by a result in Romain Brenguier's Master thesis (originally one clock, and negative/positive weights)

An interesting gadget with no Nash equilibrium



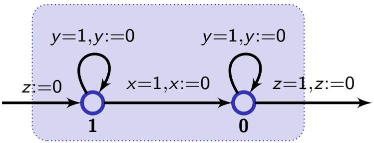
In this game, if there is a NE, then the payoff of each player is no more than 3.

$\mathsf{Add}^+(x)$

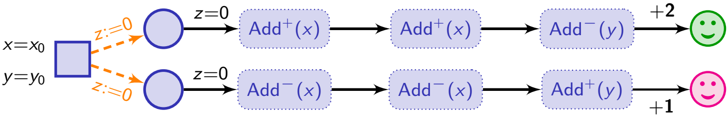


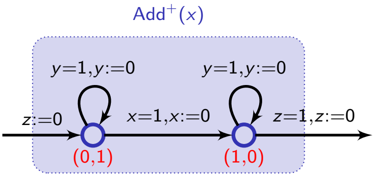
The cost is increased by x_0

$\mathsf{Add}^-(x)$

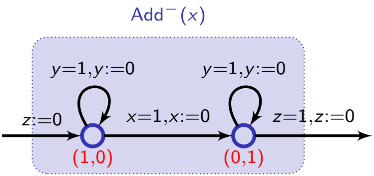


The cost is increased by $1-x_0$





cost_1 is increased by x_0
cost_2 is increased by $1 - x_0$



cost_1 is increased by $1 - x_0$
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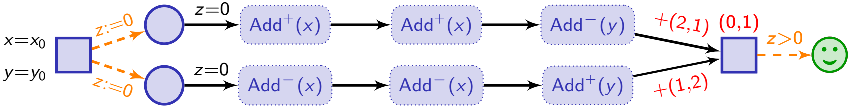
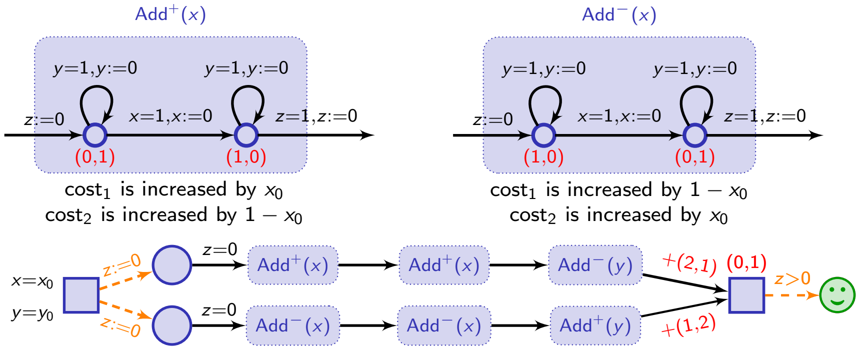
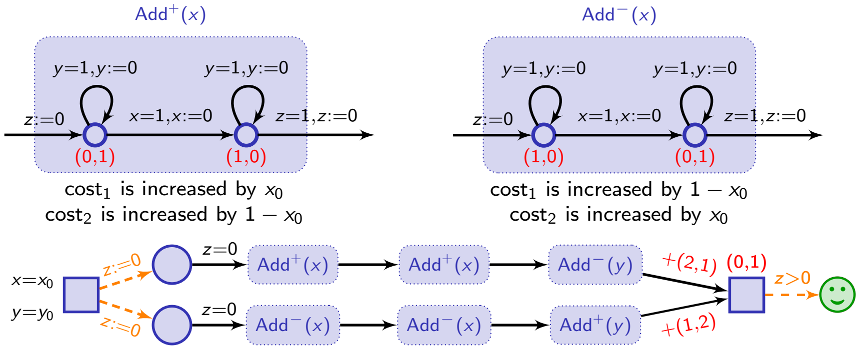




- Two possibilities: Player 2:

Player 1:	$3 + (y_0 - 2x_0)$	$3 - (y_0 - 2x_0)$
Player 2:	$3 - (y_0 - 2x_0) + \epsilon$	$3 + (y_0 - 2x_0) + \epsilon$



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Player 1:	$3 + (y_0 - 2x_0)$	$3 - (y_0 - 2x_0)$
Player 2:	$3 - (y_0 - 2x_0) + \epsilon$	$3 + (y_0 - 2x_0) + \epsilon$

- Player 2 has a strategy to get payoff $3 - |y_0 - 2x_0| + \epsilon$ (with $\epsilon > 0$) and give payoff $3 + |y_0 - 2x_0|$ to Player 1

There is a NE if and only if the two-counter machine halts.

What do we want to do?

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- ... with the help of [BBD10,BBDG12]

[BBD10] Brihaye, Bruyère, De Pril. Equilibria in quantitative reachability games *(CSR'10)*.
[BBDG12] Brihaye, Bruyère, De Pril, Gimbert. Subgame perfection for equilibria in quantitative reachability games *(FoSSaCS'12)*.

Conclusion ☺

Summary of the talk

- Quick overview of results concerning the optimal reachability problem in weighted timed games
- New insight into the value problem for this model:
  - Undecidability of this problem
  - Approximability of the optimal cost (under some conditions)

Future work

- Improve the approximation scheme $(2\mathrm{EXP}(|\mathcal{G}|) \cdot \left(1/\epsilon\right)^{|X|})$
- Extend to the whole class of weighted timed games? understand why it is not possible
- Assume stochastic uncertainty
- Is the value of any game a rational number?
- Understand the multiplayer setting