Probabilities in Timed Automata

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Based on joint works with Christel Baier (Dresden, Germany), Nathalie Bertrand (Rennes, France), Thomas Brihaye (Mons, Belgium), Marcus Größer (Dresden, Germany) and Nicolas Markey (Cachan, France)

Outline

Motivations

Our aim

Propose an alternative semantics to timed automata that measures how likely properties are satisfied.

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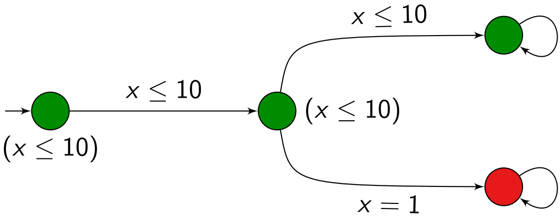
➜ Propose a new timed and probabilistic model

- Related models include continuous-time Markov chains, but also probabilistic timed automata.

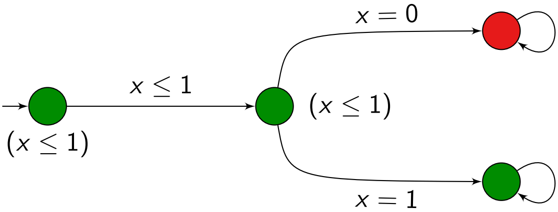
Initial example



Intuition: from the initial state,

this automaton *almost-surely* satisfies "$\mathbf{G}$ green"

A maybe less intuitive example



Does it *almost-surely* satisfy "$\mathbf{G}$ green"?

Outline

1. Introduction

2. A probabilistic semantics for timed automata

3. Solving the qualitative model-checking problem

4. Towards solutions to the quantitative model-checking problem

5. Conclusion

Our proposition

- $\pi(s \xrightarrow{e_1} \ldots \xrightarrow{e_n})$: symbolic path from s firing edges $e_1, \ldots, e_n$

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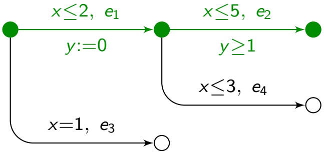

$$\pi(s_0 \xrightarrow{e_1}\xrightarrow{e_2}) = \{s_0 \xrightarrow{\tau_1, e_1} s_1 \xrightarrow{\tau_2, e_2} s_2 \mid \tau_1 \leq 2,\ \tau_1 + \tau_2 \leq 5,\ \tau_2 \geq 1\}$$

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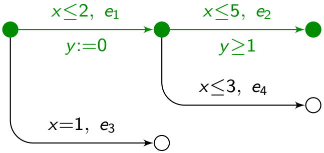

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From state s_0:

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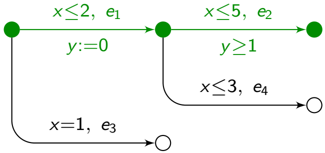

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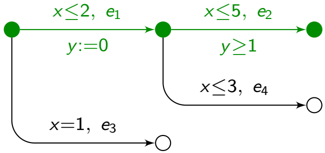

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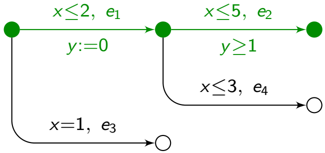

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Idea:

From state s_0:

- randomly choose a delay
- then randomly select an edge
- then continue

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Symbolic path: $\pi(s \xrightarrow{e_1} \cdots \xrightarrow{e_n}) = \{s \xrightarrow{\tau_1, e_1} s_1 \cdots \xrightarrow{\tau_n, e_n} s_n\}$

$$\mathbb{P}\Big(\pi(s \xrightarrow{e_1} \cdots \xrightarrow{e_n})\Big) = \int_{t \in I(s,e_1)} p_{s+t}(e_1)\, \mathbb{P}\Big(\pi(s_t^{e_1} \xrightarrow{e_2} \cdots \xrightarrow{e_n})\Big)\, \mathrm{d}\mu_s(t)$$

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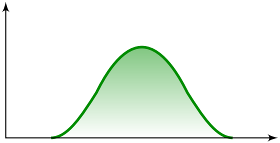

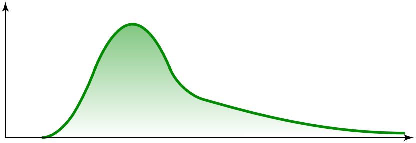

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- Easy extension to constrained symbolic paths

$$\pi_{\mathcal{C}}(s \xrightarrow{e_1} \cdots \xrightarrow{e_n}) = \{s \xrightarrow{\tau_1, e_1} s_1 \cdots \xrightarrow{\tau_n, e_n} s_n \mid (\tau_1, \cdots, \tau_n) \models \mathcal{C}\}$$

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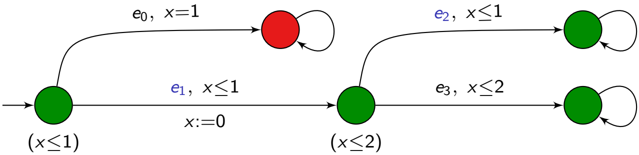
- Example:
  - $\mathsf{Zeno}(s) \;=\; \bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{(e_1, \cdots, e_n) \in E^n} \mathsf{Cyl}(\pi_{\Sigma_i \tau_i \leq M}(s \xrightarrow{e_1} \cdots \xrightarrow{e_n}))$

An example of computation (with uniform distributions)



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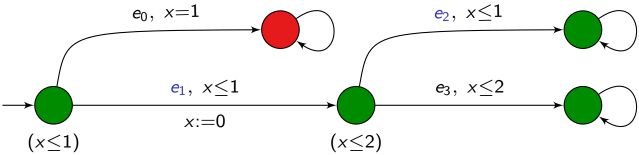

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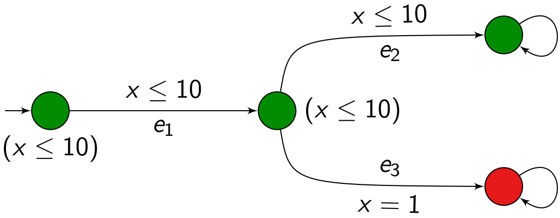
$$\mathbb{P}\Big(\pi\big(s_0 \xrightarrow{e_1}\xrightarrow{e_2}\big)\Big) = \int_0^1 \mathbb{P}\Big(\pi\big(s_1 \xrightarrow{e_2}\big)\Big)\mathrm{d}\mu_{s_0}(t) + \int_1^1 \frac{\mathbb{P}\Big(\pi\big(s_1 \xrightarrow{e_2}\big)\Big)}{2}\mathrm{d}\mu_{s_0}(t)$$

$$= \int_0^1 \int_0^1 \left(\frac{\mathbb{P}\big(\pi(s_2)\big)}{2}\mathrm{d}\mu_{s_1}(u)\right) \mathrm{d}\mu_{s_0}(t)$$

$$= \int_0^1 \int_0^1 \left(\frac{1}{2}\frac{\mathrm{d}u}{2}\right)\mathrm{d}t \quad = \frac{1}{4}$$

Back to the first example



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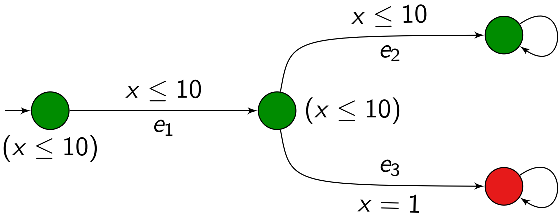

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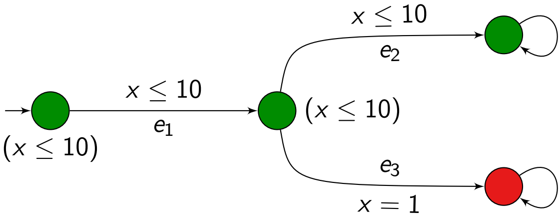

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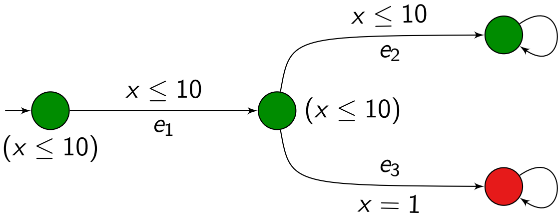

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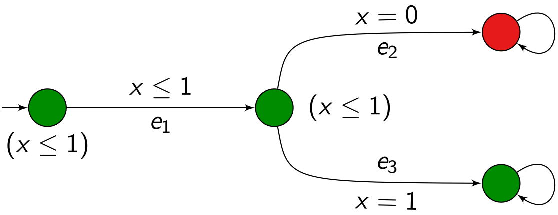

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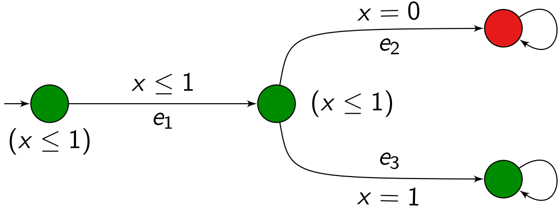

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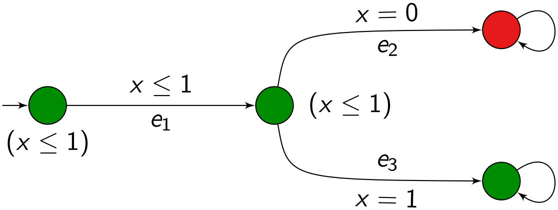

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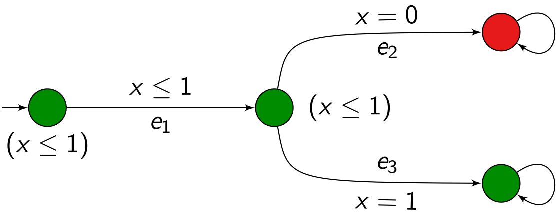

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Almost-sure satisfaction

If φ is an LTL (or ω-regular) property,

$$s \mathrel{|\!\!\approx} \varphi \quad \overset{\text{def}}{\Leftrightarrow} \quad \underbrace{\mathbb{P}\Big(\{\varrho \in \mathsf{Runs}(s) \mid \varrho \models \varphi\}\Big)}_{\overset{\text{def}}{=}\ \mathbb{P}\left(s \models \varphi\right)} = 1$$

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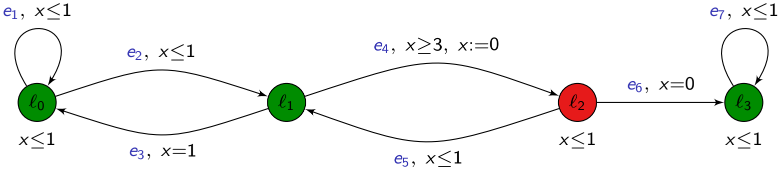
Qualitative model-checking question: $s \mathrel{|\!\!\approx} \varphi$?

Outline

An example


e_1, $x\leq 1$
e_2, $x\leq 1$
e_4, $x\geq 3$, $x:=0$
e_7, $x\leq 1$
e_6, $x=0$
ℓ_0
ℓ_1
ℓ_2
ℓ_3
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e_3, $x=1$
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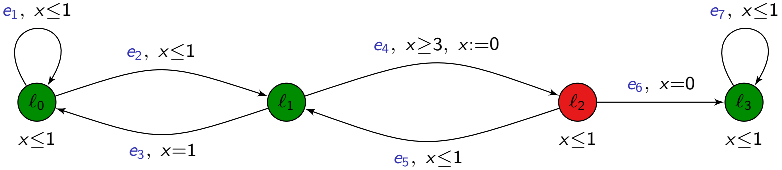

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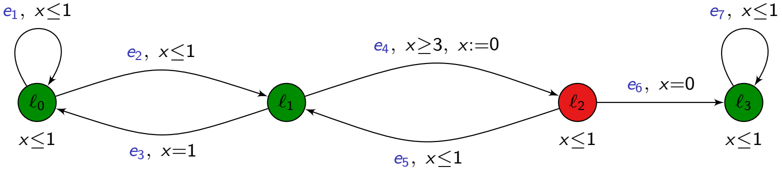

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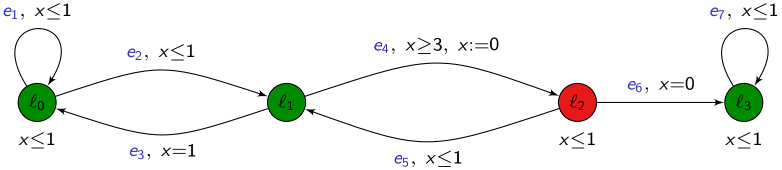

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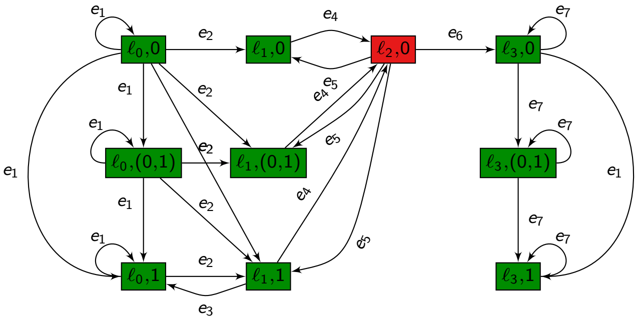
Indeed, almost surely, paths are of the form $e_1^* e_2 \Big(e_4 e_5 \Big)^\omega$

The classical region automaton



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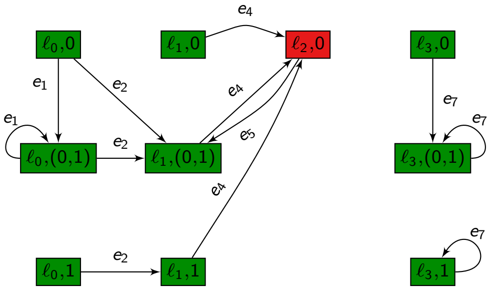

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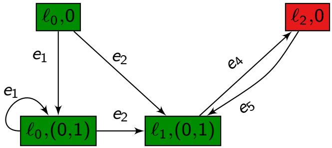

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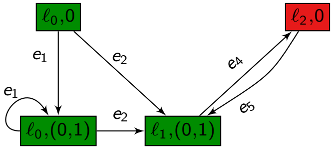

... viewed as a finite Markov chain $MC(\mathcal{A})$

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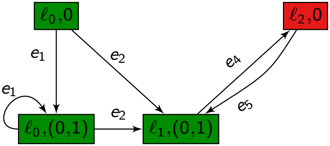

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Theorem

For single-clock timed automata,

$$\mathcal{A} \mathrel{|\!\!\approx} \varphi \quad \text{iff} \quad \mathbb{P}(MC(\mathcal{A}) \models \varphi) = 1$$

Result

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For single-clock timed automata, the almost-sure model-checking

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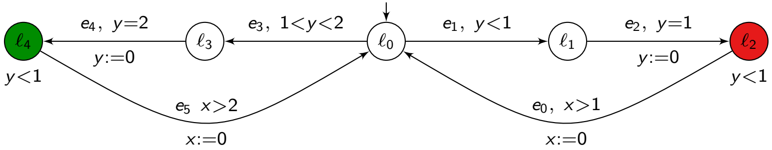
- Complexity:
  - size of single-clock region automata = polynomial [LMS04]
  - apply result of [CSS03] to the finite Markov chain
- Correctness: the proof is rather involved
  - requires the definition of a topology over the set of paths
  - notions of largeness (for proba 1) and meagerness (for proba 0)
  - link between probabilities and topology thanks to the topological games called Banach-Mazur games

An example with two clocks



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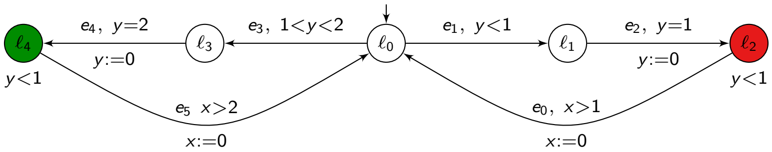

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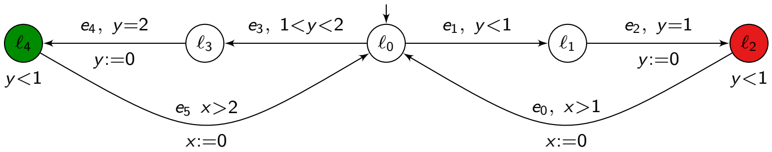

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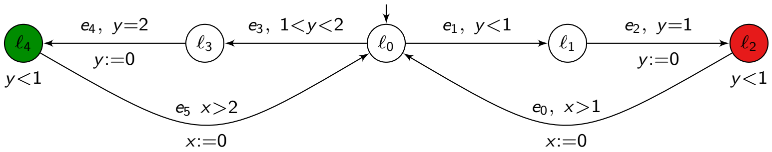

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- However, we can prove that $\mathbb{P}\Big(\mathbf{G}\,\neg\text{red}\Big) > 0$
- There is a *strange* convergence phenomenon: along an execution, if $\delta_i > 0$ is the delay in location ℓ_4, then we have that $\sum_i \delta_i \leq 1$

A note on Zeno behaviours

- The set of Zeno behaviours is measurable:

$$\mathsf{Zeno}(s) \;=\; \bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{(e_1,\cdots,e_n) \in E^n} \mathsf{Cyl}(\pi(s \xrightarrow{e_1} \cdots \xrightarrow{e_n}))$$

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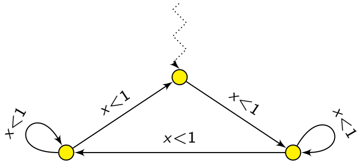

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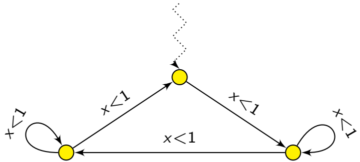
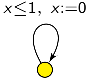


  - an interesting notion of non-Zeno timed automata



Outline

Quantitative model-checking

How likely an automaton will satisfy a property?
I.e., what is the value $\mathbb{P}\big(s \models \varphi\big)$?

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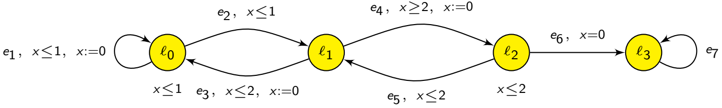
  - we will be able to decide the threshold problem:
    "Given $\mathcal{A}$, φ, $c \in \mathbb{Q}$, and $\sim \in \{<, \leq, =, \geq, >\}$, does $\mathbb{P}\big(s_0 \models \varphi\big) \sim c$ in $\mathcal{A}$?"

An example



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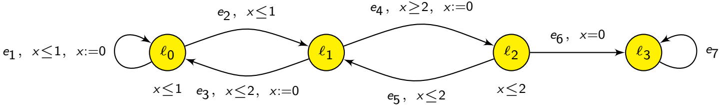

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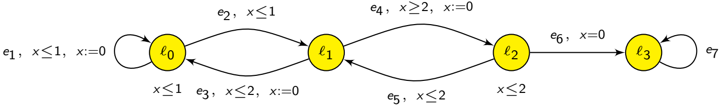

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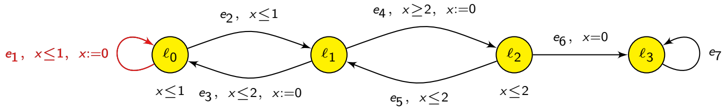

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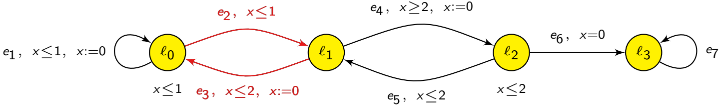

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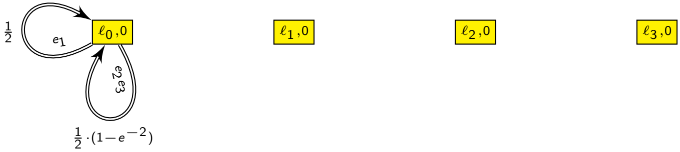

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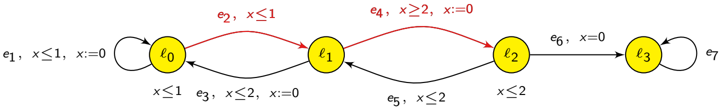

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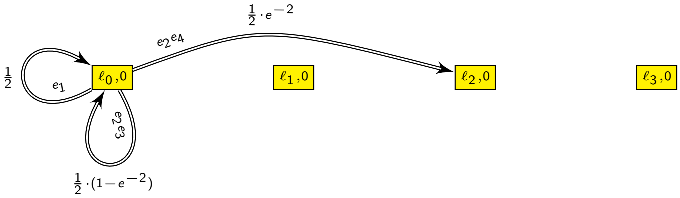

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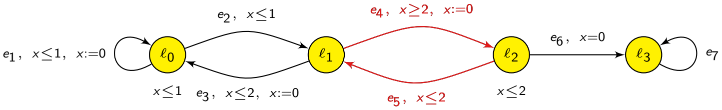

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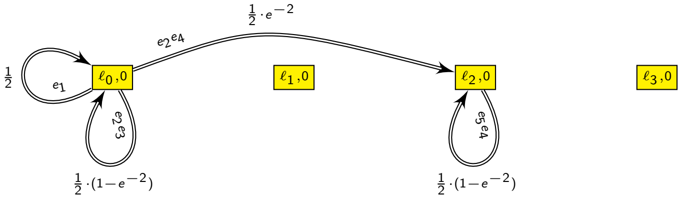

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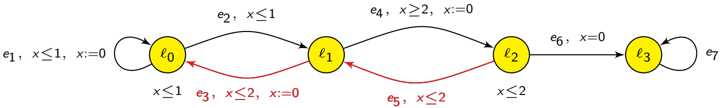

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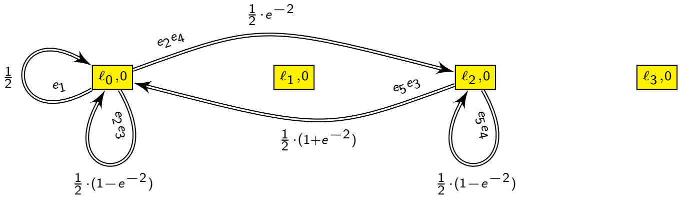

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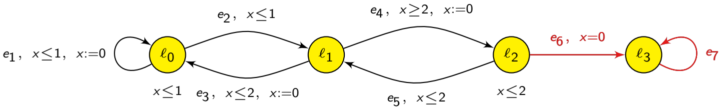

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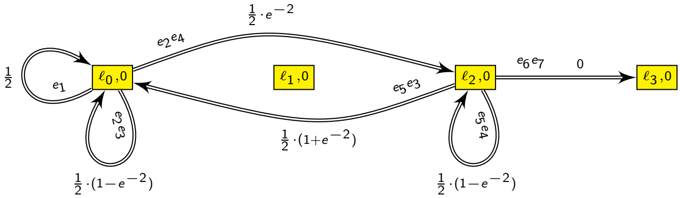

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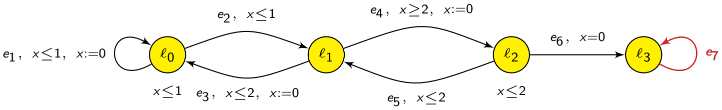

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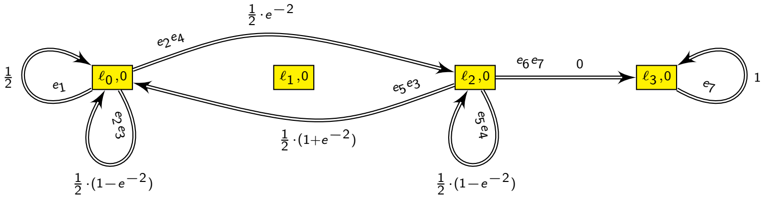

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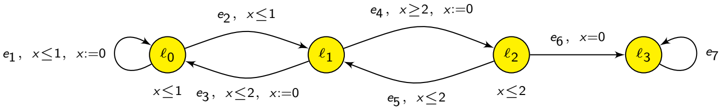

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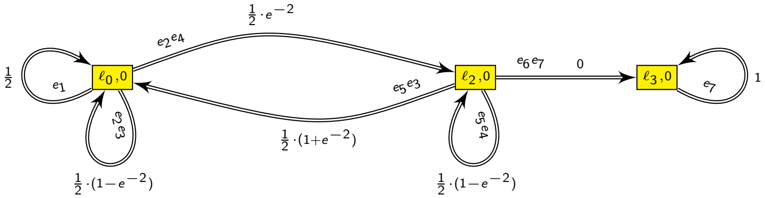

Correctness of the abstraction

Theorem

Under some hypotheses, for single-clock automaton $\mathcal{A}$ and property φ,

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- **Limits of the abstraction:** there may be no closed form for the values labelling the edges of $MC'(\mathcal{A})$.

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☞ Note: the hypothesis "reset all bounded cycles" is necessary to get this form.

Approximating the probability

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- When $(a_N, b_N) \subseteq (\alpha, \beta)$, the two sequences $(f(a_i))_{i \geq N}$ and $(f(b_i))_{i \geq N}$ are monotonic and converge to $f\left(e^{-r}\right)$

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  - stop when the under- and the over-approximations are on the same side of the threshold c

Outline

Conclusions

- a probabilistic semantics for timed automata which removes "unlikely" (sequences of) events
  $\rightsquigarrow$ extend continuous-time Markov chains
- qualitative model-checking has a topological interpretation
- abstraction and algorithm for qualitative model-checking of ω-regular and LTL properties (one clock)
- quantitative model-checking of ω-regular and LTL properties (restrictive framework)

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- quantitative model-checking of ω-regular and LTL properties (restrictive framework)

Ongoing works

- better understand the framework with several clocks
- our semantics can be viewed as a $\frac{1}{2}$-player game, hence extend to $1\frac{1}{2}$- and $2\frac{1}{2}$-player games
  $\rightsquigarrow$ further interesting (un)decidability results