Laboratoire Méthodes Formelles

Memory complexity for winning games on graphs

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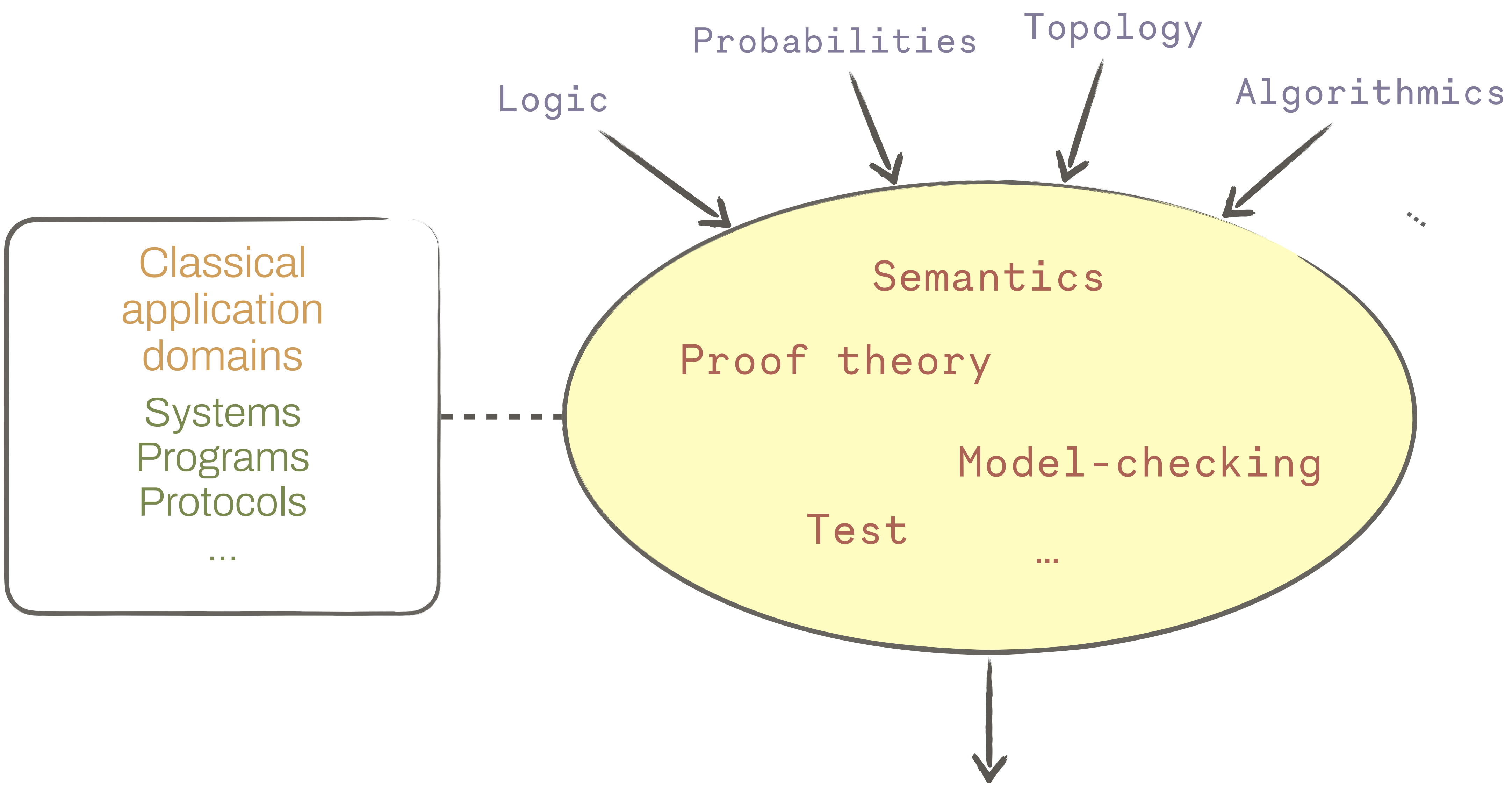
Based on joined work with Stéphane Le Roux, Youssouf Oualhadj, Michael Randour, Pierre Vandenhove. Thanks to Pierre for his slides

Motivation

—

The setting

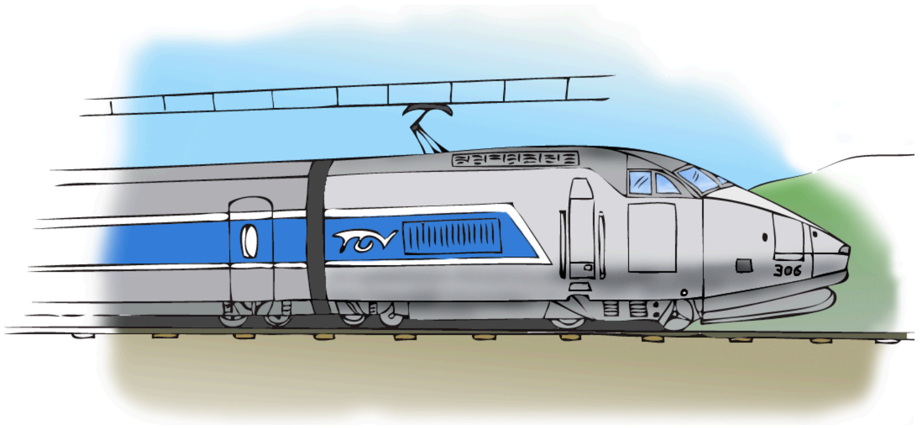
My field of research: Formal methods



Give guarantees (+ certificates) on functionalities or performances

Model-checking

System



Model-checking

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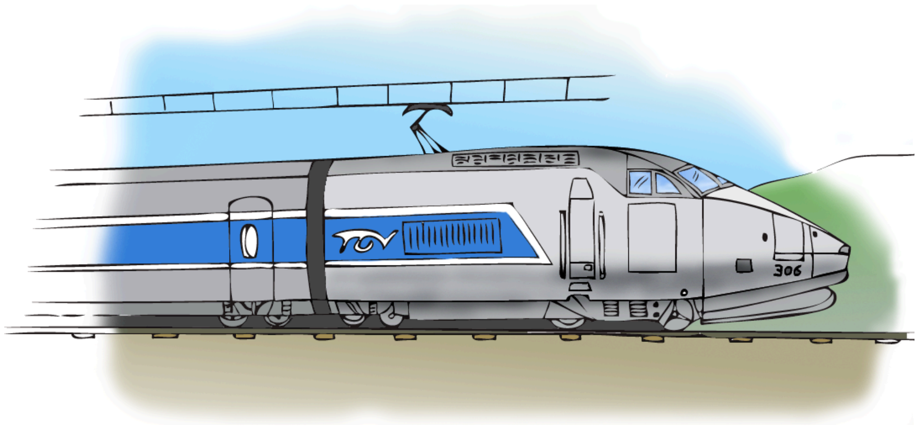

Properties

Model-checking

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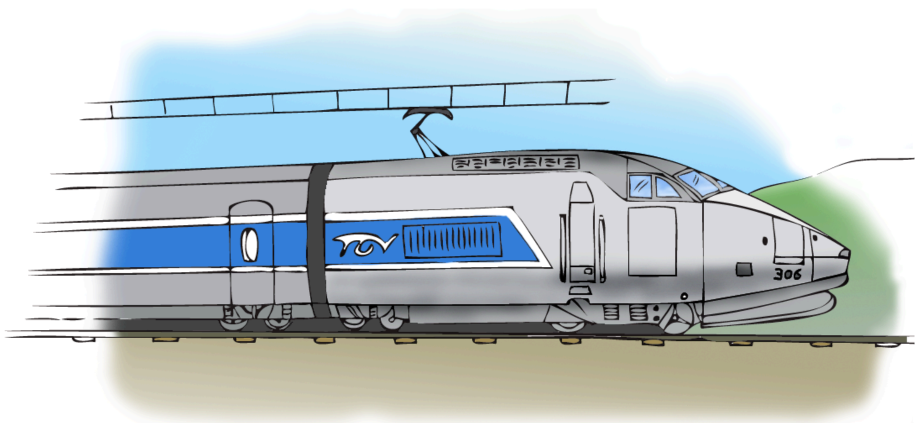

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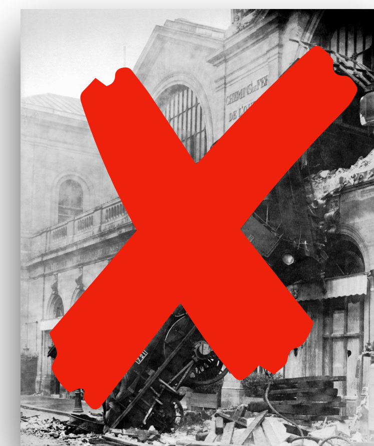

Model-checking

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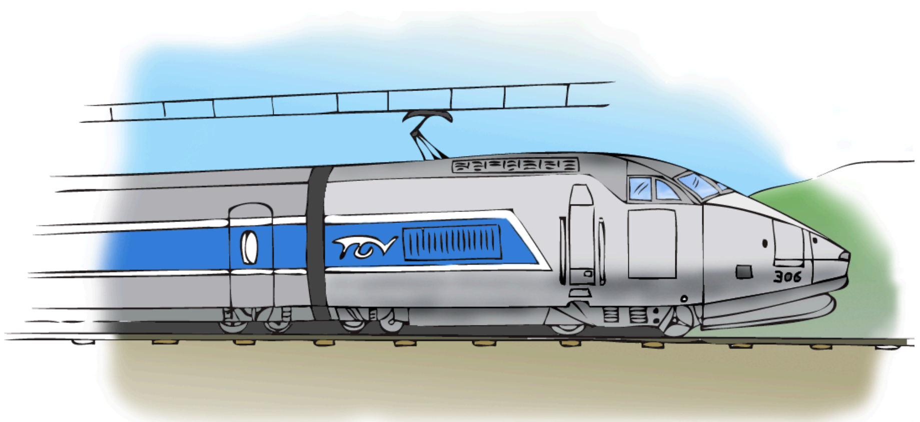

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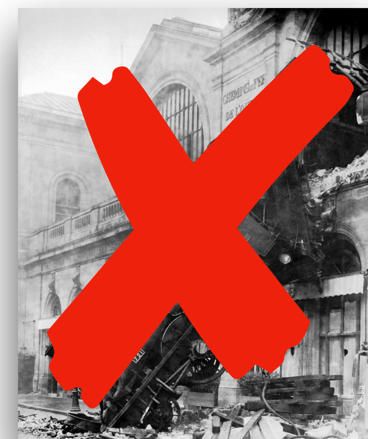

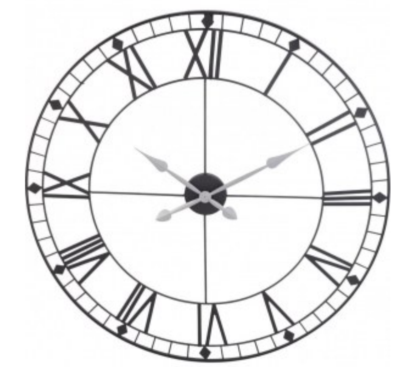

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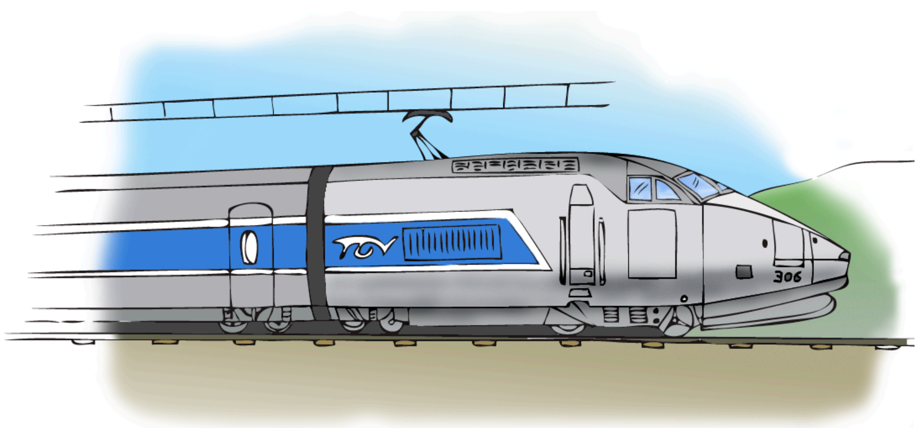

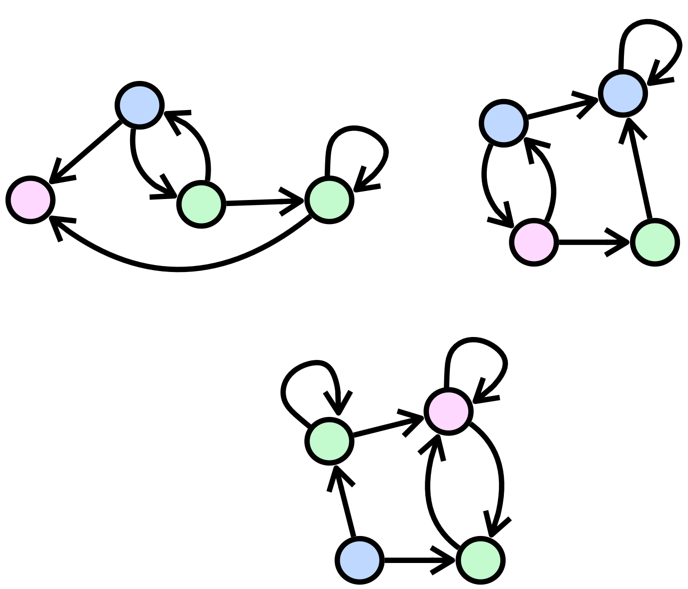

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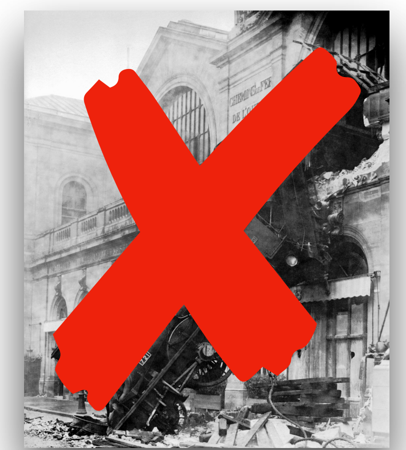

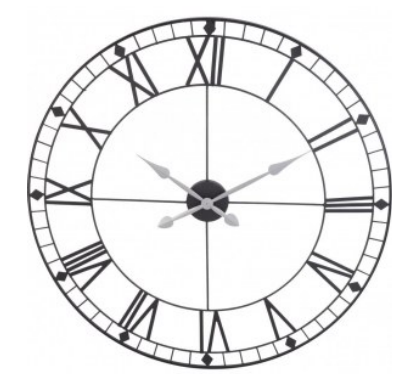

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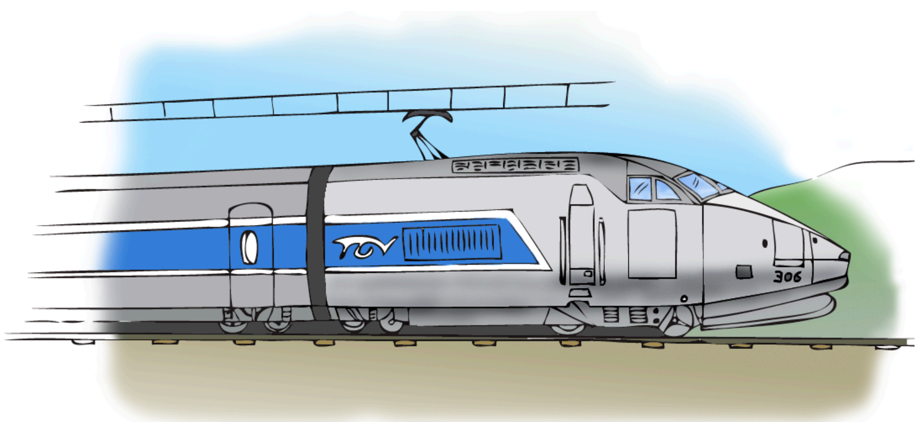

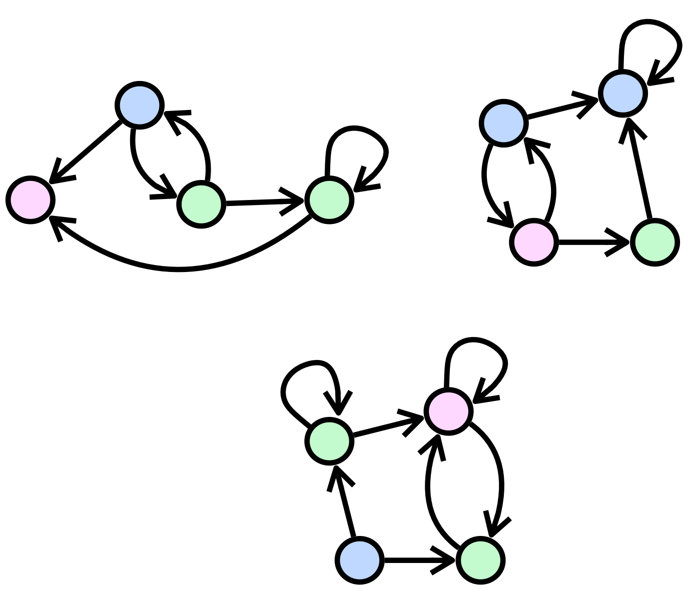

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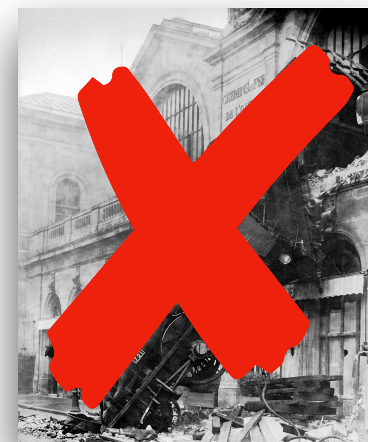

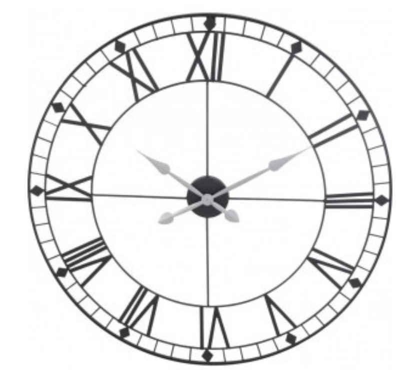

$$\varphi = \mathbf{AG}\,\neg\text{crash} \wedge \left(\mathbb{P}(\mathbf{F}_{\leq 2\text{h}}\,\text{arr}) \geq 0{,}9\right)$$

Model-checking

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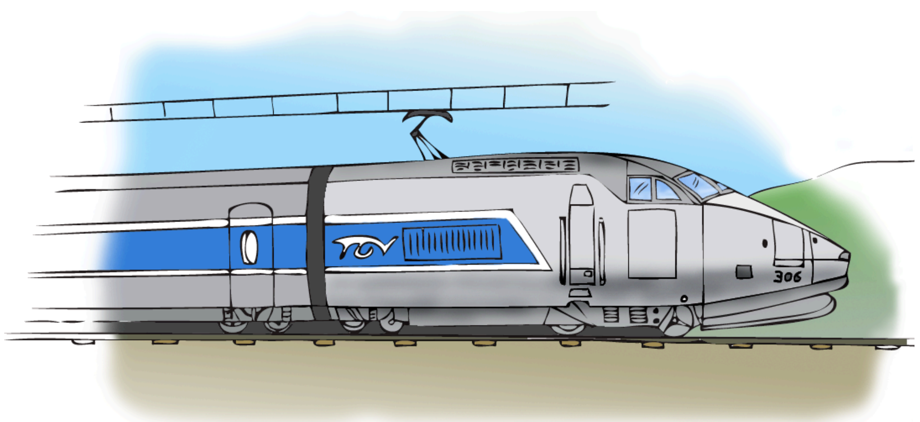

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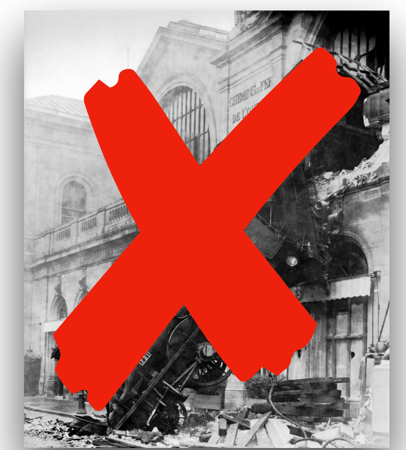

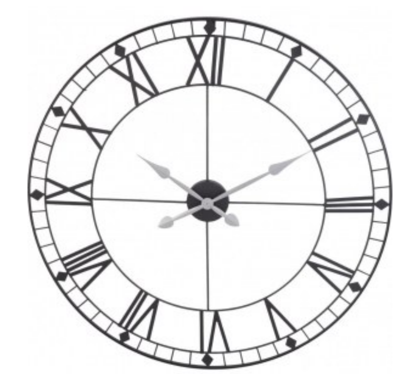

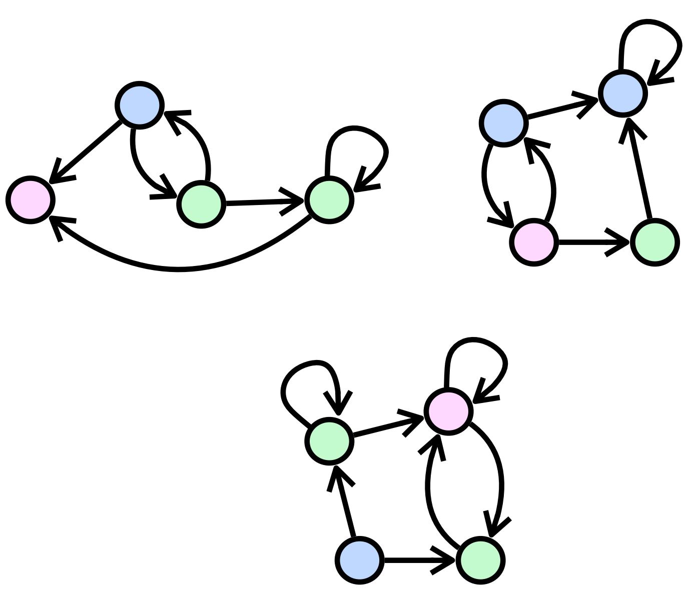

Model-checking algorithm

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Model-checking

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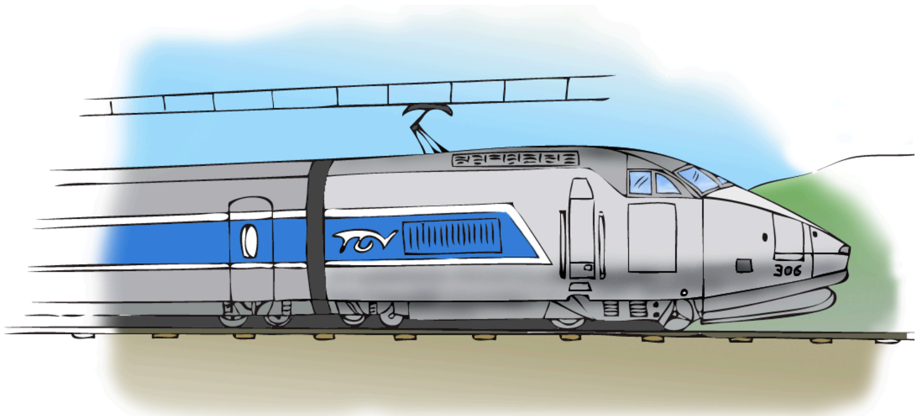

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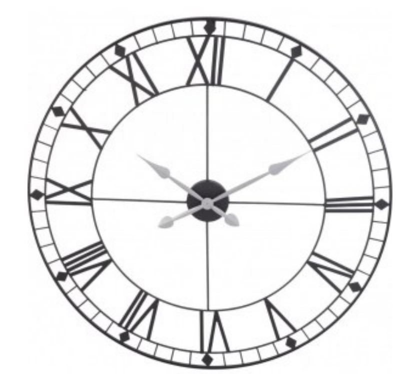

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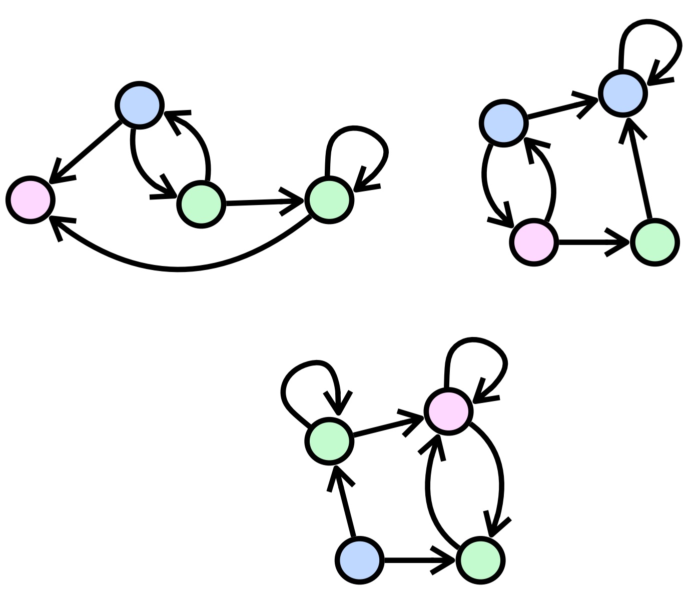


Model-checking algorithm

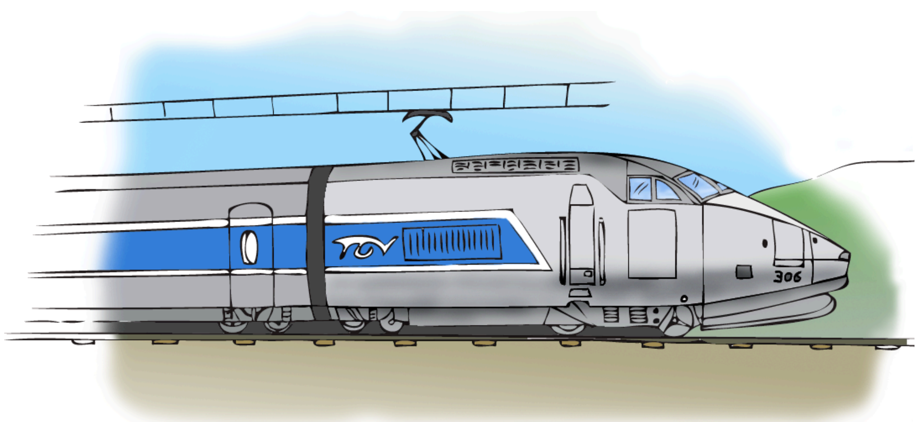
↓

Yes/No/Why?

$$\varphi = \mathbf{AG}\,\neg\text{crash} \wedge \left(\mathbb{P}(\mathbf{F}_{\leq 2\text{h}}\text{arr}) \geq 0{,}9\right)$$

Control or synthesis

System



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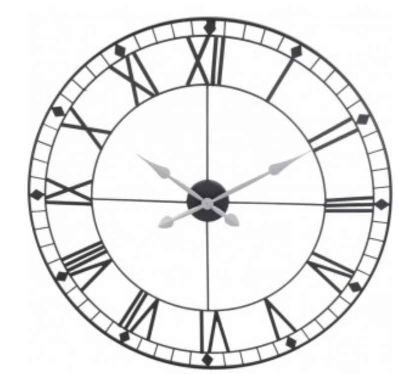

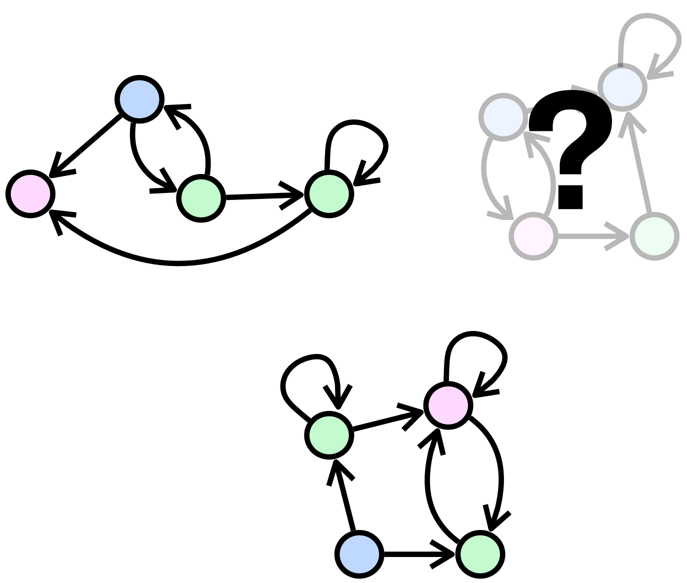

Control/synthesis algorithm

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No/Yes/How?

The talk in one slide

Strategy synthesis for two-player games

Find good and simple controllers for systems interacting with an antagonistic environment

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The talk in one slide

Strategy synthesis for two-player games

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When are simple strategies sufficient to play optimally?

Our general approach

[Tho95] On the synthesis of strategies in infinite games (STACS'95).
[Tho02] Thomas. Infinite games and verification (CAV'02).
[GU08] Grädel, Ummels. Solution concepts and algorithms for infinite multiplayer games (New Perspectives in Games and Interactions, 2008).
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- Use **graph-based game models** (state machines) to represent the system and its evolution

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Our general approach

- Use **graph-based game models** (state machines) to represent the system and its evolution
- Use **game theory concepts** to express admissible situations
  - Winning strategies
  - (Pareto-)Optimal strategies
  - Nash equilibria
  - Subgame-perfect equilibria
  - …

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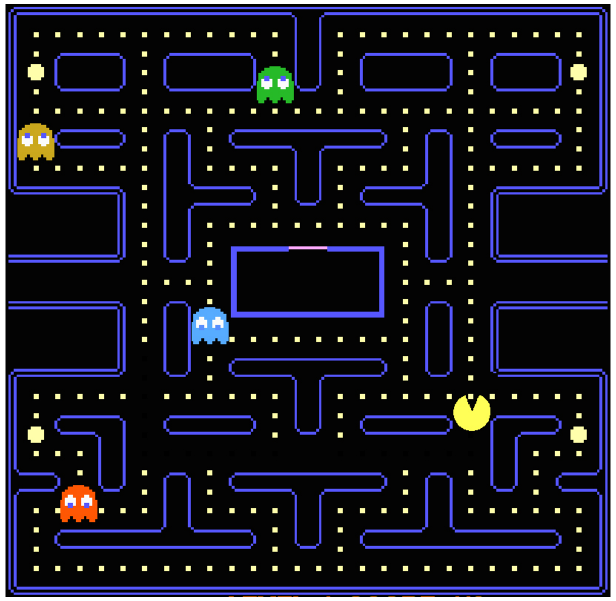
Games
What they often are



Games
A broader sense

Interaction

Goal

- Model and analyze (using math. tools) situations of interactive decision making

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Ingredients

- Several decision makers (players)
- Possibly each with different goals
- The decision of each player impacts the outcome of all

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Wide range of applicability

« [...] it is a context-free mathematical toolbox. »

- Social science: e.g. social choice theory
- Theoretical economics: e.g. models of markets, auctions
- Political science: e.g. fair division
- Biology: e.g. evolutionary biology
- ...

[MSZ13] Maschler, Solan, Zamir. Game theory (2013).

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+ Computer science

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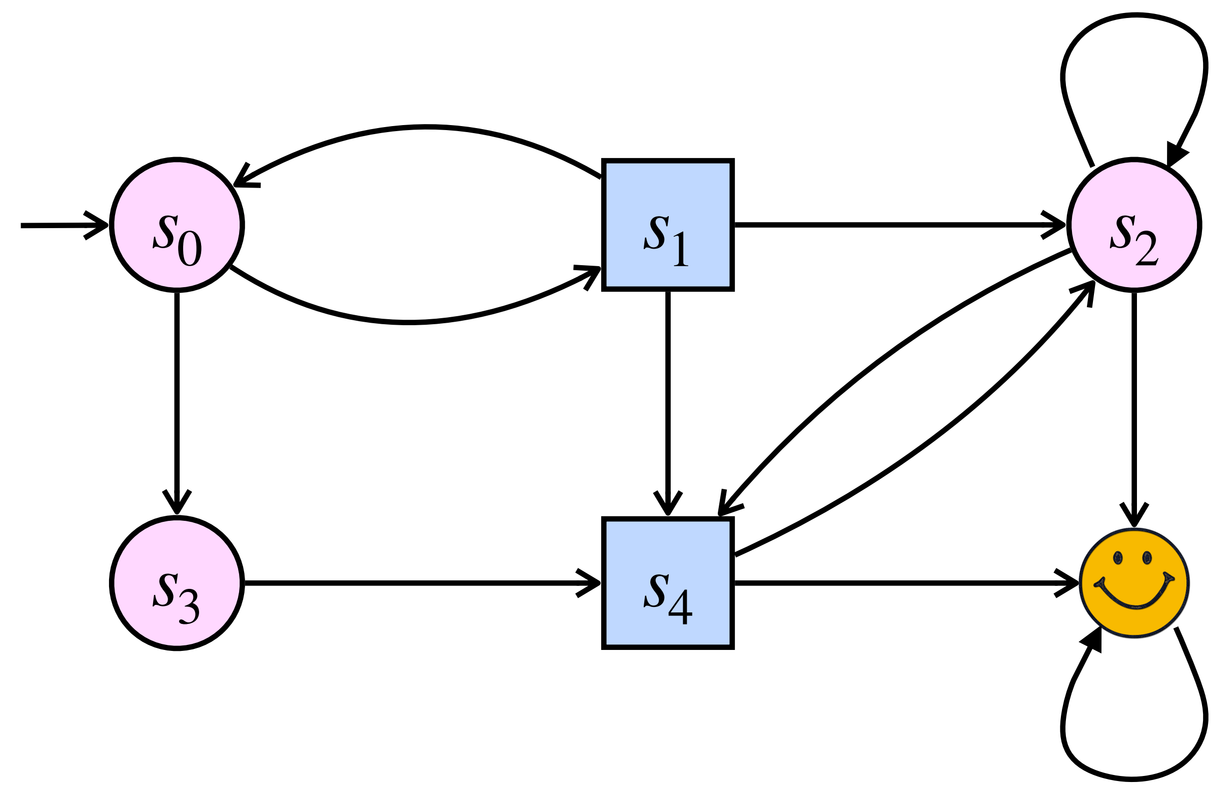
Games on graphs

States, Edges

$$\mathcal{G} = (S, s_0, S_1, S_2, E)$$

○ : player P_1

□ : player P_2



Games on graphs

States → S, Edges → E

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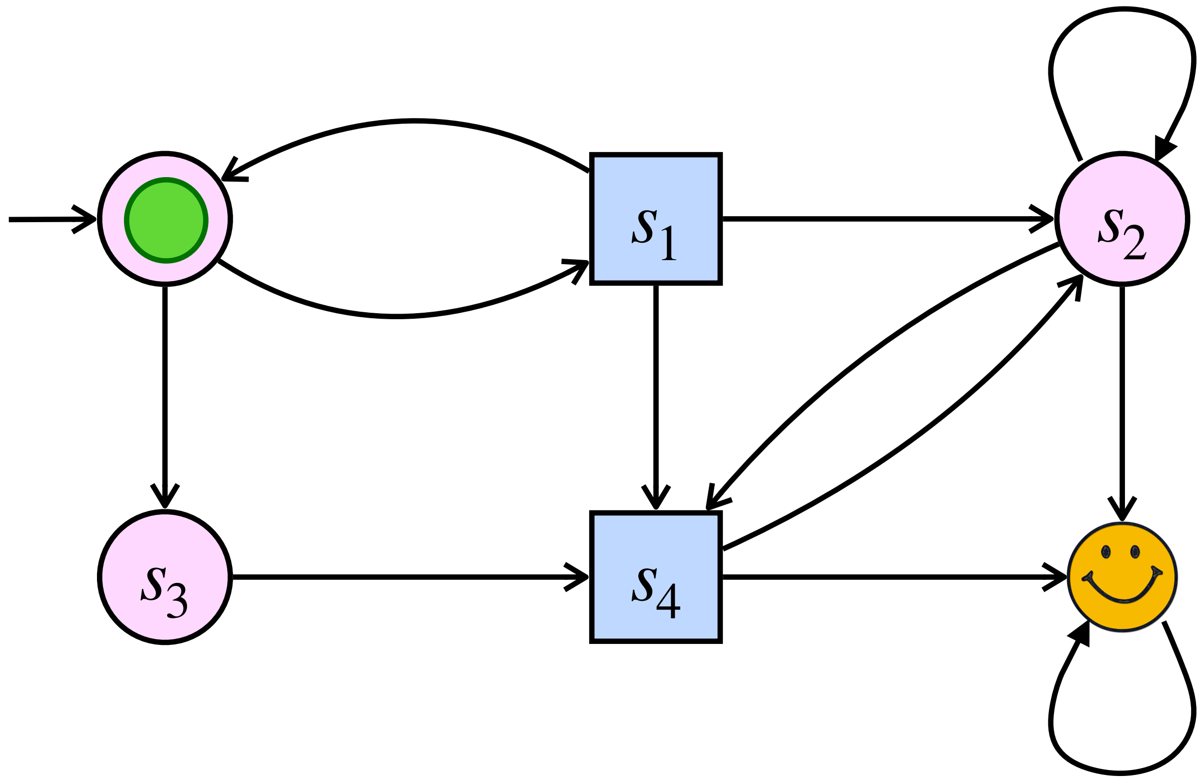

s_0

Games on graphs

States Edges

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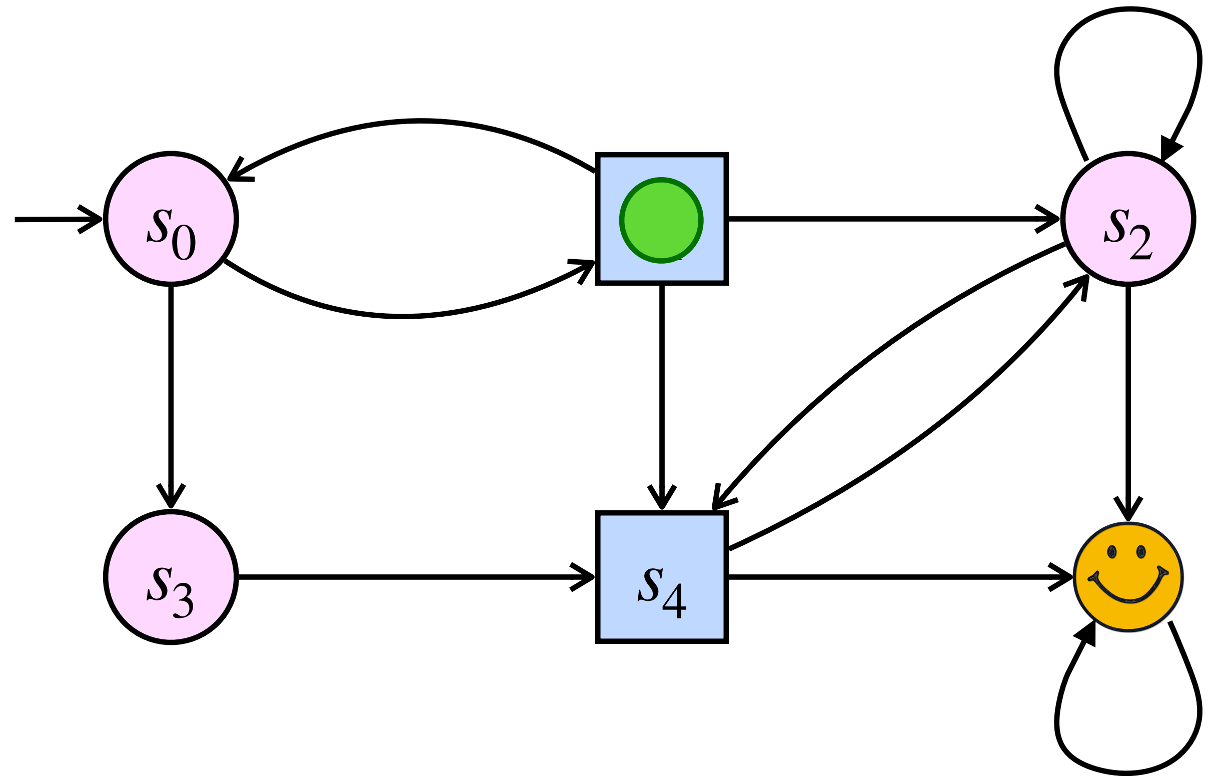
: player P_1

: player P_2



$$s_0 \rightarrow s_1$$

1. P_1 chooses the edge (s_0, s_1)

Games on graphs

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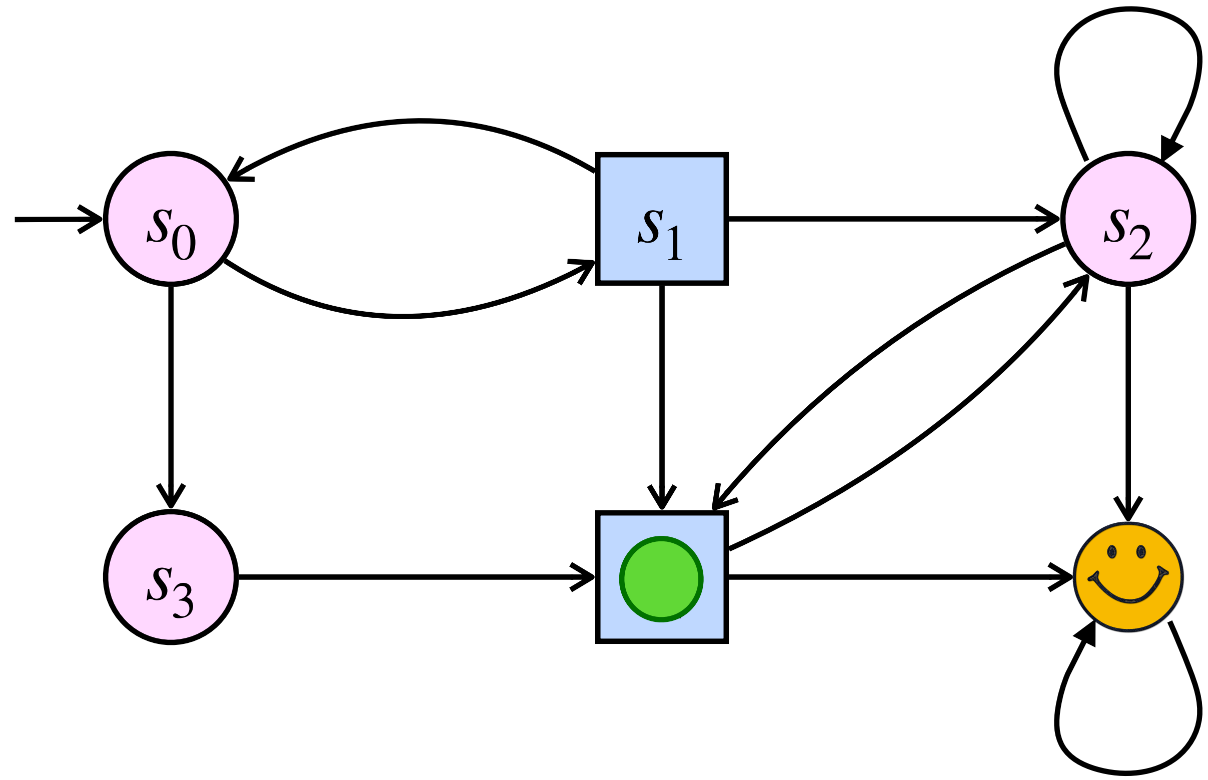

$$s_0 \to s_1 \to s_4$$

1. P_1 chooses the edge (s_0, s_1)
2. P_2 chooses the edge (s_1, s_4)

Games on graphs

States → S, Edges → E

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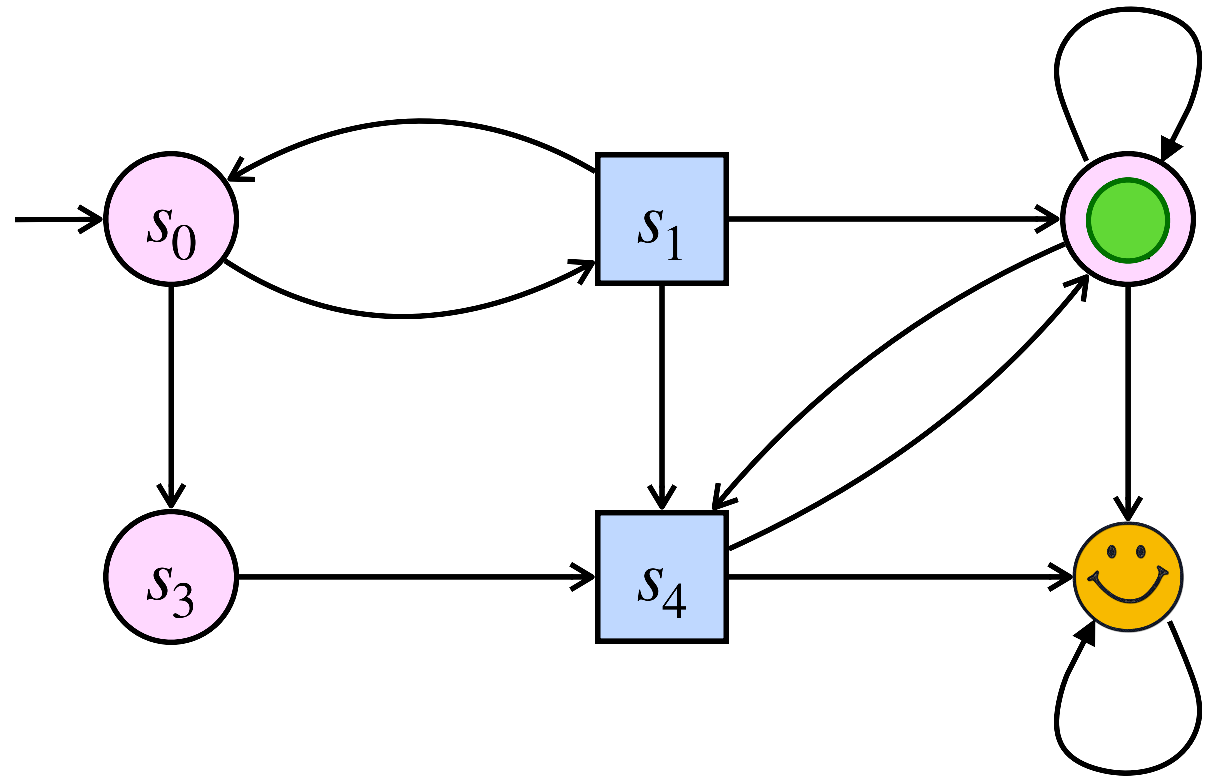

$$s_0 \to s_1 \to s_4 \to s_2$$

1. P_1 chooses the edge (s_0, s_1)
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Games on graphs

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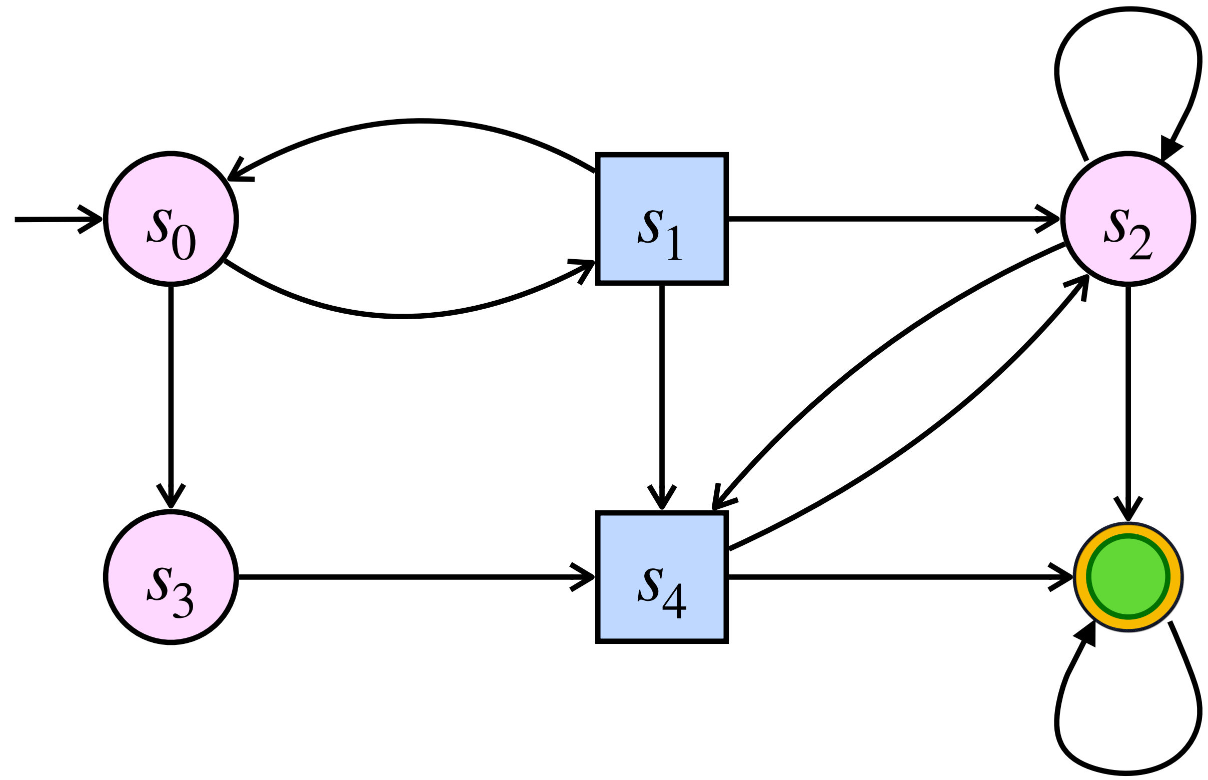

$s_0 \to s_1 \to s_4 \to s_2 \to$ 

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4. P_1 chooses the edge (s_2, )

Games on graphs

States Edges

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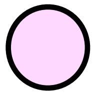
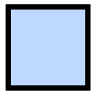
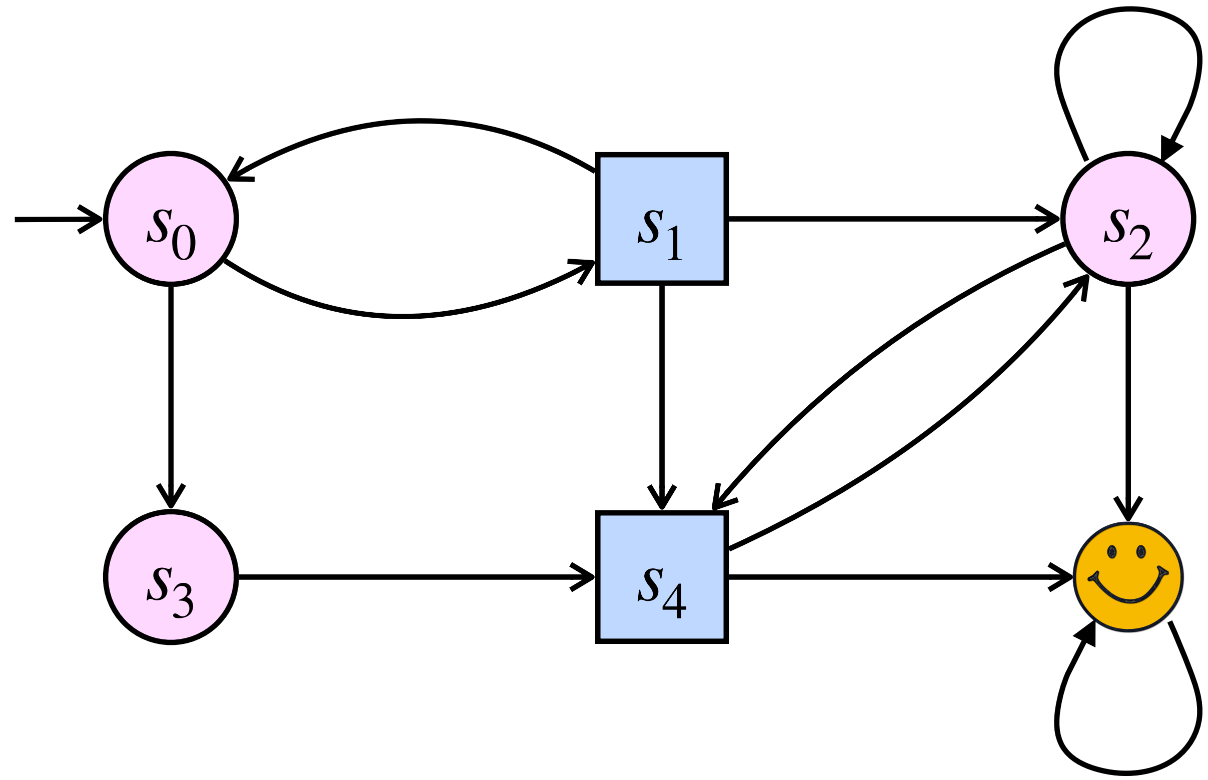
 : player P_1

 : player P_2



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States, Edges

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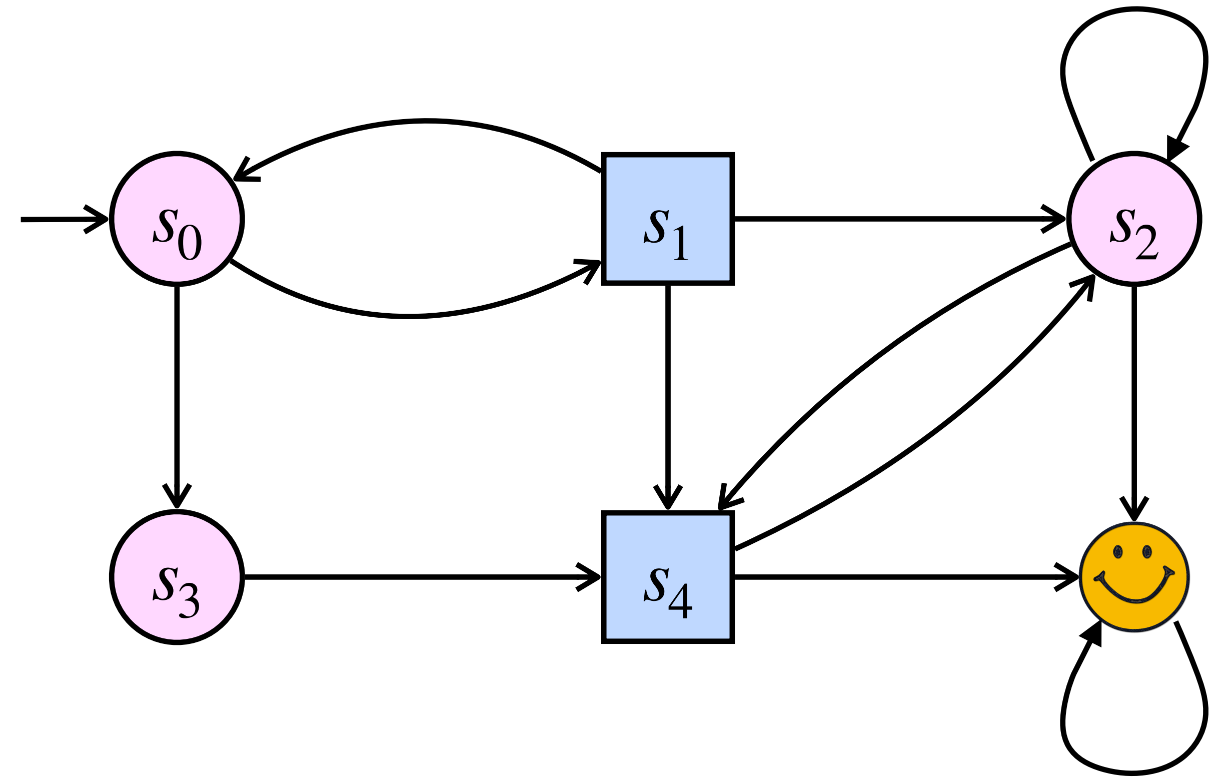

$$s_0 \rightarrow s_1 \rightarrow s_4 \rightarrow s_2 \rightarrow$$ ☺

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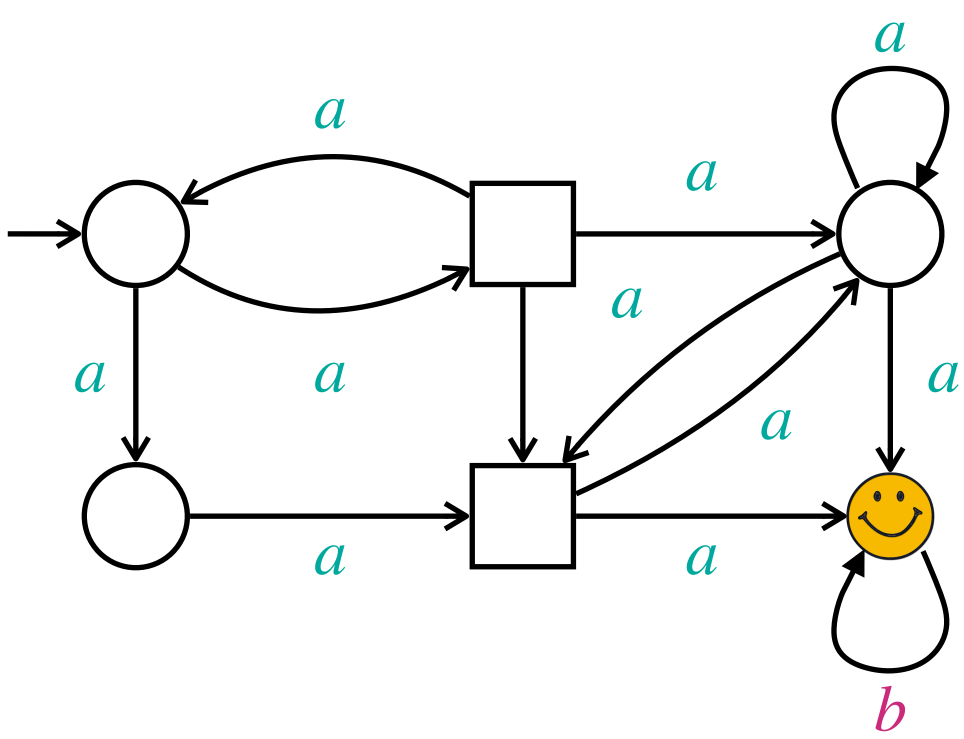
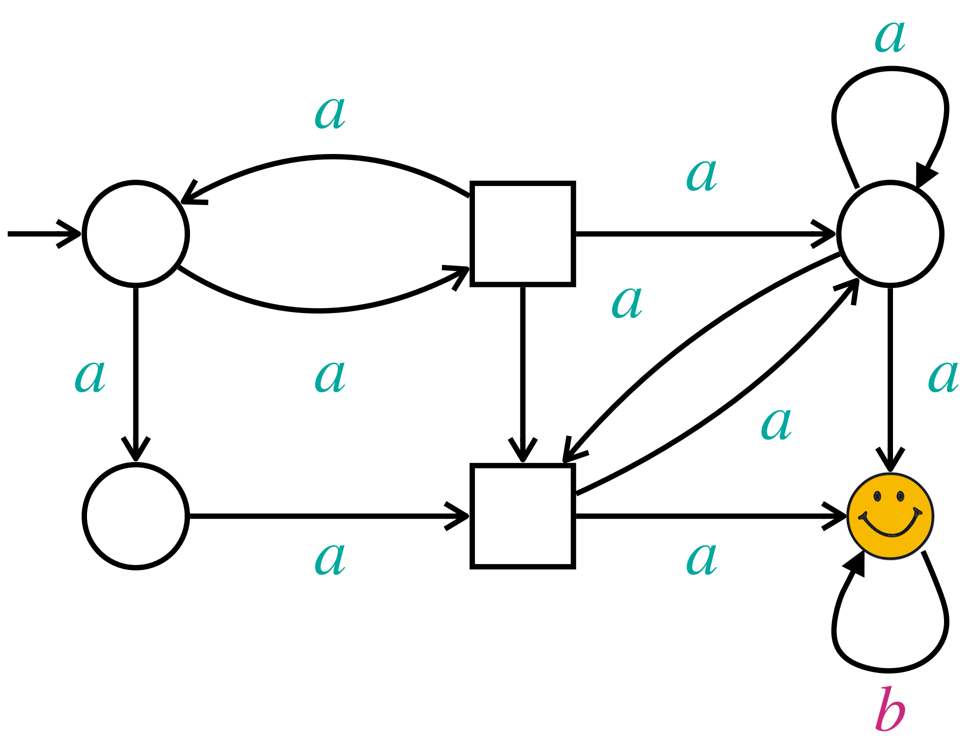
Players use **strategies** to play.
A strategy for P_i is $\sigma_i : S^*S_i \rightarrow E$

Objectives for the players



$$C = \{a, b\}$$
$$E \subseteq S \times C \times S$$

Objectives for the players



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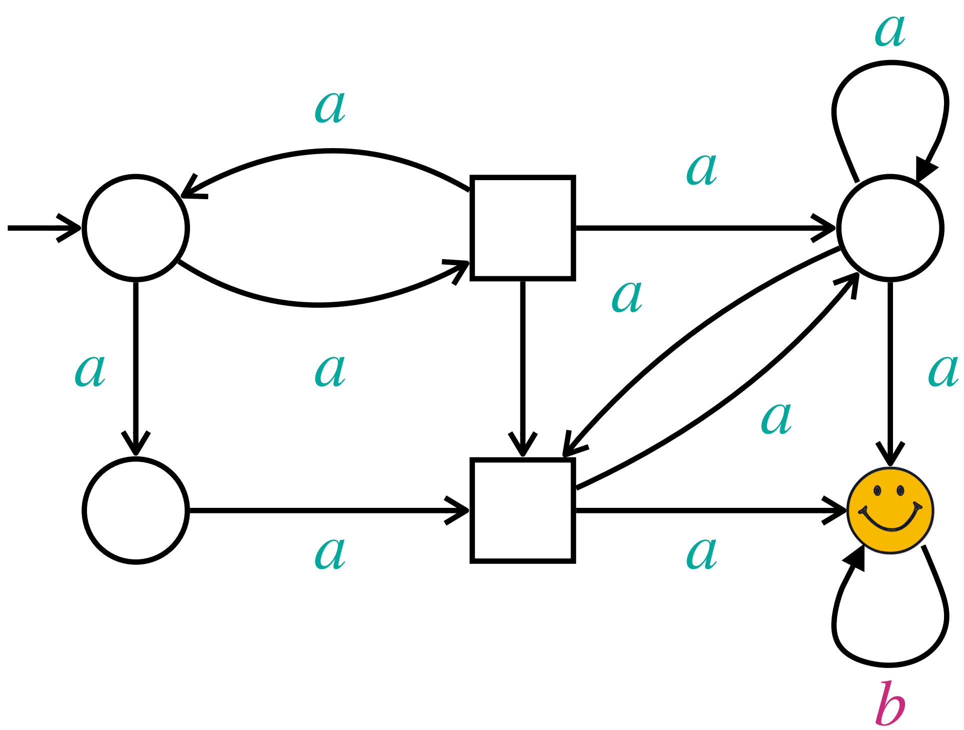
- Winning objective for P_i: $W_i \subseteq C^\omega$, e.g. $W_1 = C^* \cdot b \cdot C^\omega$

Objectives for the players



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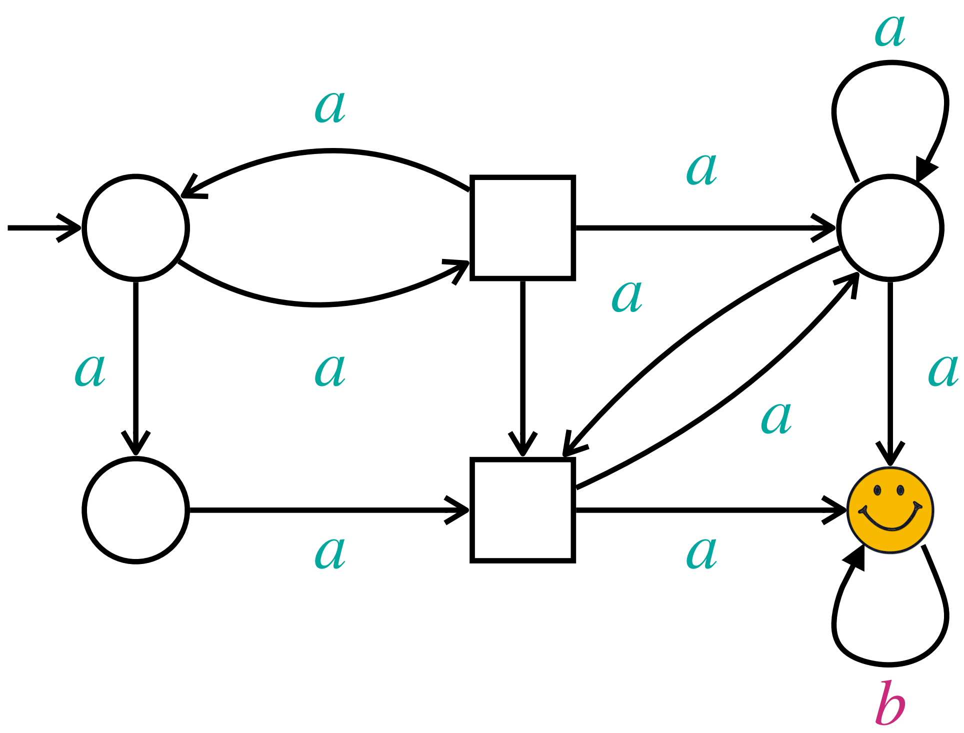

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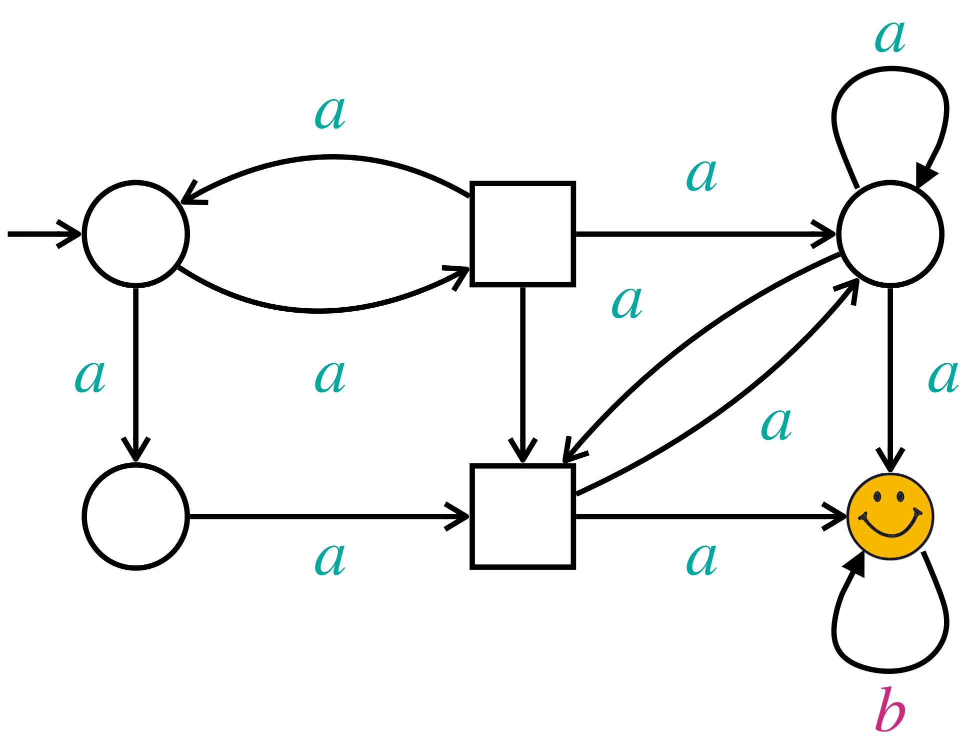
Objectives for the players



Zero-sum hypothesis

$C = \{a, b\}$
$E \subseteq S \times C \times S$

- Winning objective for P_i: $W_i \subseteq C^\omega$, e.g. $W_1 = C^* \cdot b \cdot C^\omega$

$W_2 = W_1^c$

- Payoff function: $p_i : C^\omega \to \mathbb{R}$, e.g. mean-payoff

$p_1 + p_2 = 0$

- Preference relation: $\sqsubseteq_i \subseteq C^\omega \times C^\omega$ (total preorder)

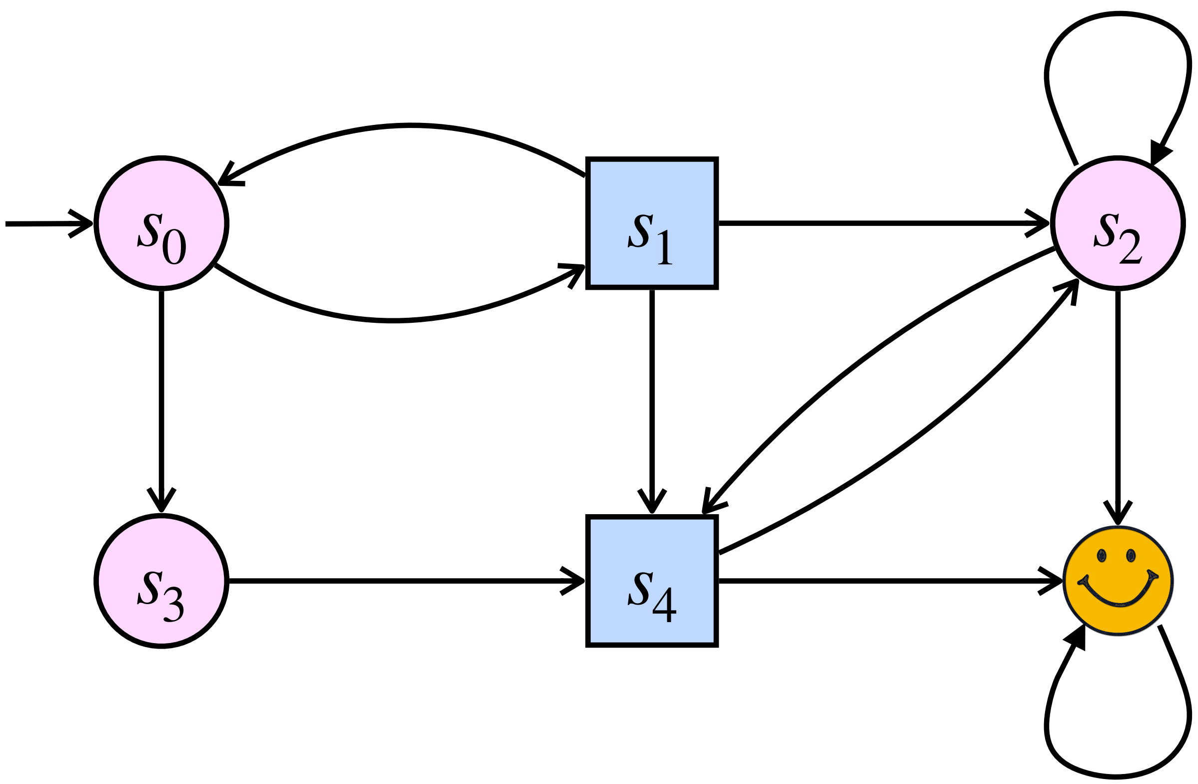
$\sqsubseteq_2 = \sqsubseteq_1^{-1}$

What does it mean to win a game?

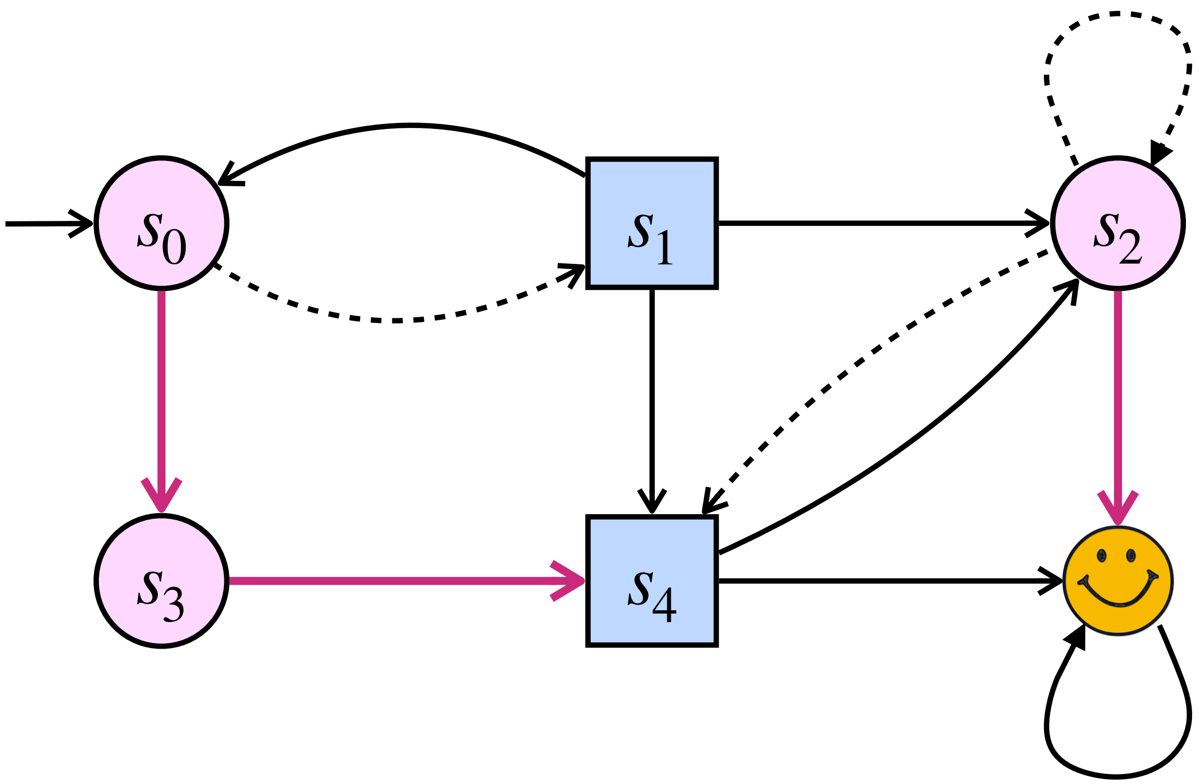
What does it mean to win a game?

- Play $\rho = s_0 s_1 s_2 \ldots$ is compatible with σ_i whenever $s_j \in S_i$ implies $(s_j, s_{j+1}) = \sigma_i\left(s_0 s_1 \ldots s_j\right)$. We write Out$(\sigma_i)$.

Outcomes of a strategy

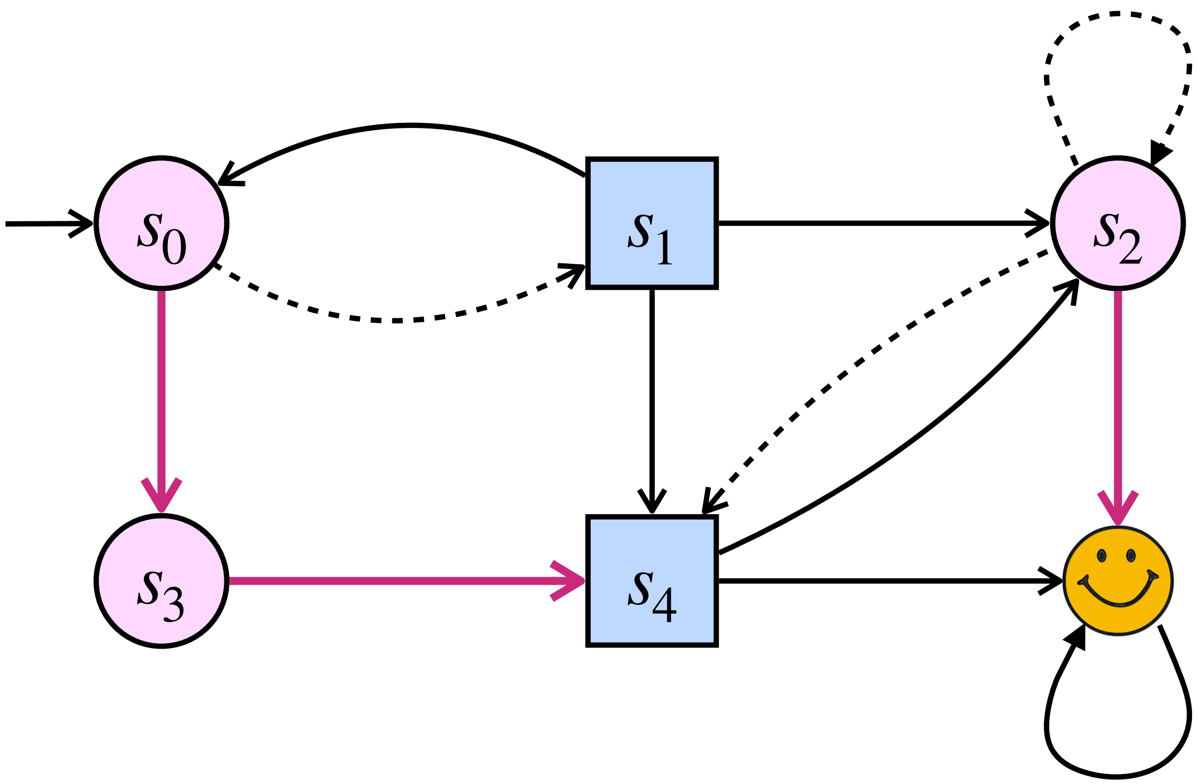


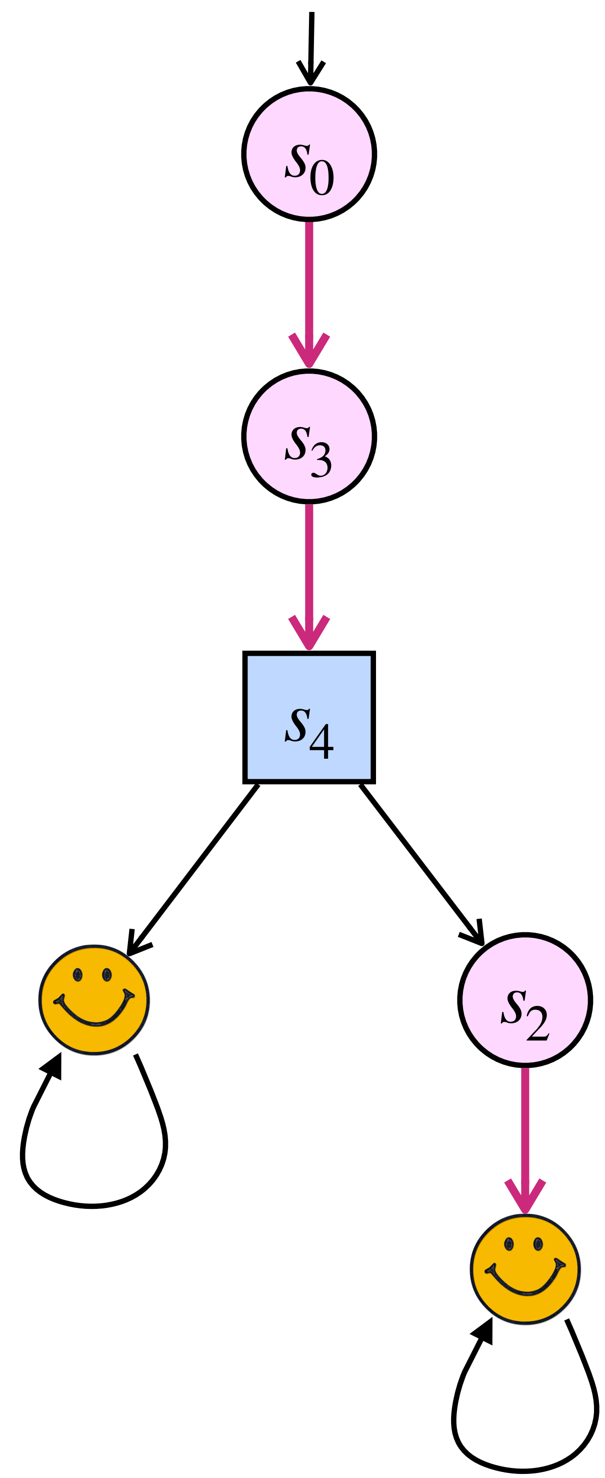
Outcomes of a strategy



- Strategy σ

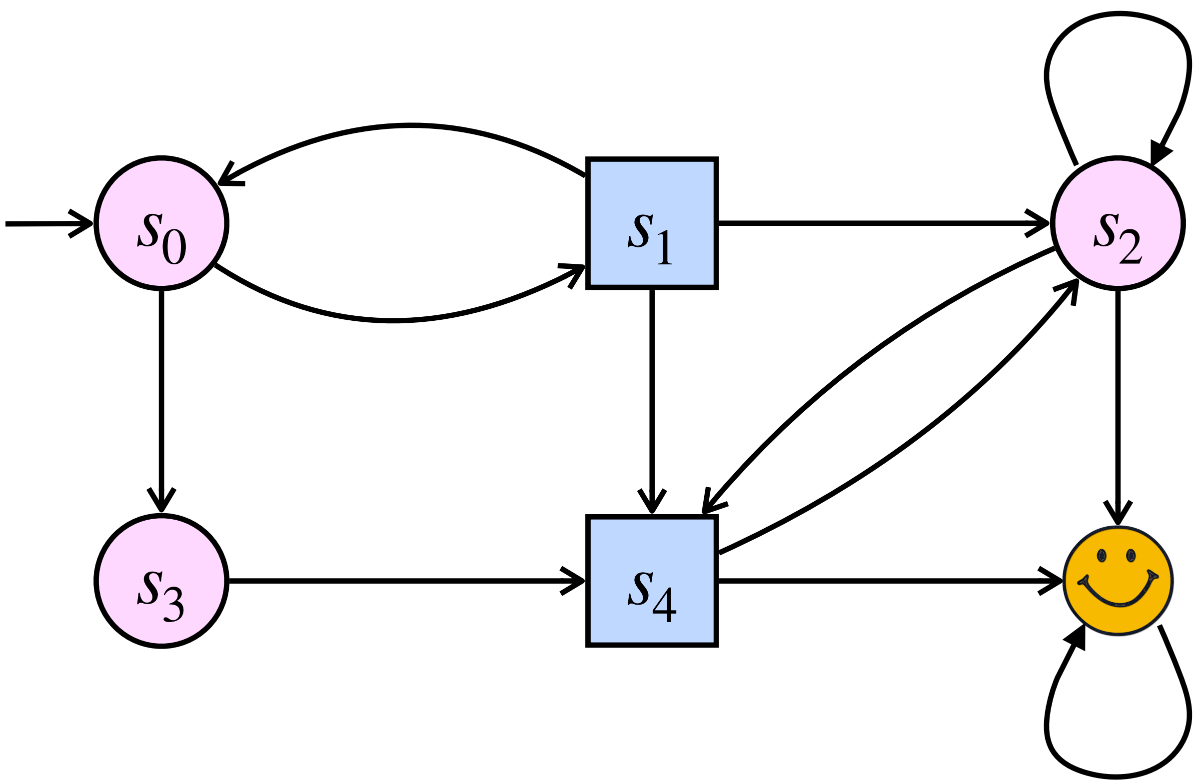
Outcomes of a strategy



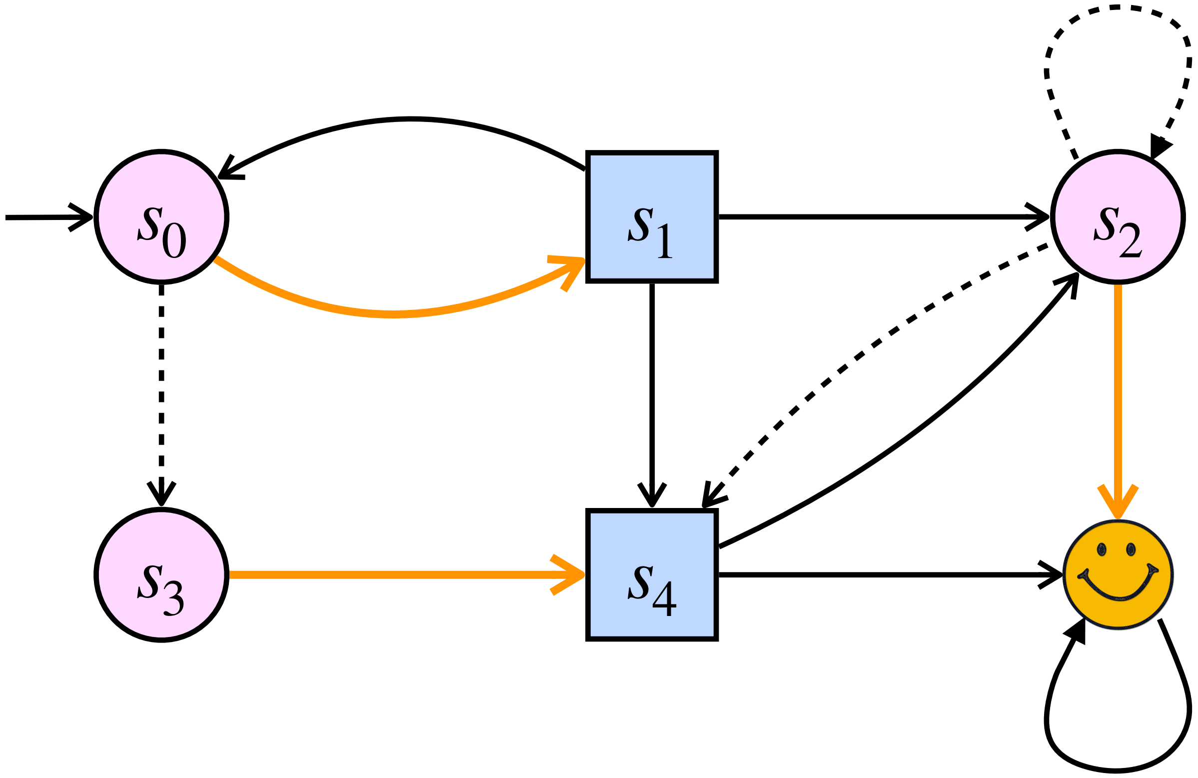


- Strategy σ
- Out(σ) has two plays, which are both winning

Outcomes of a strategy

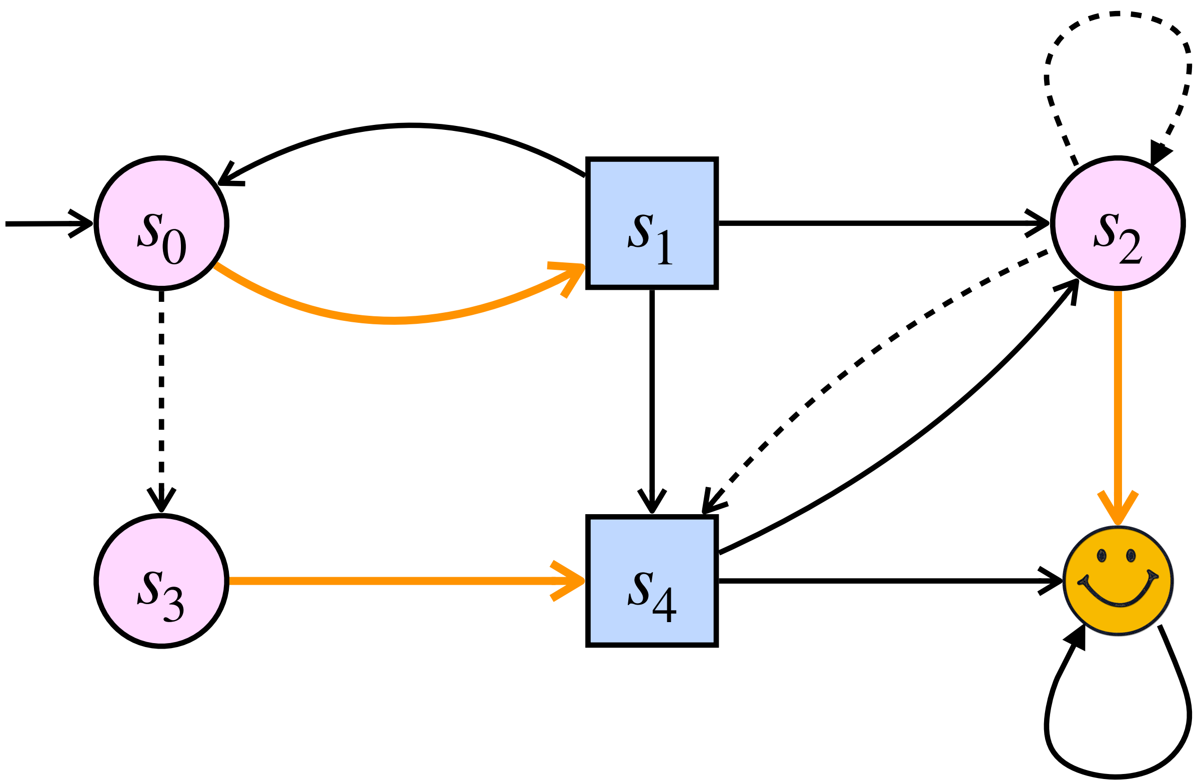


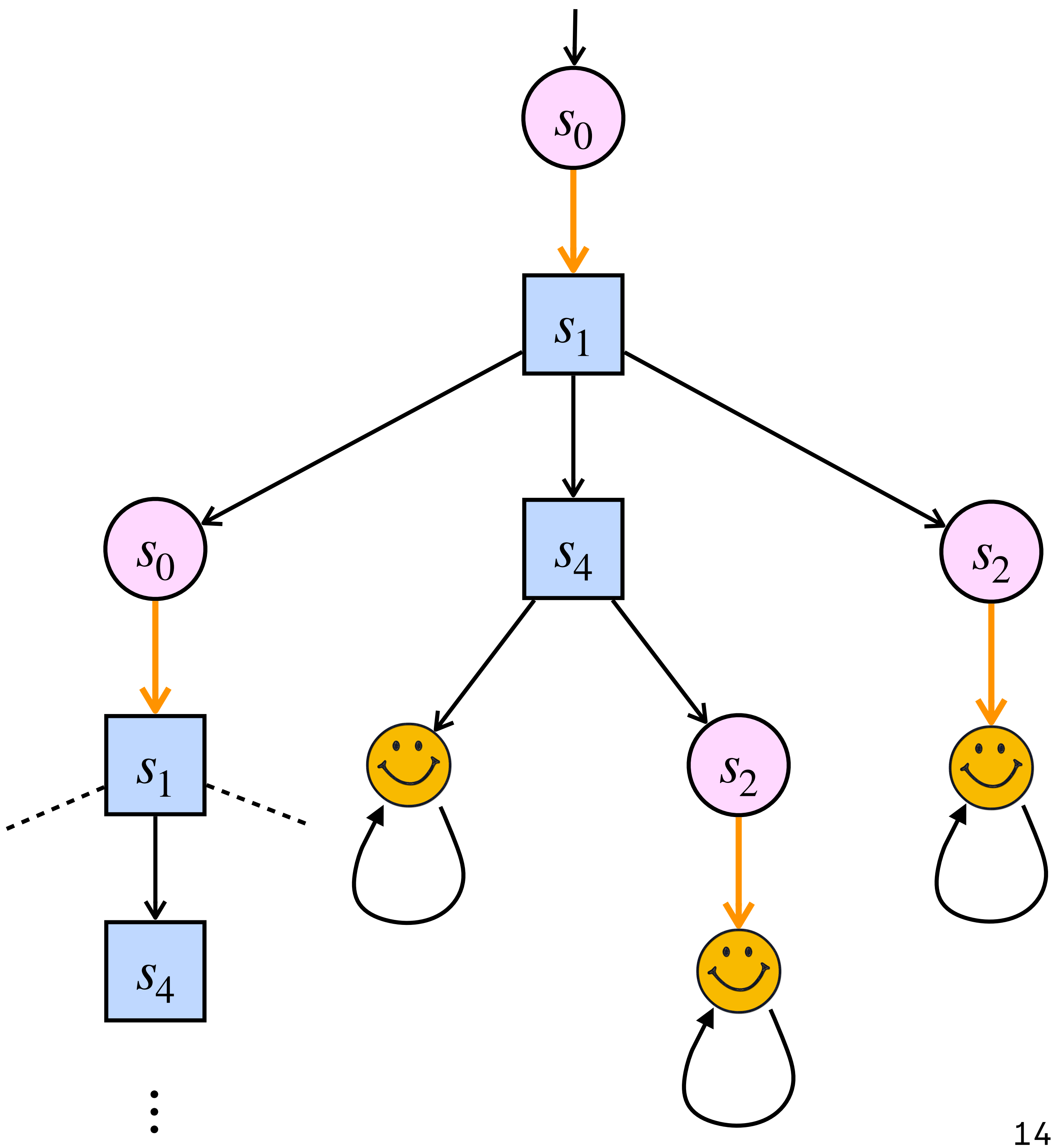
Outcomes of a strategy



- Strategy σ

Outcomes of a strategy





- Strategy σ
- Out(σ) has infinitely many plays, some of them are not winning

What does it mean to win a game?

- Play $\rho = s_0 s_1 s_2 \ldots$ is compatible with σ_i whenever $s_j \in S_i$ implies $(s_j, s_{j+1}) = \sigma_i\left(s_0 s_1 \ldots s_j\right)$. We write Out$(\sigma_i)$.

- σ_i is **winning** if all plays compatible with σ_i belong to W_i

[Mar75] Martin. Borel determinacy (Annals of Mathematics).

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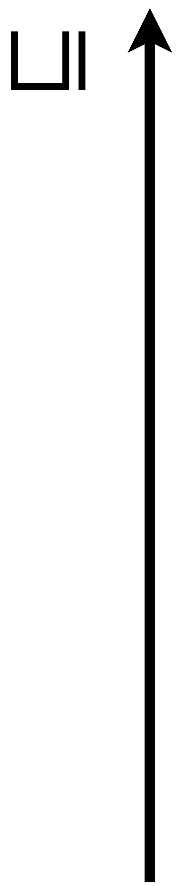
Martin's determinacy theorem

Turn-based zero-sum games are determined for Borel winning objectives: in every game, either P_1 or P_2 has a winning strategy.

[Mar75] Martin. Borel determinacy (Annals of Mathematics).

Optimality of strategies



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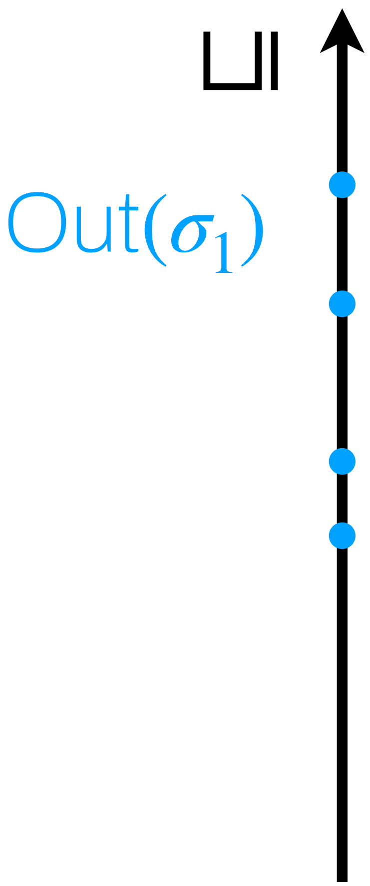

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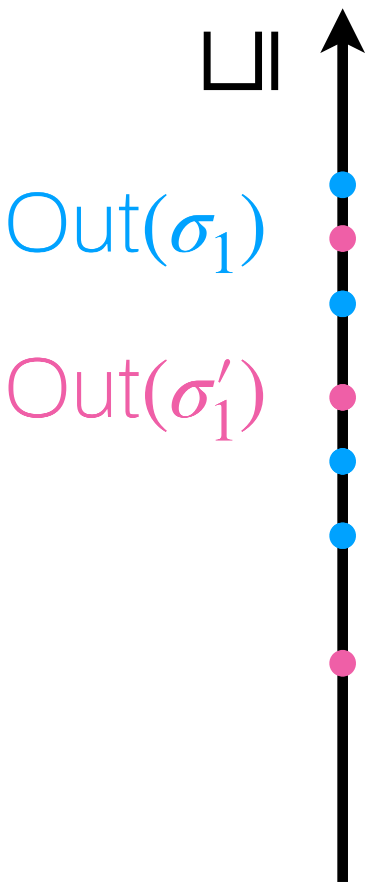

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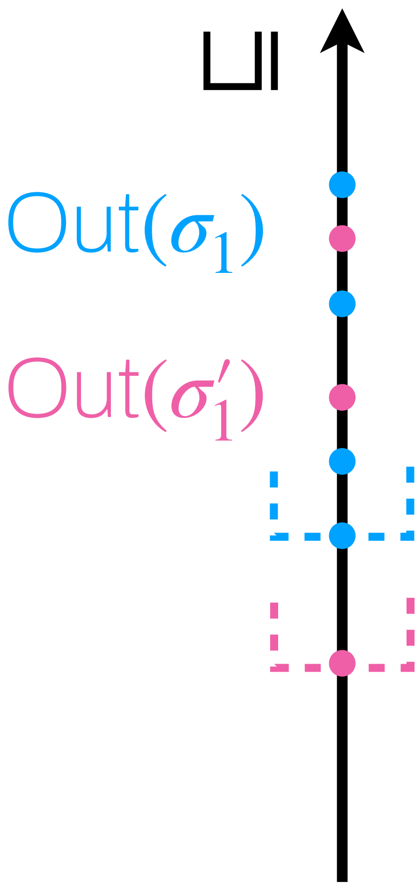

- σ_1 is better than σ_1' whenever $\mathrm{Out}(\sigma_1)^{\uparrow} \subseteq \mathrm{Out}(\sigma_1')^{\uparrow}$

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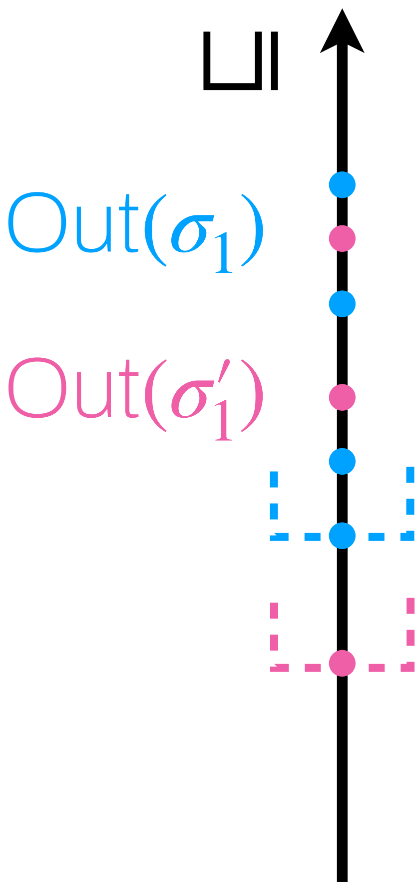

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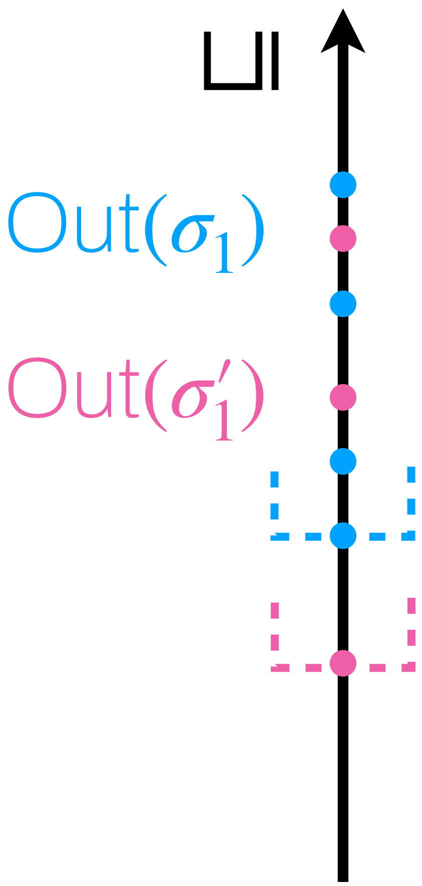

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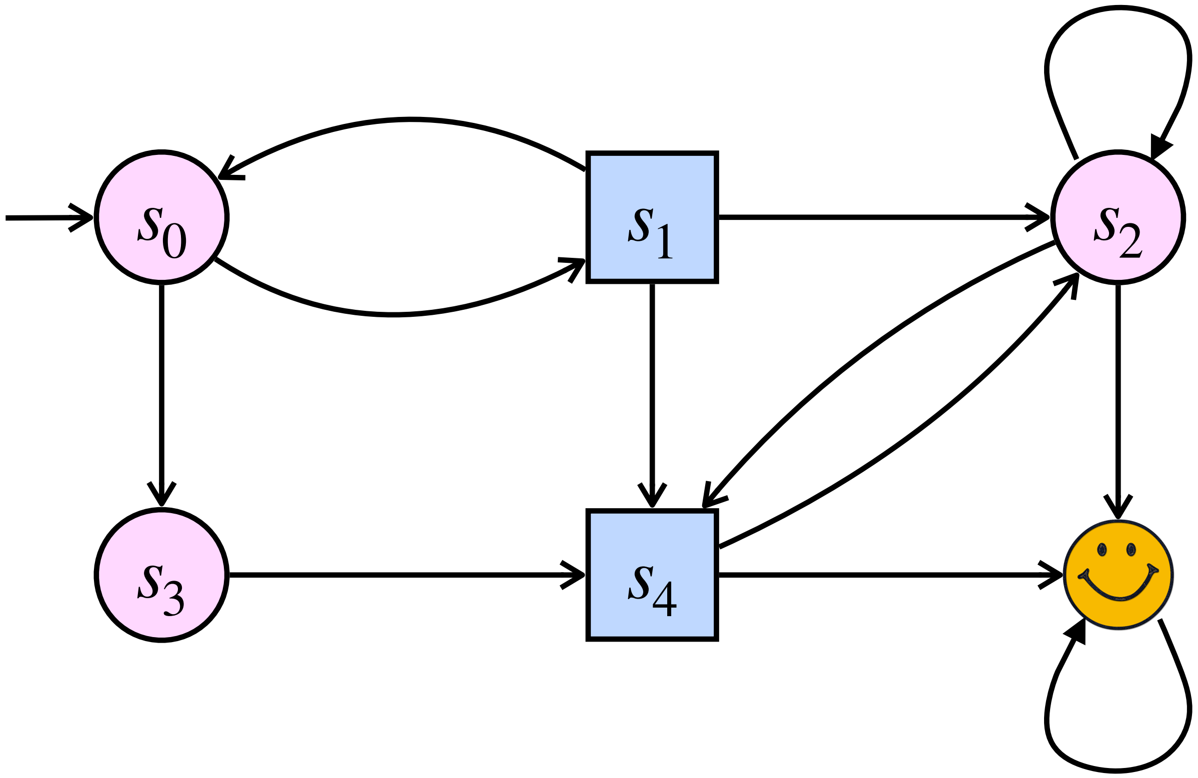
Remark

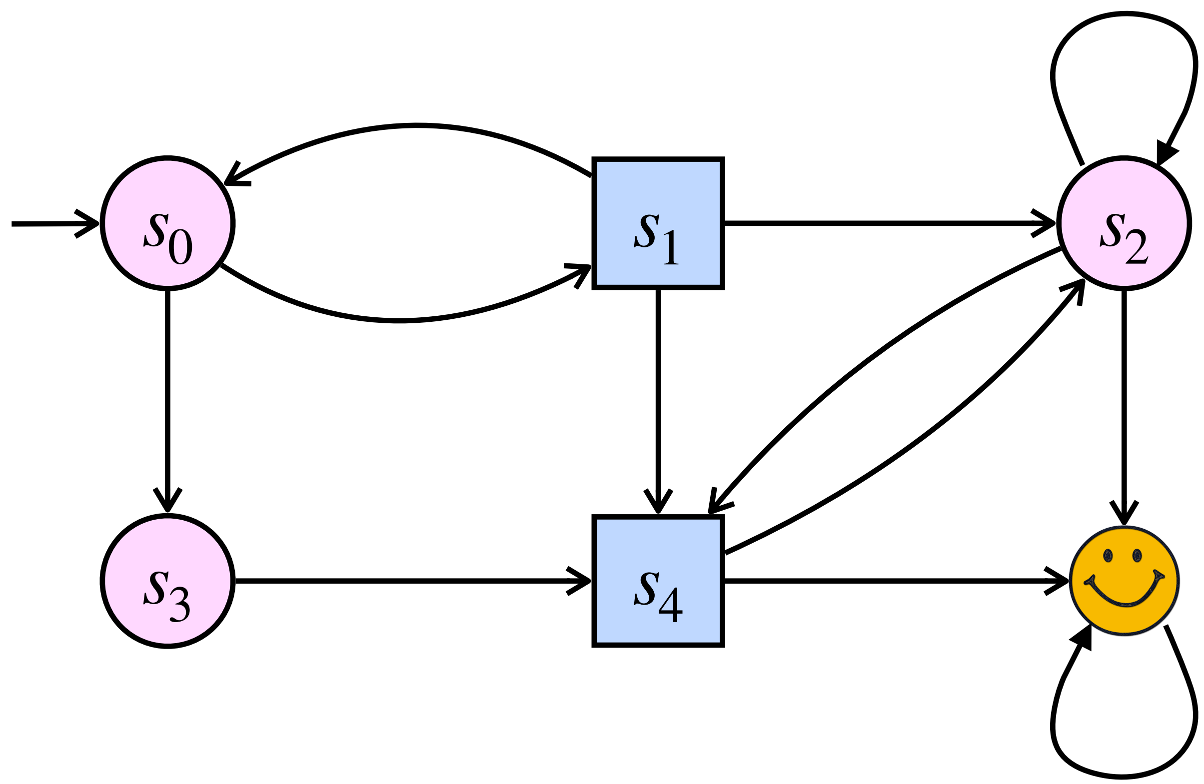
- Optimal strategies might not exist
- If $\sqsubseteq$ given by a payoff function, notion of ε-optimal strategies
- Optimality vs subgame-optimality

Relevant questions



φ = Reach(🙂)

Relevant questions



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- Can P_1 win the game, i.e. does P_1 have a winning strategy?
  Can P_1 play optimally?

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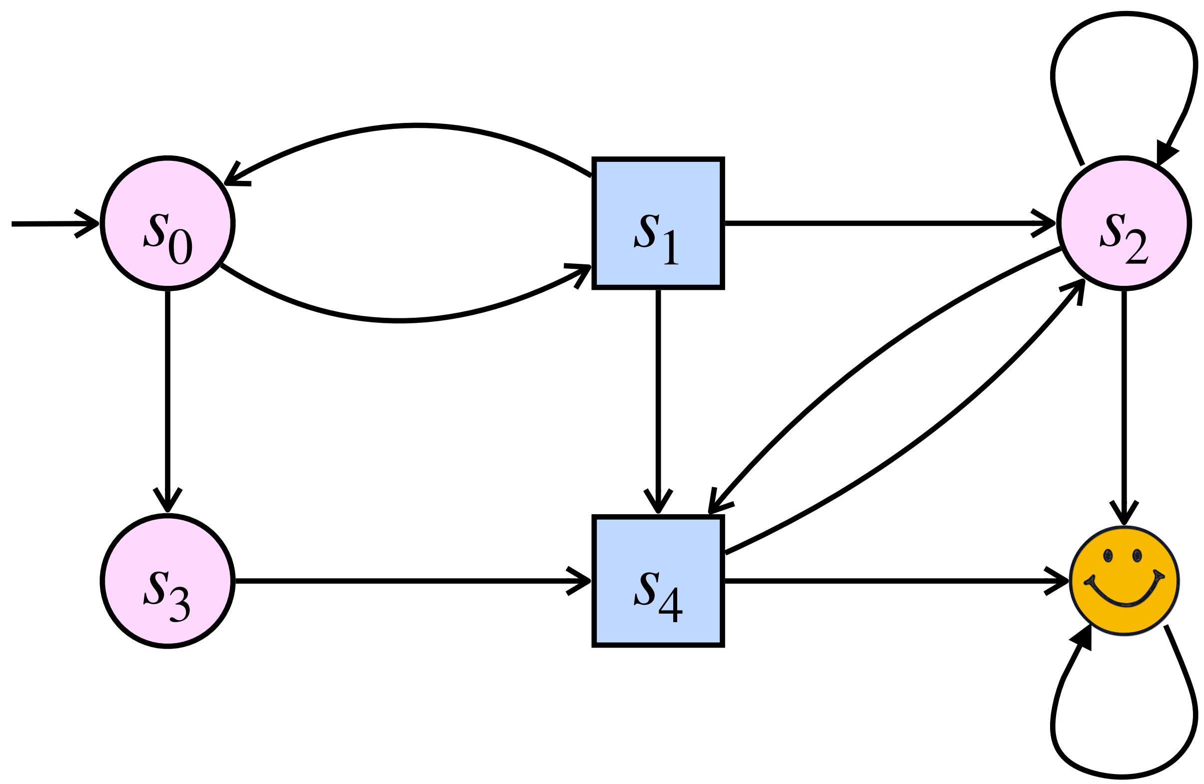

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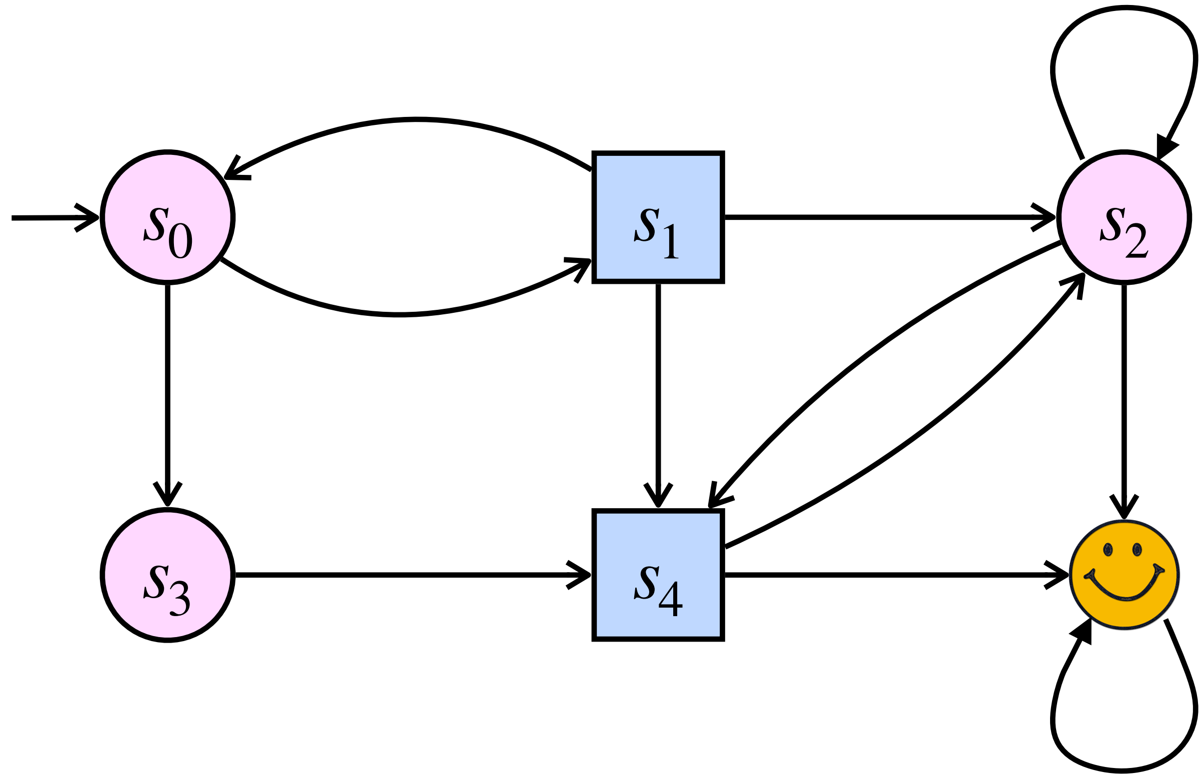

$\varphi =$ Reach(☺)

- Can P_1 win the game, i.e. does P_1 have a winning strategy?
  Can P_1 play optimally?
- Is there an effective (efficient) way of winning?
- How complex is it to win?

Example: the Nim game

- Players alternate
- Each player can take one or two sticks
- The player who takes the last one wins
- P_1 starts

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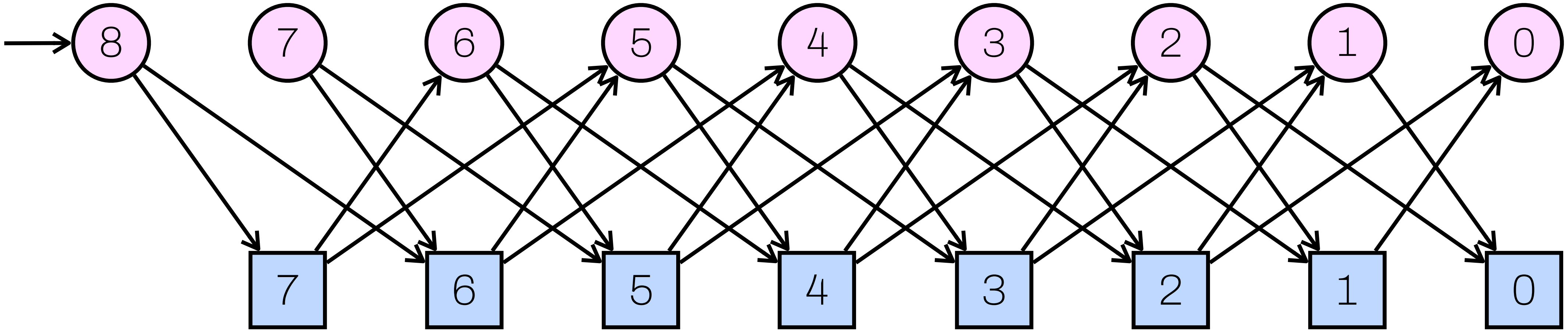

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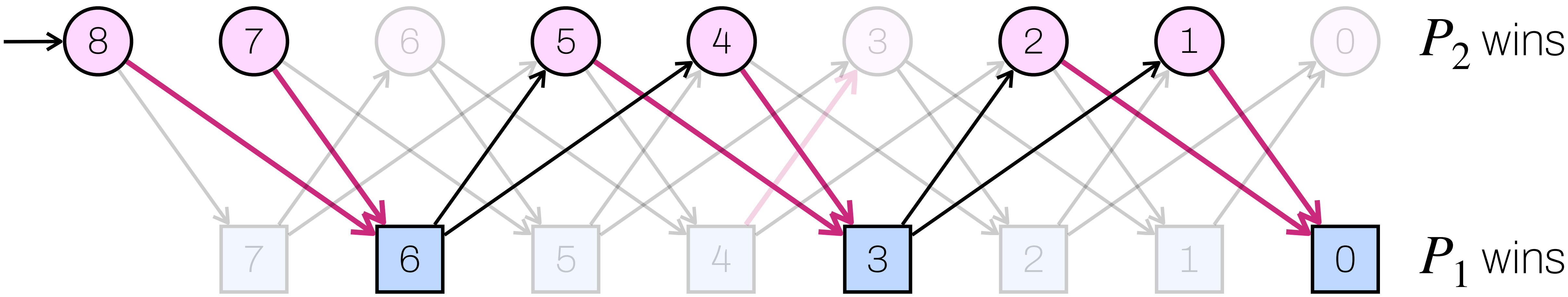

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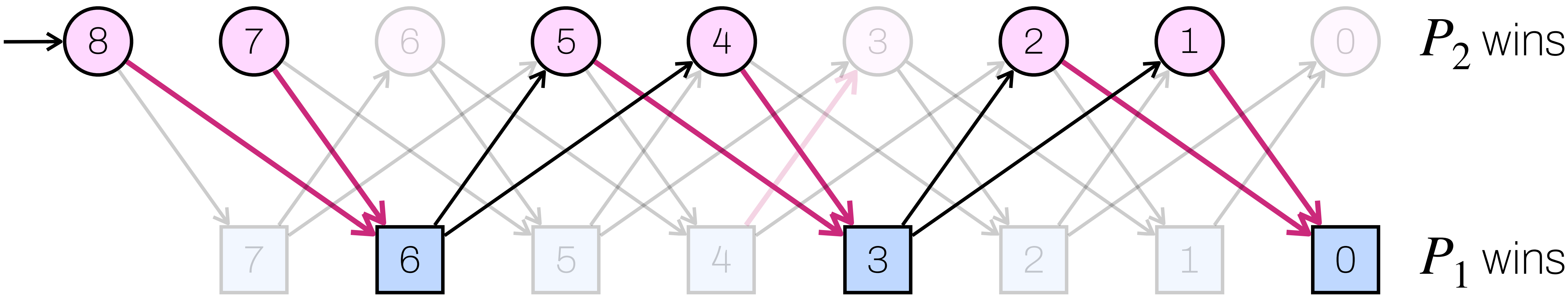

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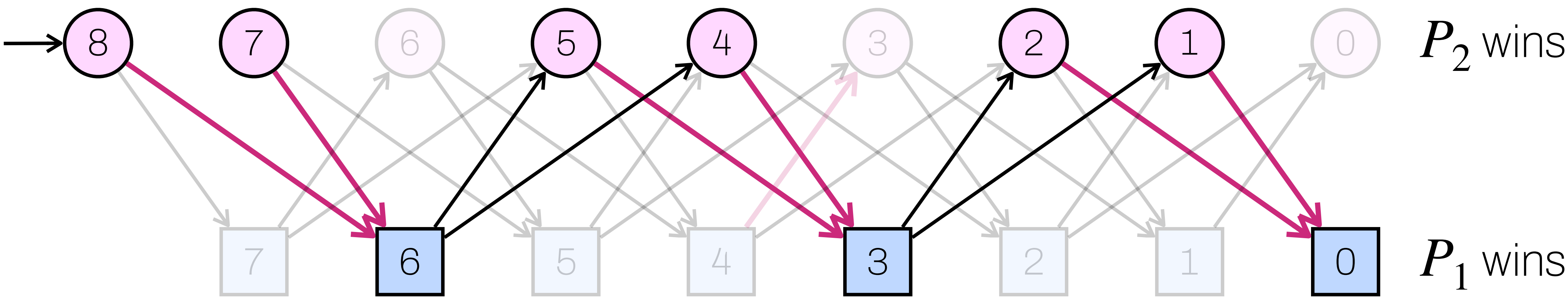


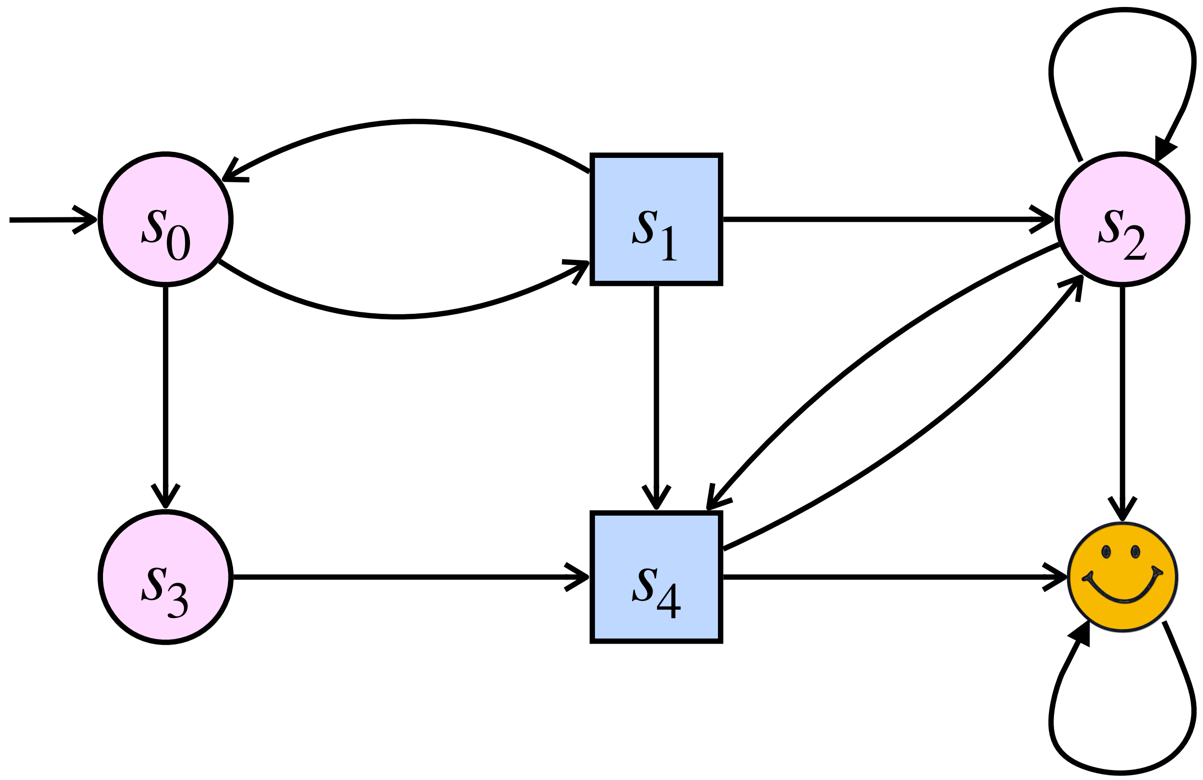
P_1 wins

- from all ○ $\equiv 1$ or $2 \mod 3$
- from all □ $\equiv 0 \mod 3$

P_2 wins

- from all ○ $\equiv 0 \mod 3$
- from all □ $\equiv 1$ or $2 \mod 3$

Computation of winning states in the running example



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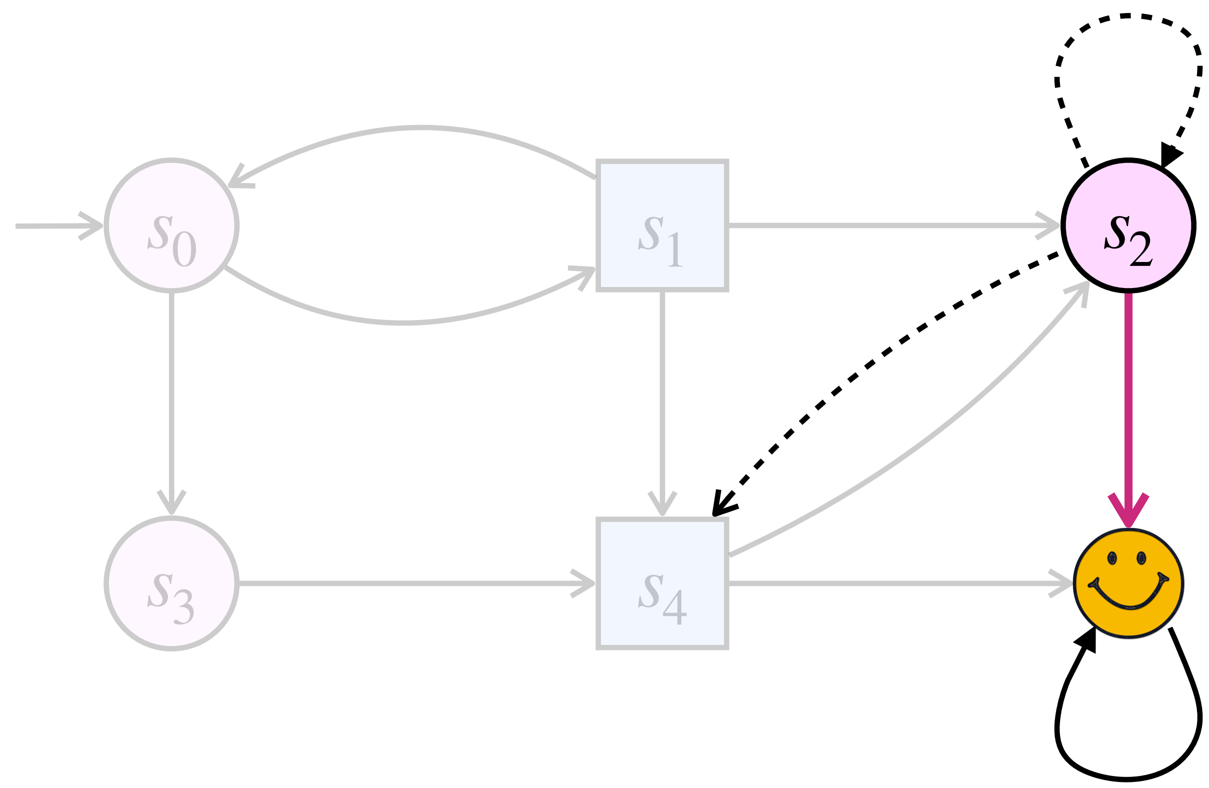

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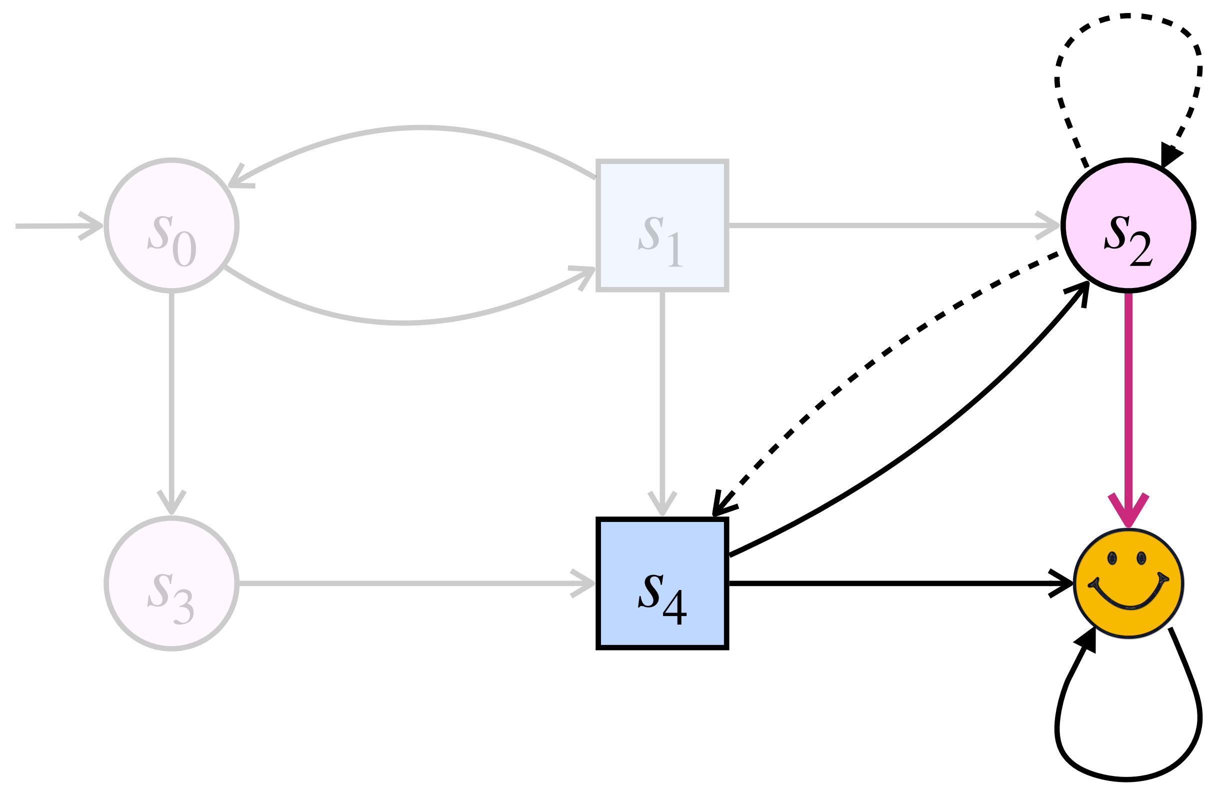

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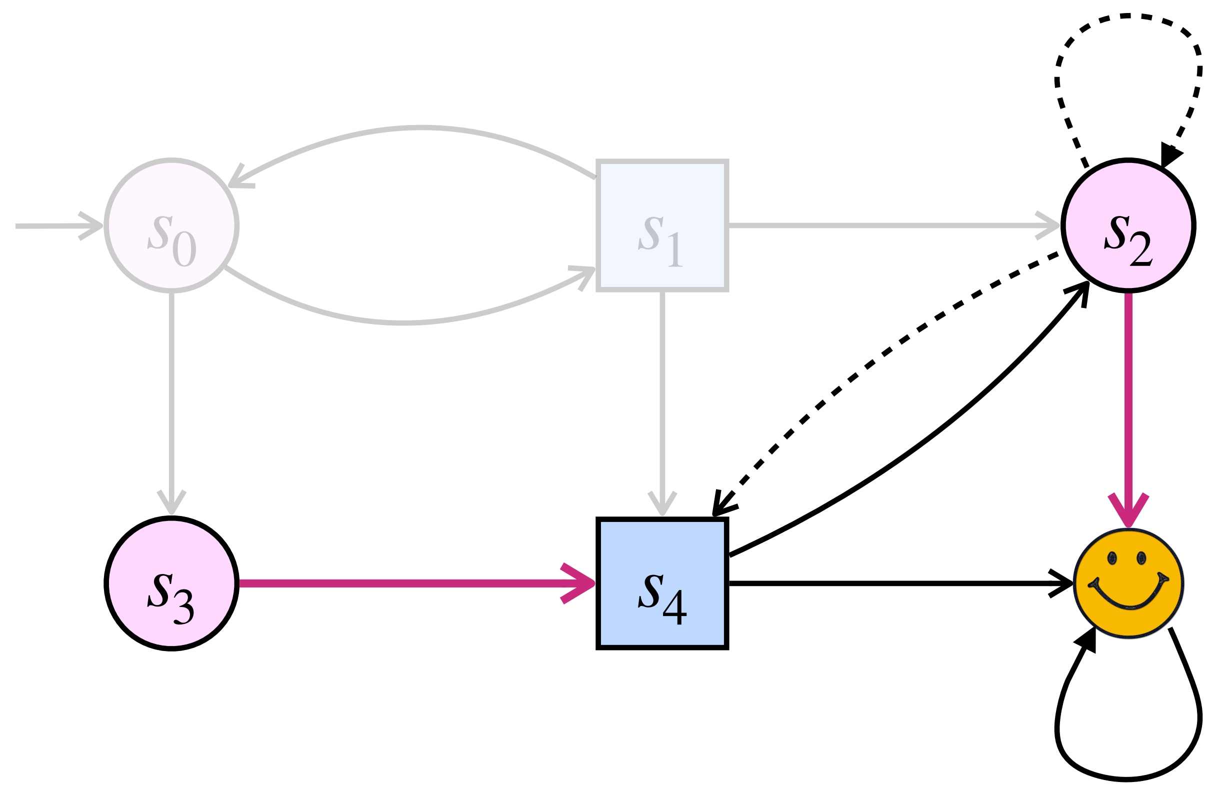

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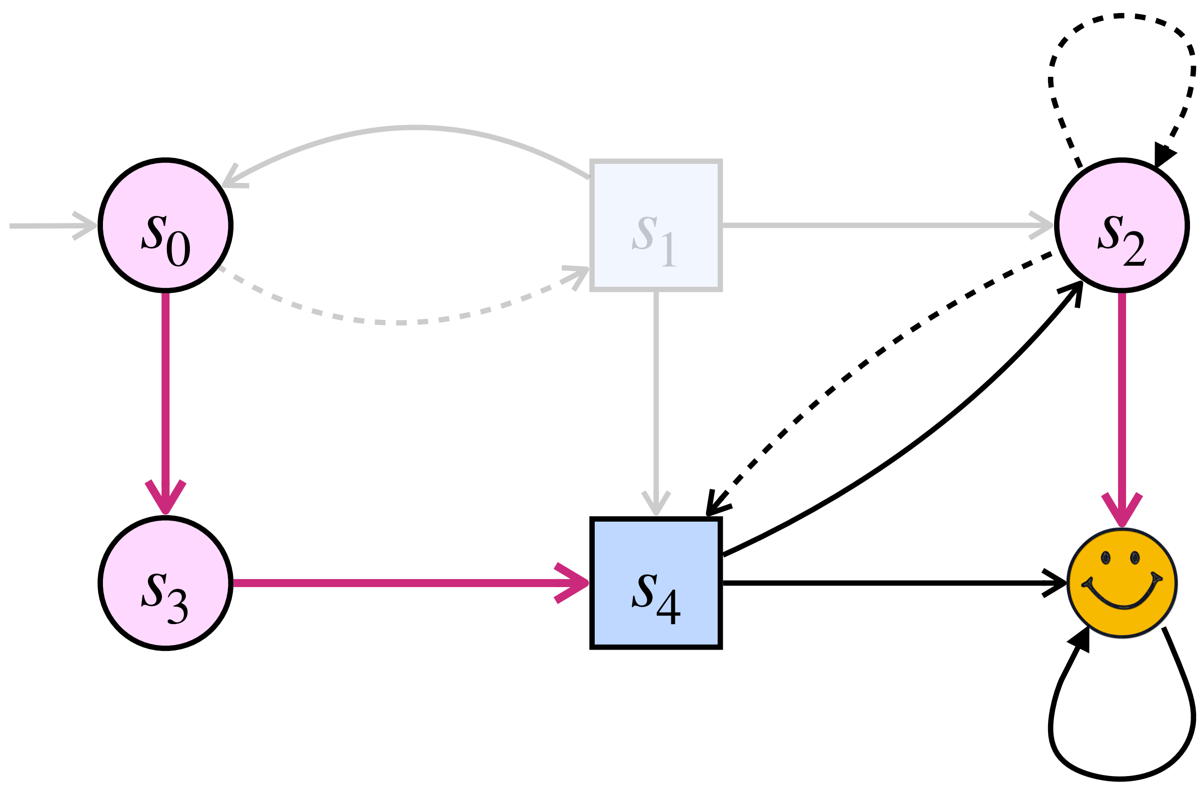

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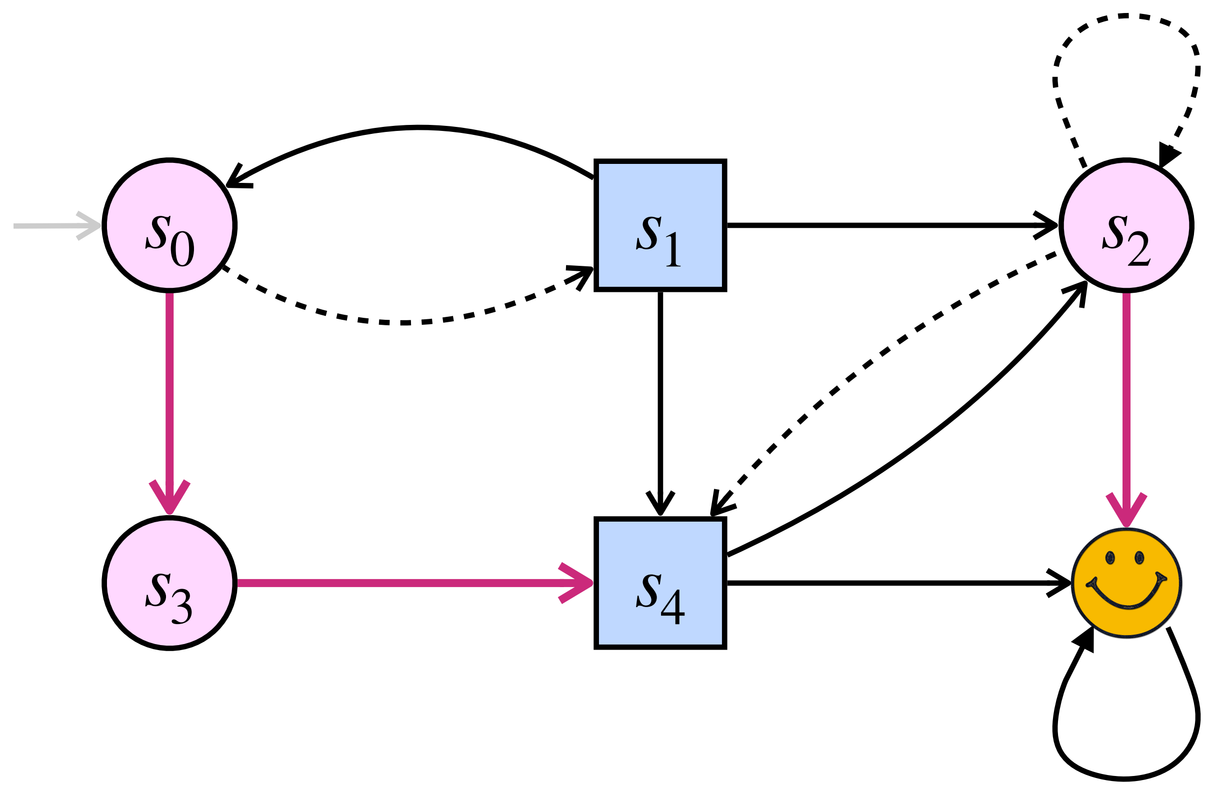

All states are winning for P_1

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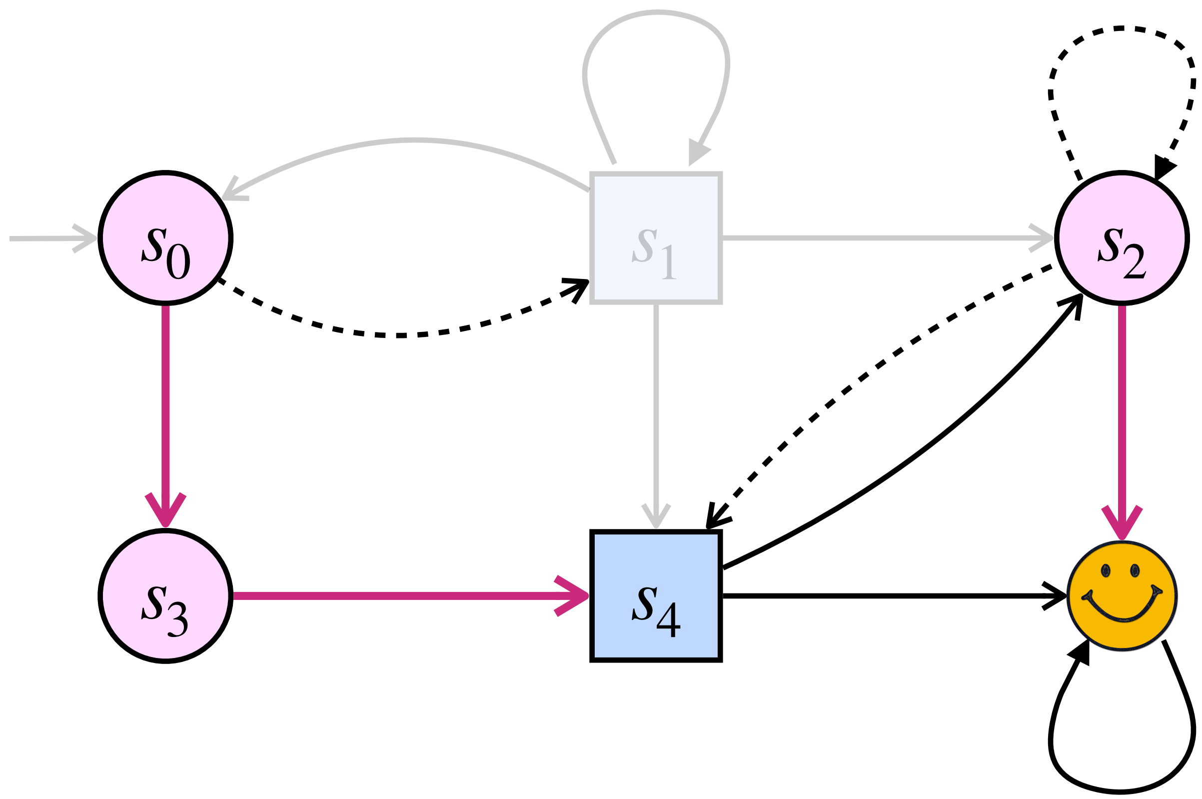

One state is not winning for P_1
It is winning for P_2

Chess game

[Zer13] Zermelo. Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels (Congress Mathematicians, 1912).
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Chess game

Zermelo's Theorem

From every position, either White can force a win, or Black can force a win, or both sides can force at least a draw.

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- According to Claude Shannon, there are 10^{43} legit positions in chess

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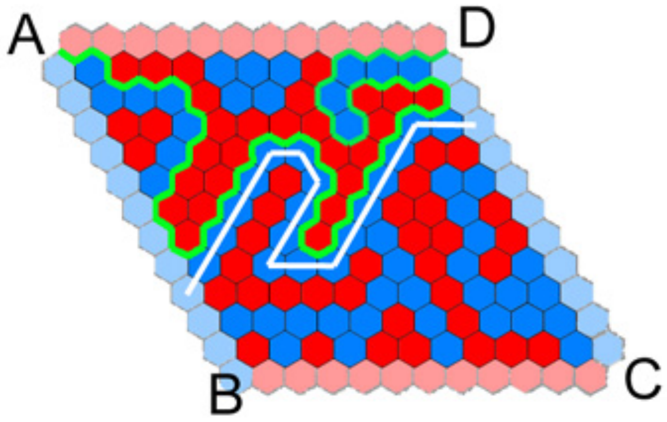
Hex game



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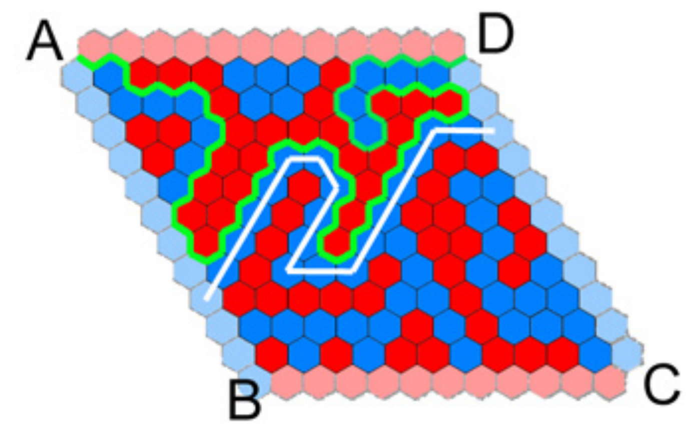
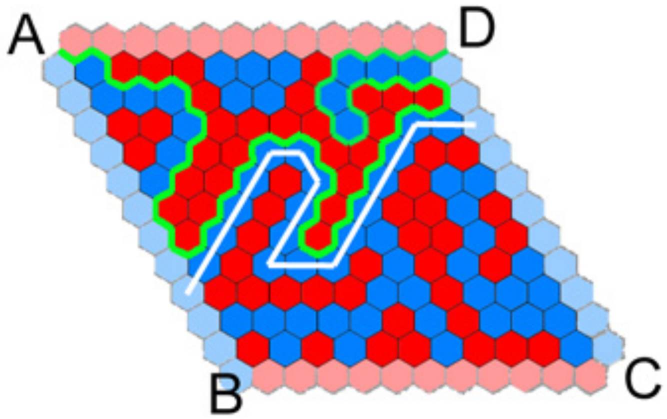


Solving the Hex game

First player has always a winning strategy.

Hex game



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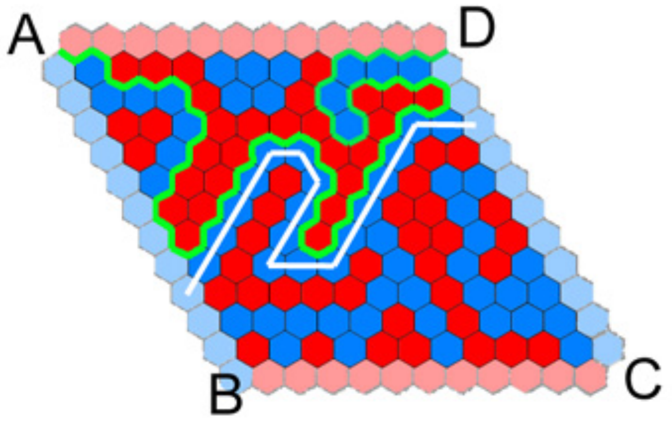


Solving the Hex game

First player has always a winning strategy.

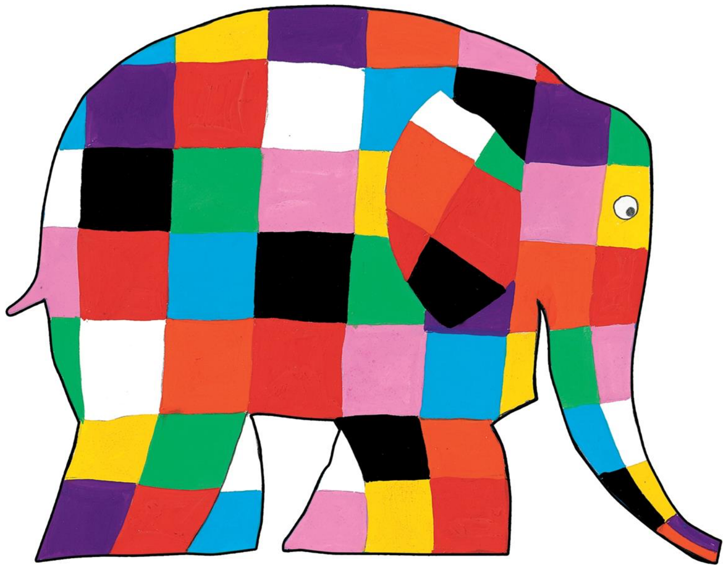
- Determinacy results (no tie is possible) + strategy stealing argument
- A winning strategy is not known yet.

What we do not consider

- Concurrent games
- Stochastic games and strategies
- Partial information
- Values
- Determinacy of Blackwell games

Families of strategies

Families of strategies



General strategies

$$\sigma_i : S^* S_i \rightarrow E$$

- May use any information of the past execution
- Information used is therefore potentially infinite
- Not adequate if one targets implementation

On the simplest side: positional strategies

From $\sigma_i : S^*S_i \rightarrow E$ to $\sigma_i : S_i \rightarrow E$

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- Reachability, parity, mean-payoff, positive energy, ...
  $\rightarrow$ positional strategies are sufficient to win

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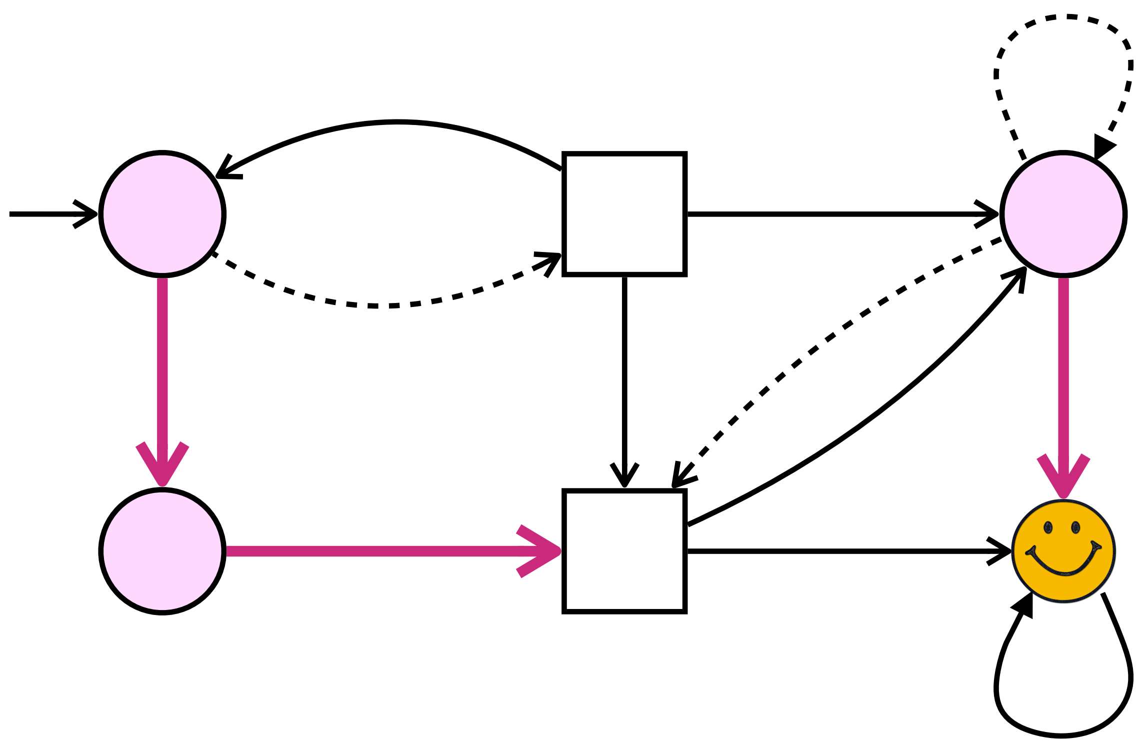
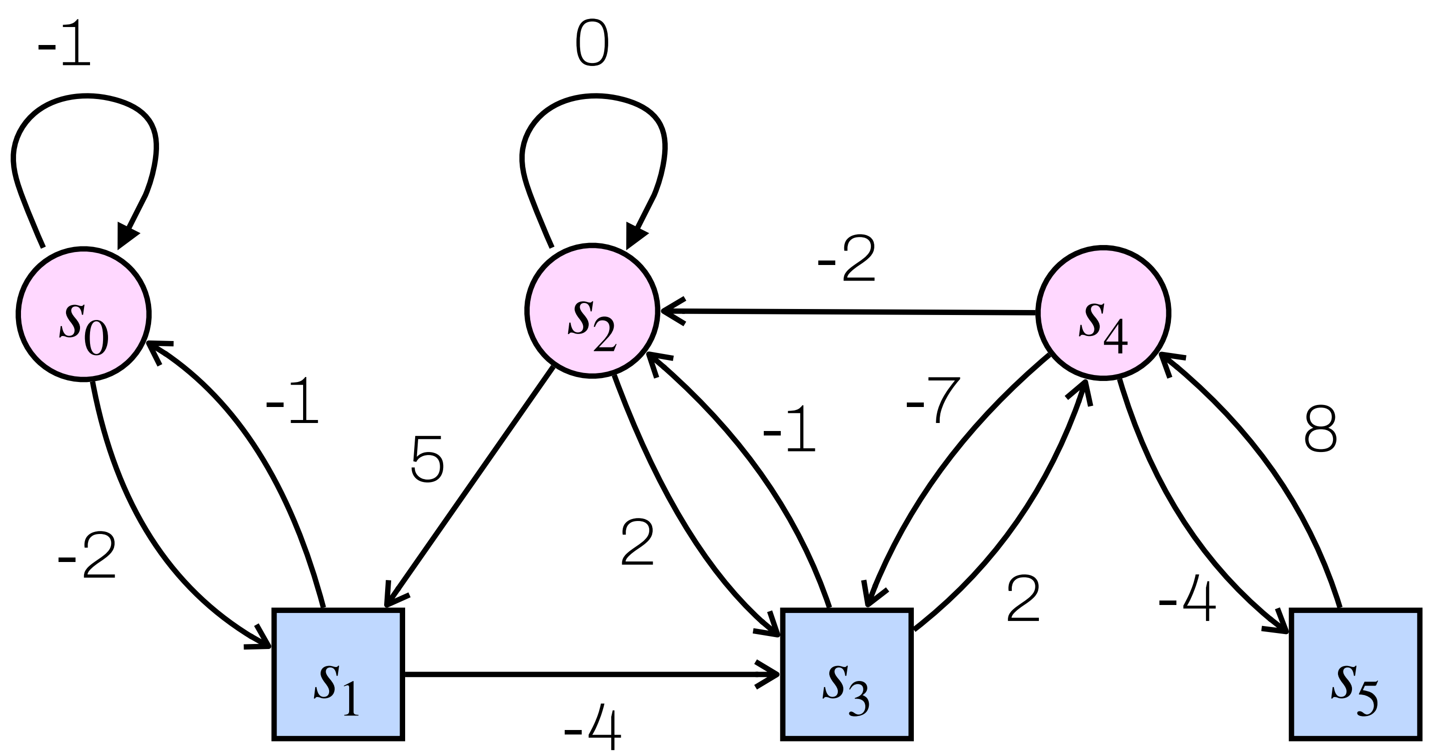


Example: mean-payoff

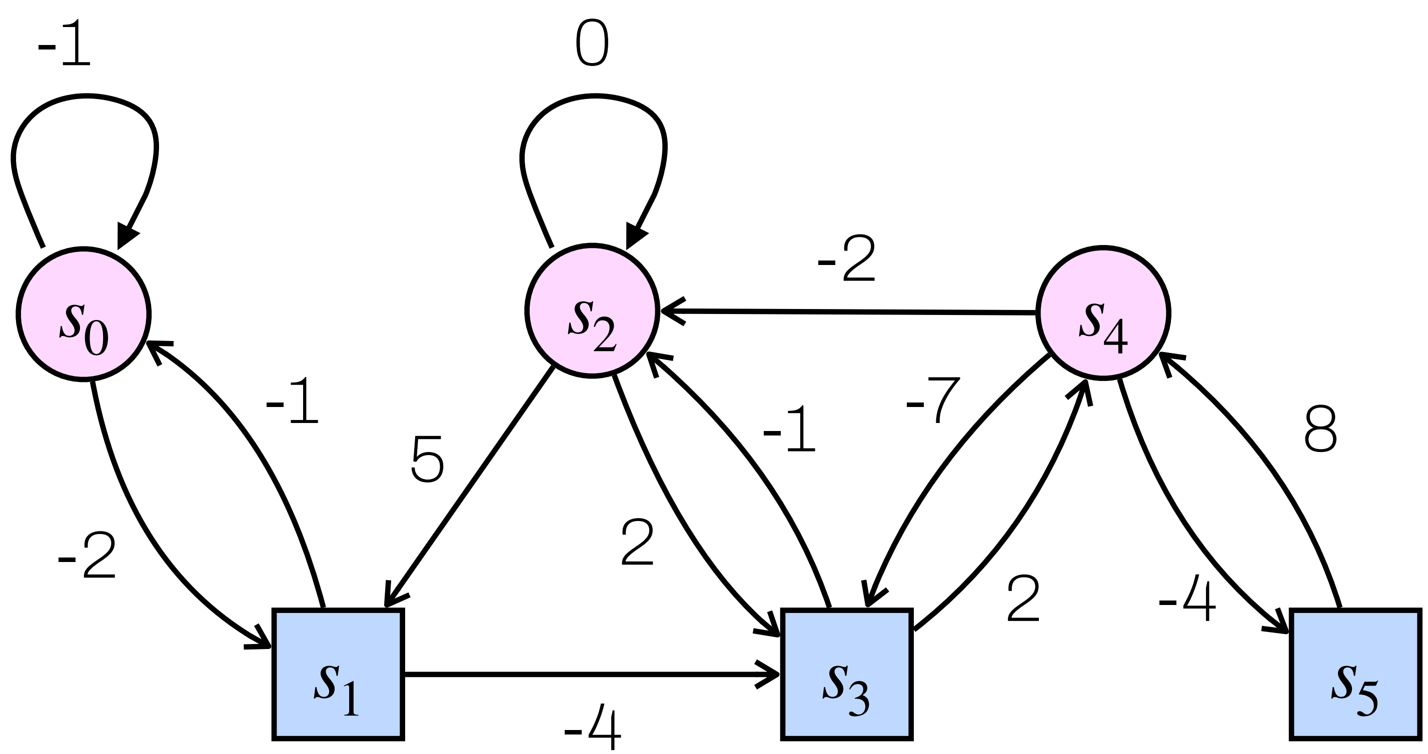


[Ohl21] Ohlmann. Monotonic graphs for Parity and Mean-Payoff games (PhD thesis).

Example: mean-payoff

- P_1 maximizes, P_2 minimizes

$$\overline{\mathsf{MP}} = \limsup_{n} \frac{\sum_{i \neq n} c_i}{n}$$



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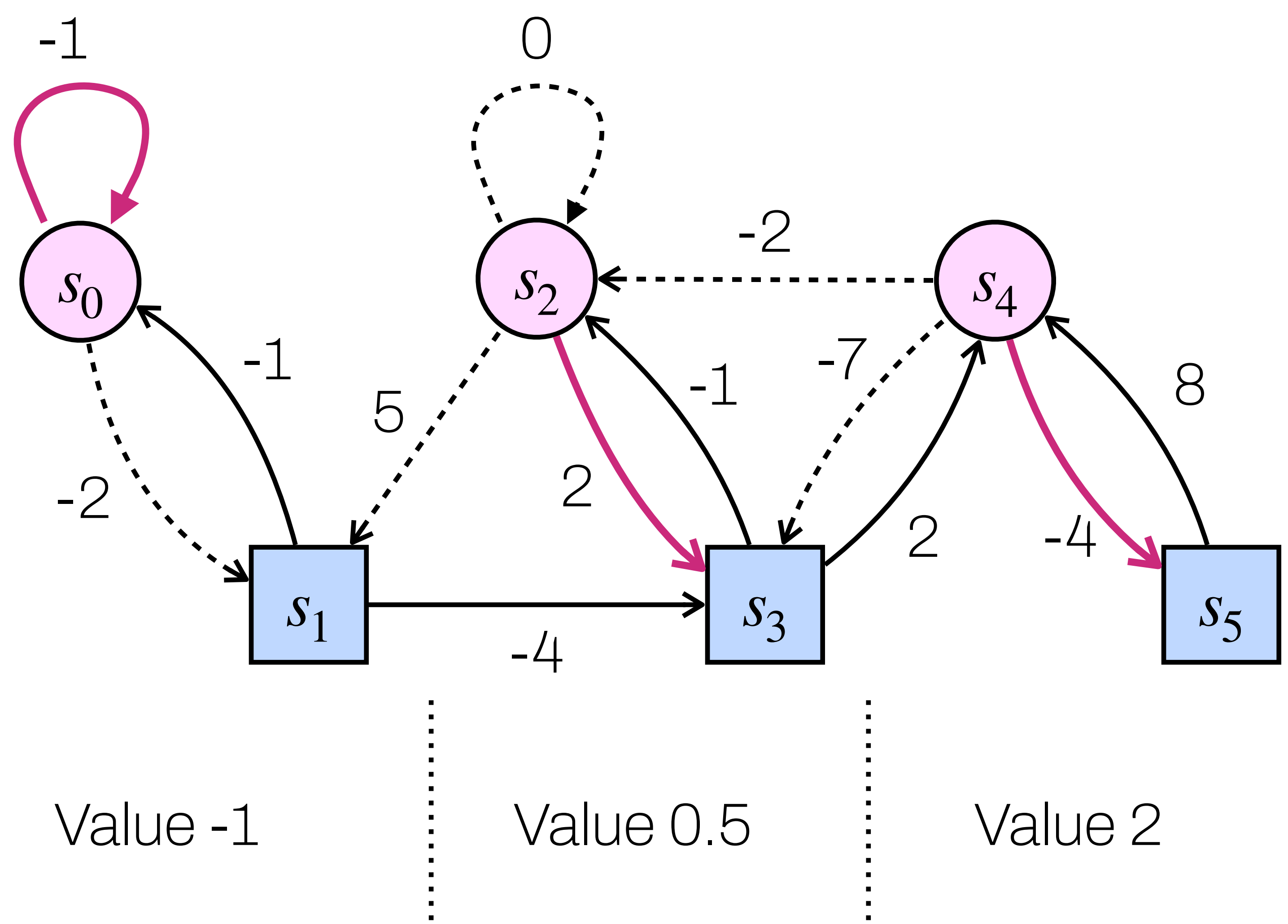

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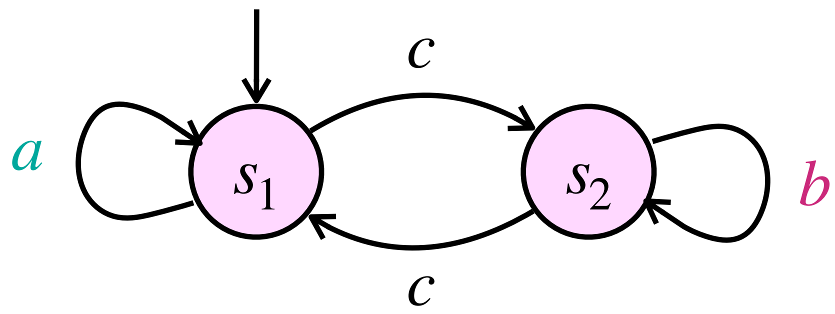
Do we need more?

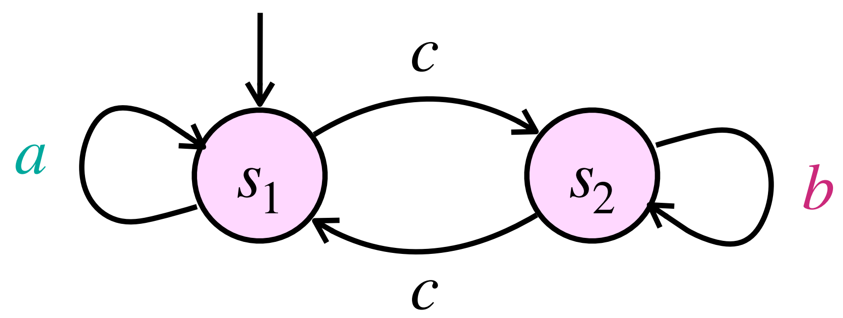
Examples



« See infinitely often both a and b »

Büchi(a) ∧ Büchi(b)

Examples



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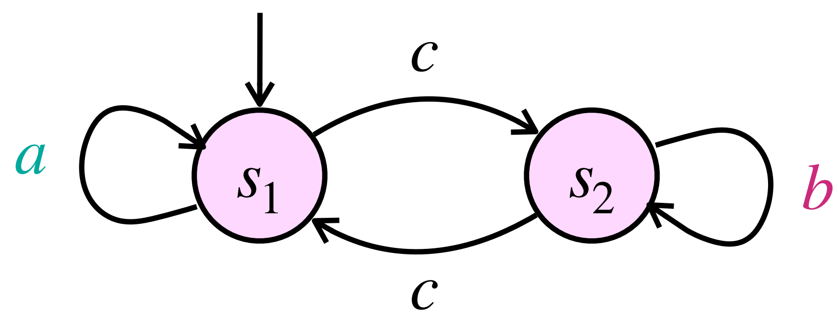
Winning strategy

- At each visit to s_1, loop once in s_1 and then go to s_2
- At each visit to s_2, loop once in s_2 and then go to s_1
- Generates the sequence $(acbc)^\omega$

Examples



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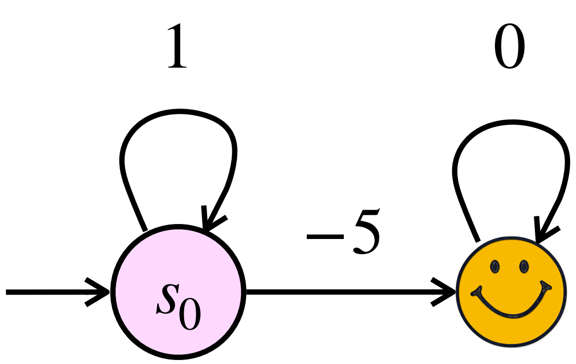

« Reach the target with energy level 0 »

$\mathbf{FG}\ (\mathsf{EL} = 0)$

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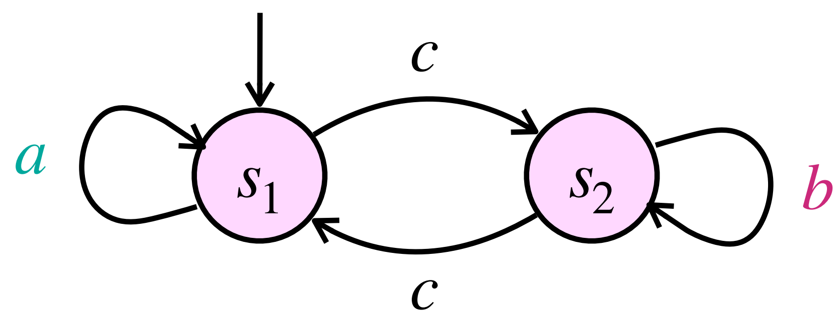

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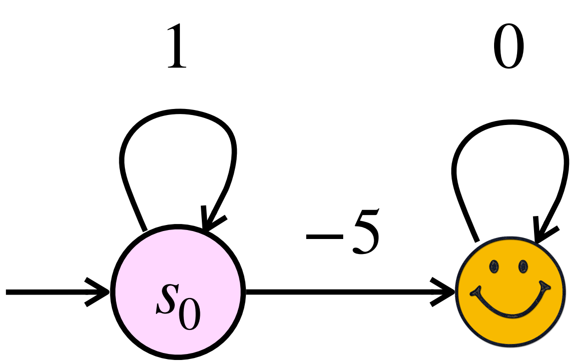

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- Loop five times in s_0
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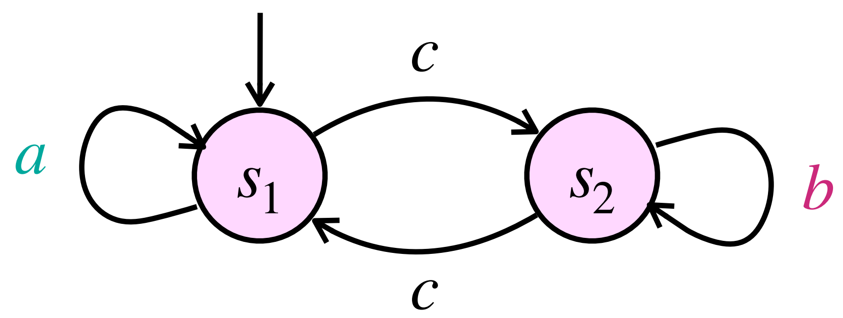
$1\ 1\ 1\ 1\ 1\ \ -5\ 0\ 0\ 0 \ldots$

Examples



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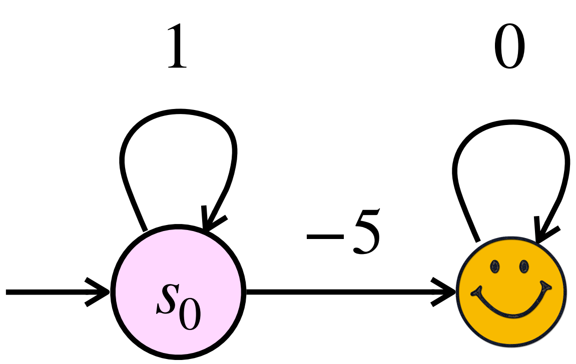

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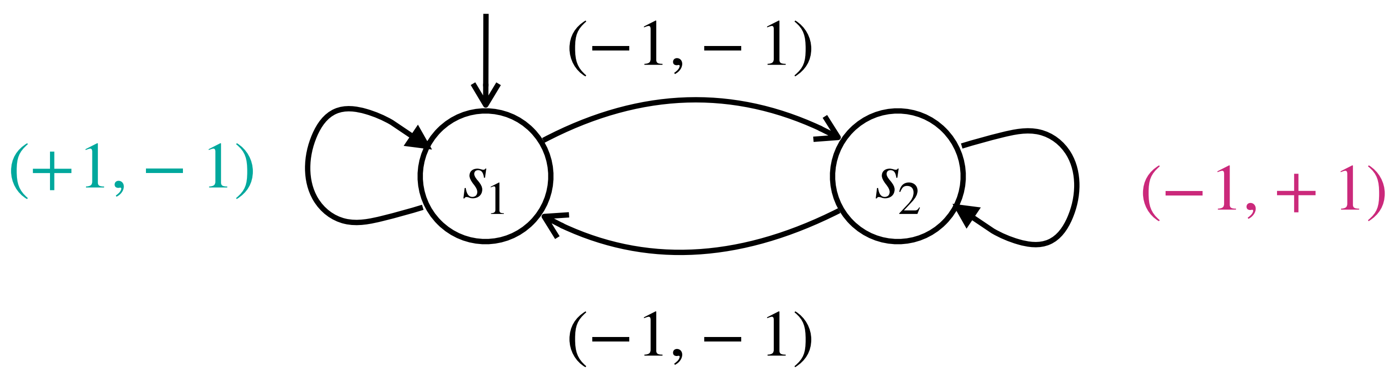
These two strategies require only **finite** memory

Example: multi-dimensional mean-payoff



« Have a (limsup) mean-payoff ≥ 0 on both dimensions »

So-called *multi-dimensional mean-payoff*

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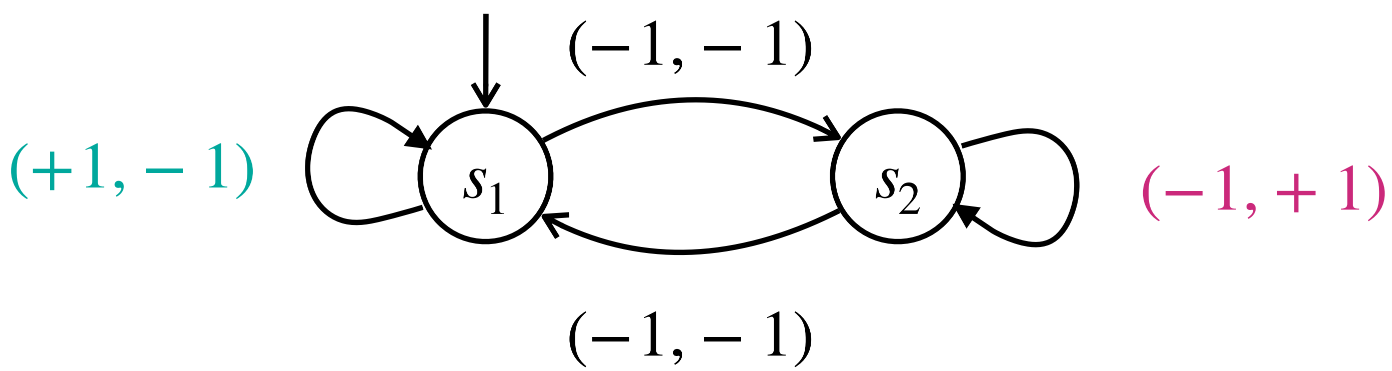

« Have a (limsup) mean-payoff ≥ 0 on both dimensions »
So-called *multi-dimensional mean-payoff*

Winning strategy

- After k-th switch between s_1 and s_2, loop $2k-1$ times and then switch back
- Generates the sequence
  $(-1,-1)\,(-1,+1)\,(-1,-1)(+1,-1)(+1,-1)(+1,-1)\,(-1,-1)$
  $(-1,+1)(-1,+1)(-1,+1)(-1,+1)(-1,+1)\,(-1,-1)$
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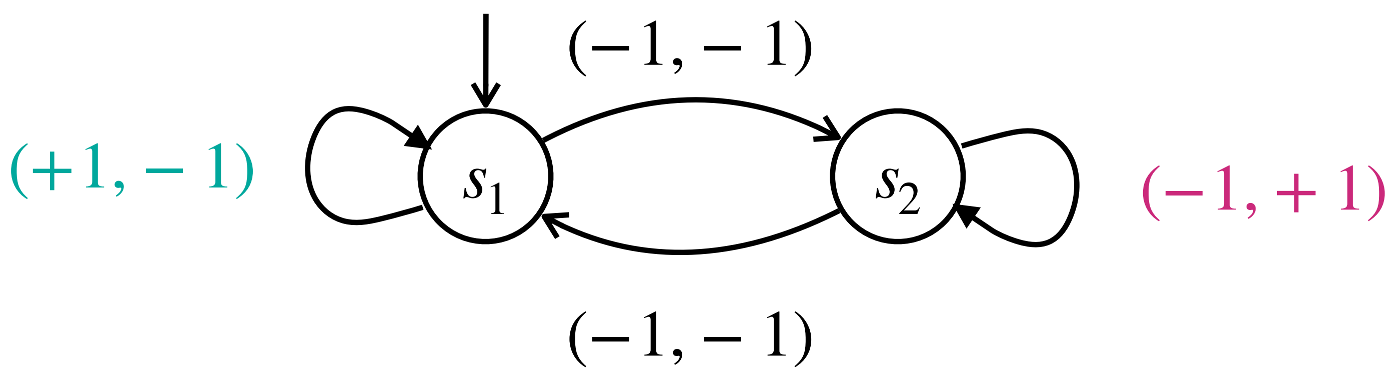

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$(+1,-1)(+1,-1)(+1,-1)(+1,-1)(+1,-1)(+1,-1)(+1,-1)\,(-1,-1)$...

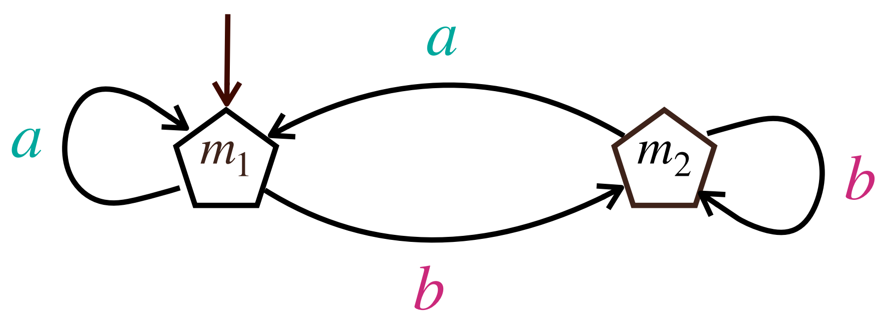
This strategy requires **infinite** memory, and this is unavoidable

We focus on finite memory!

Chromatic* memory

Memory skeleton

$$\mathscr{M} = (M, m_{\text{init}}, \alpha_{\text{upd}}) \text{ with } m_{\text{init}} \in M \text{ and } \alpha_{\text{upd}} : M \times C \to M$$



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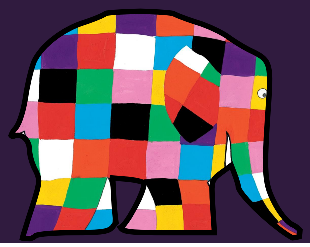

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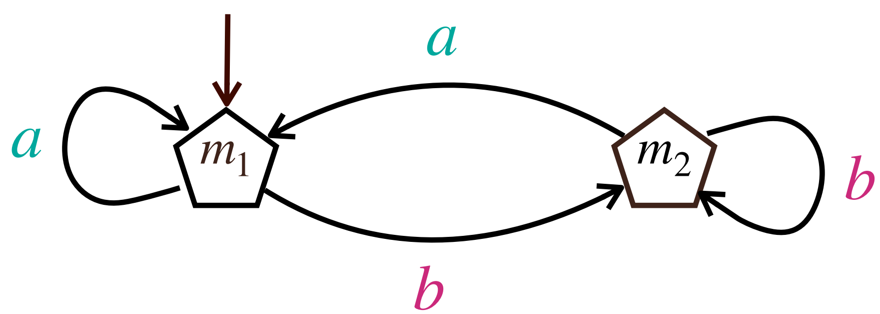

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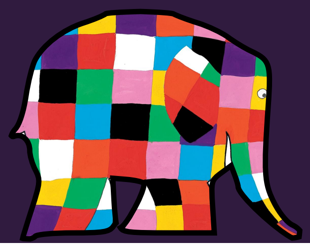

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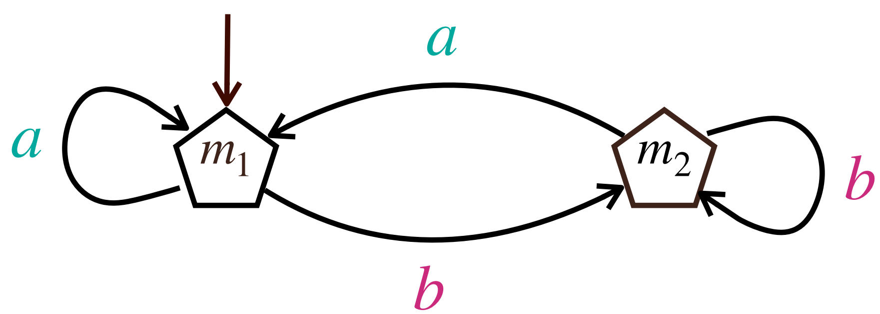

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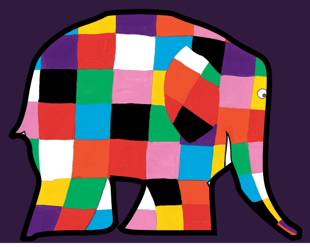

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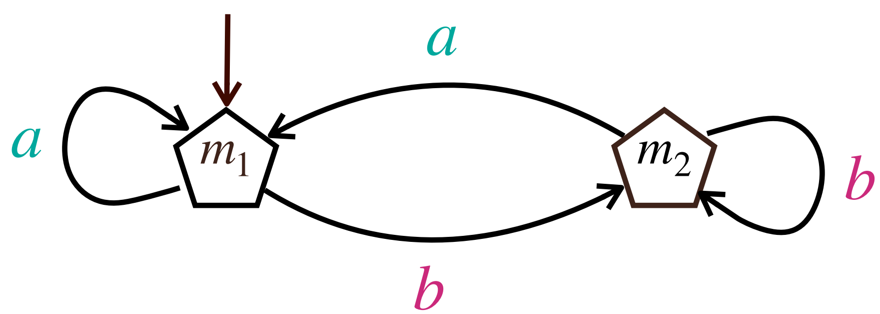

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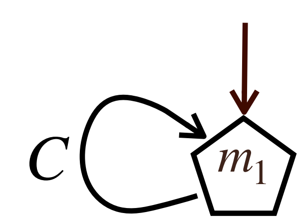

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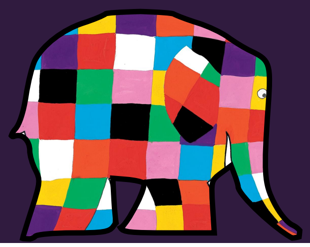

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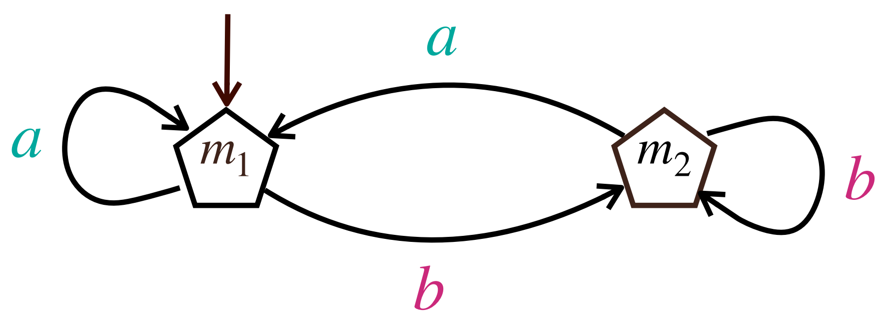

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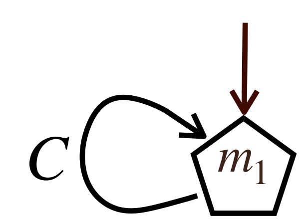

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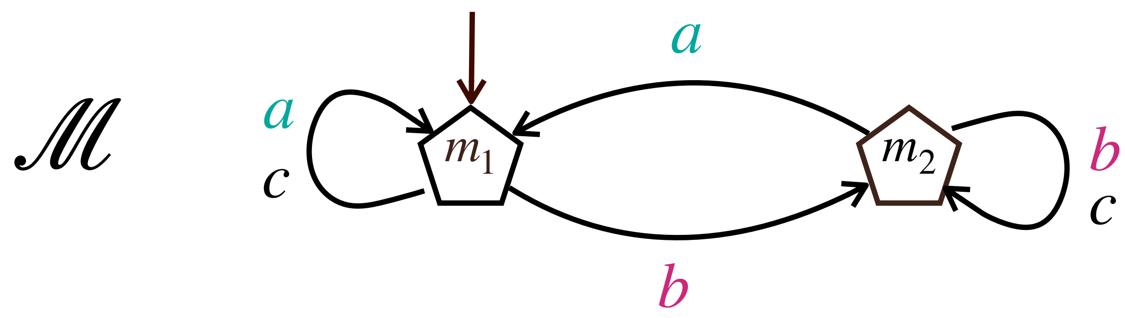
Example of chromatic memory



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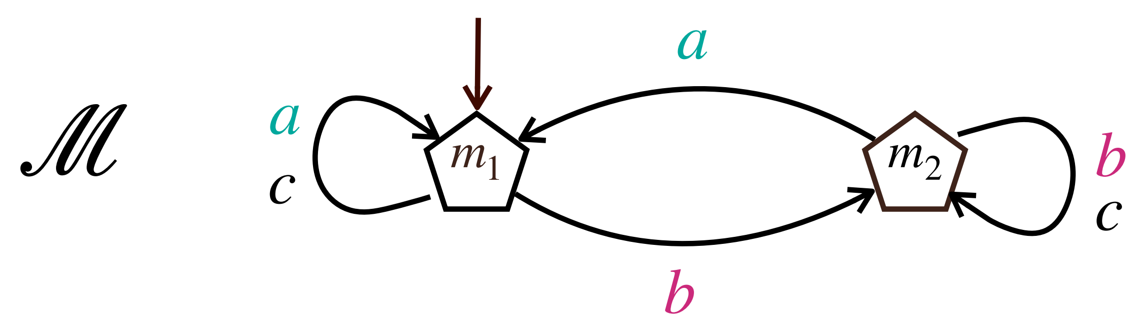

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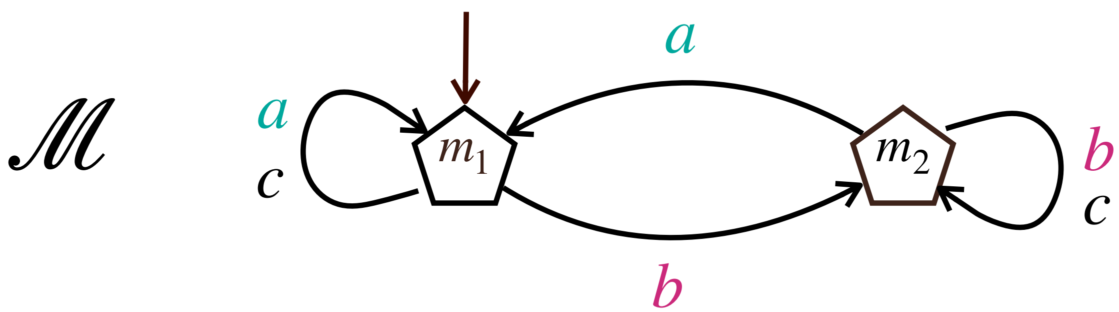

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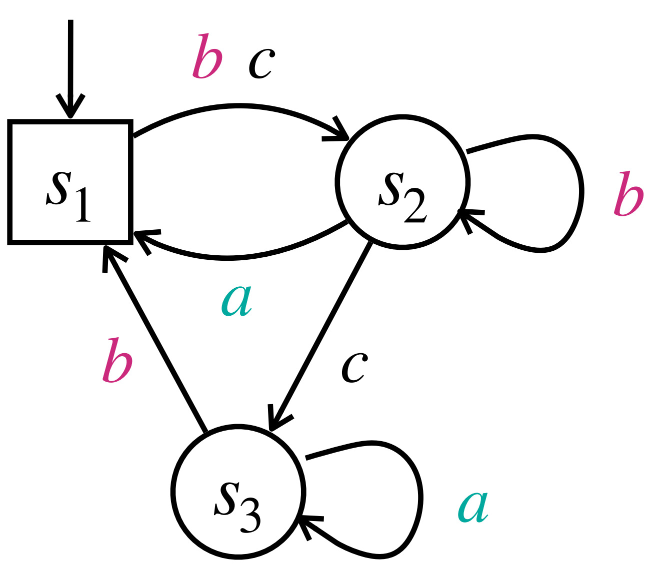

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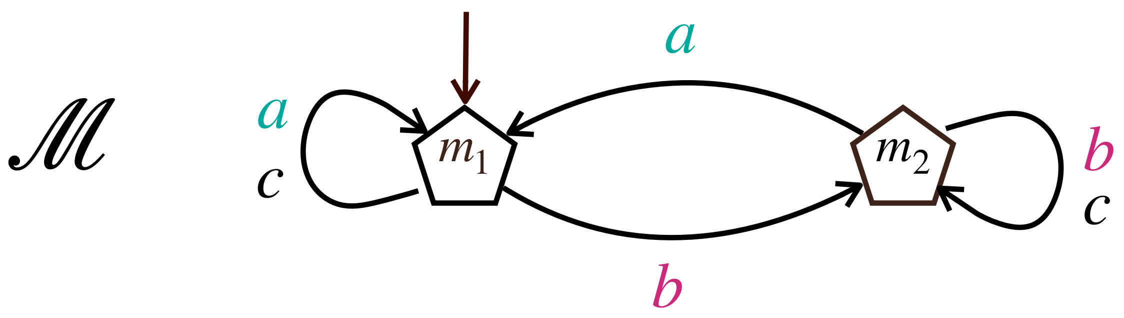

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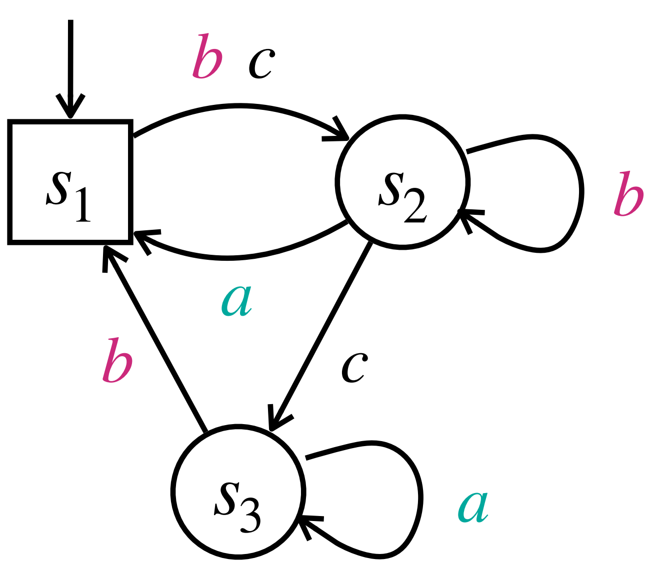

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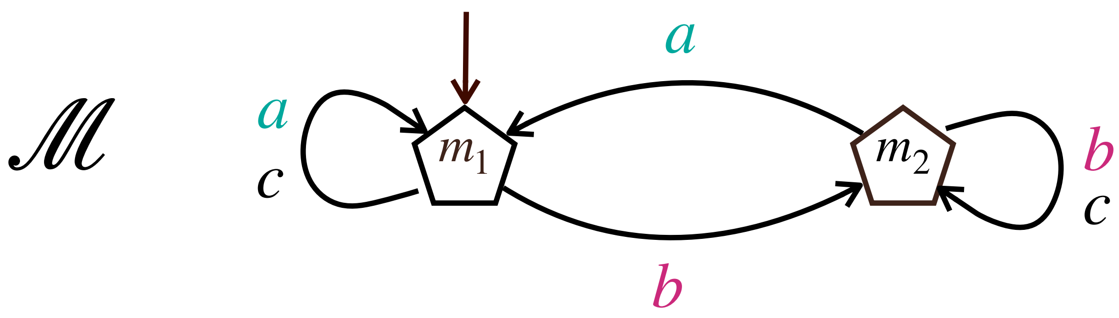

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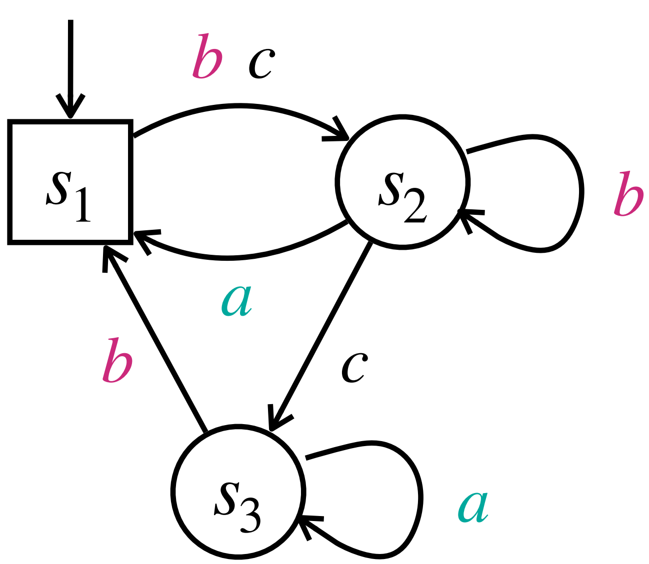

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Understand well low-memory specifications

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Positional / finite-memory determinacy

Is it the case that positional (resp. finite-memory) strategies suffice to win/be optimal when winning/optimal strategies exist?

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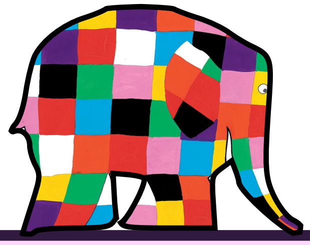

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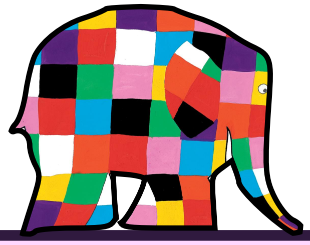

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- Finite vs infinite games

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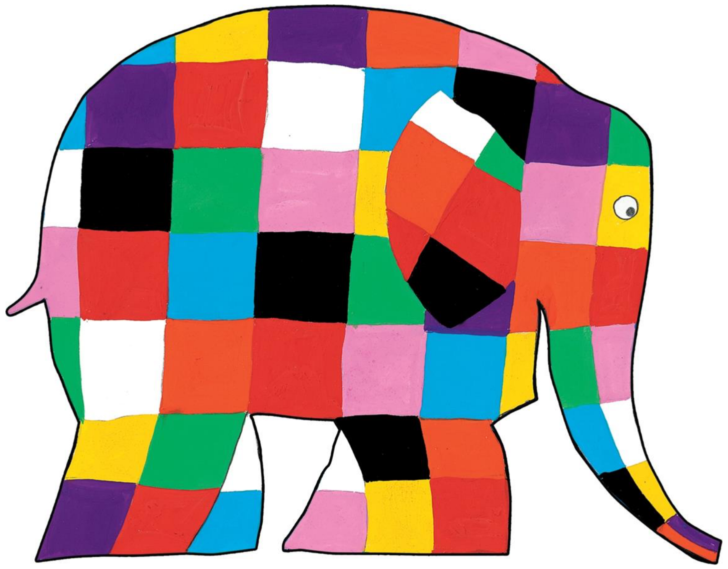
Characterizing positional and chromatic finite-memory determinacy in finite games



The approach

[GZ05] Gimbert, Zielonka. Games Where You Can Play Optimally Without Any Memory (CONCUR'05).

The approach

- Characterize winning objectives ensuring **memoryless determinacy**, that is, the existence of positional winning strategies (for both players) in all finite games

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If this is in W then one of those is in W

Two characterizations

Let $\sqsubseteq$ be a preference relation (for P_1).

Characterization - Two-player games

The two following assertions are equivalent:

1. All finite games have positional optimal strategies for both players;
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Applications

Lifting theorem

P_i has positional optimal strategies in all finite P_i-games

$$\Downarrow$$

Both players have positional optimal strategies in all finite 2-player games.

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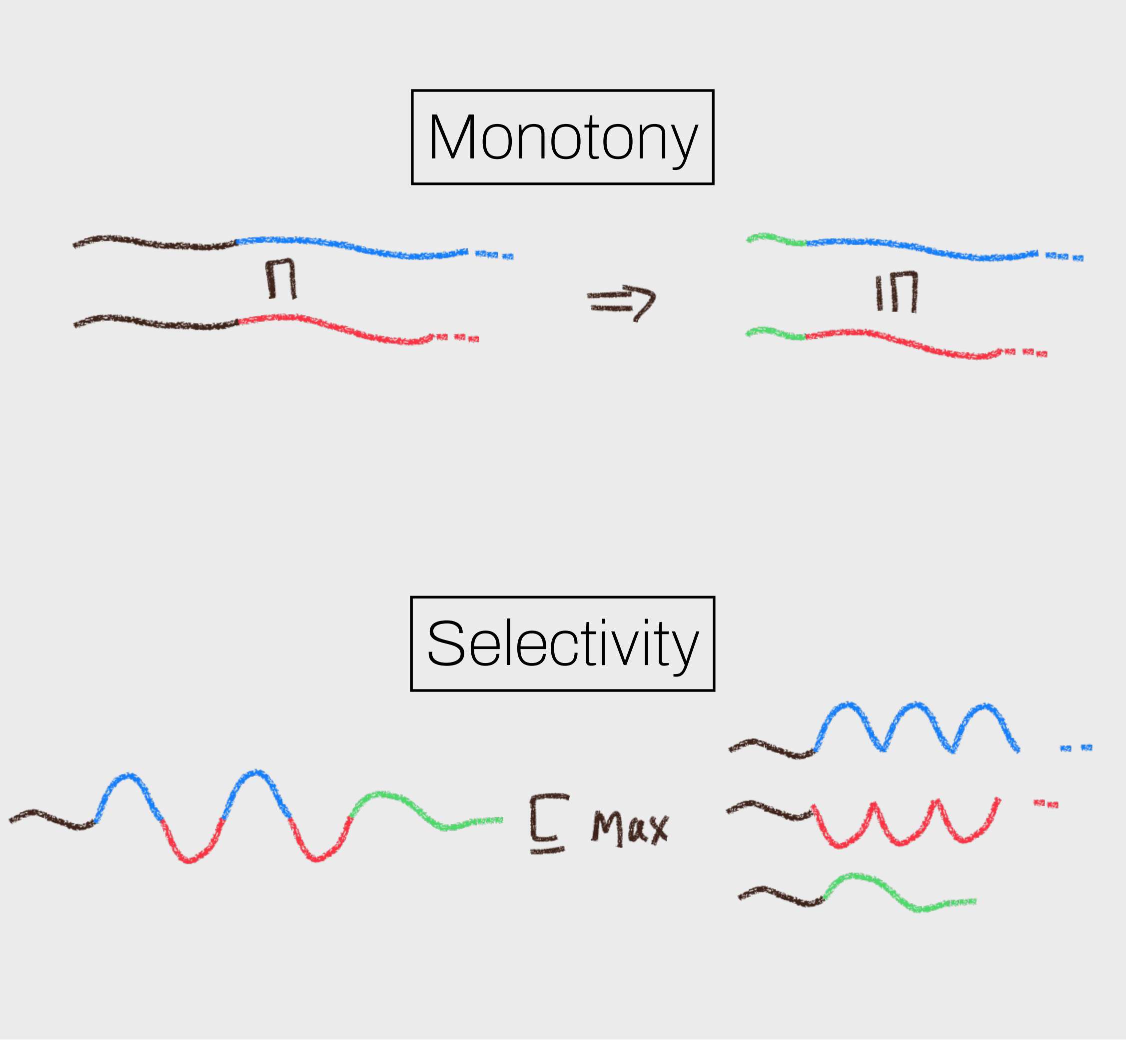
Very powerful and extremely useful in practice

- Easy to analyse the one-player case (graph analysis)
  - Mean-payoff, average-energy [BMRLL15]

[BMRLL15] Bouyer, Markey, Randour, Larsen, Laursen. Average-energy games (GandALF'15).

Discussion of examples

- Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective
- Parity, mean-payoff:
  - Prefix-independent hence monotone
  - Selective
- Average-energy games [BMRLL15]
  - Lifting theorem!!



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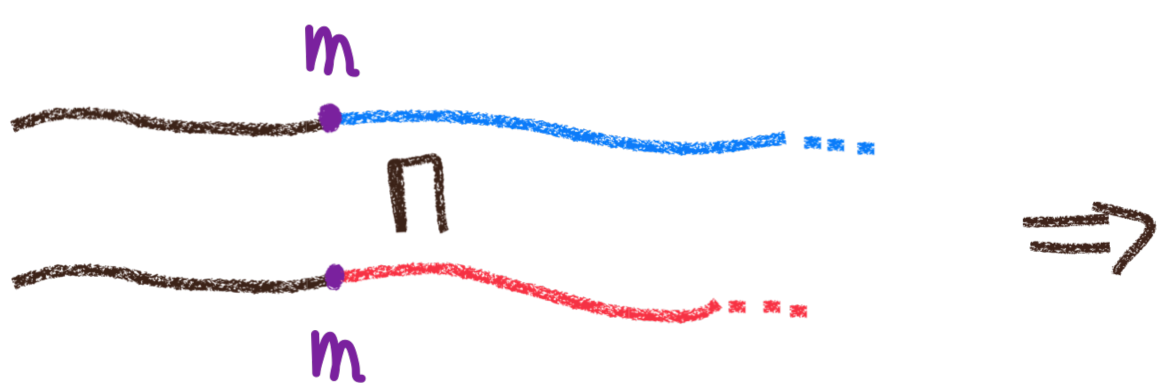

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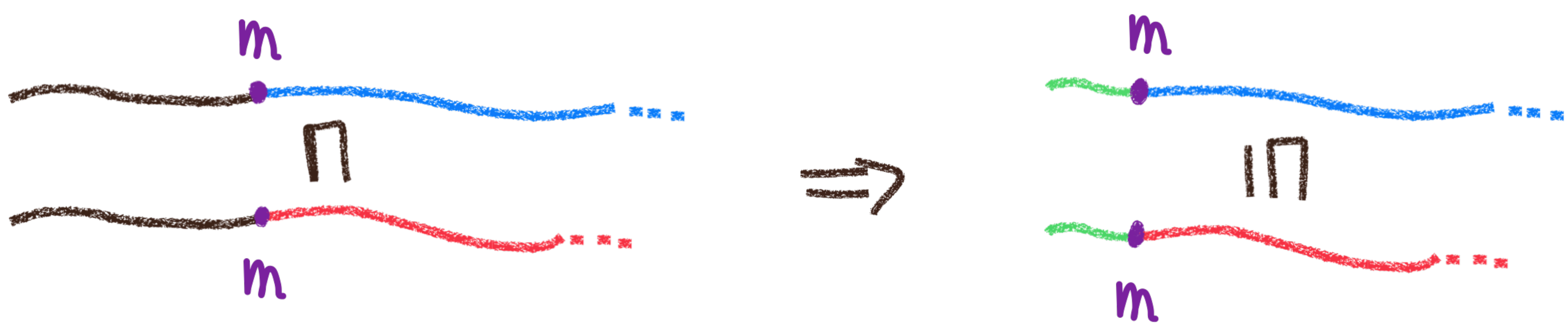

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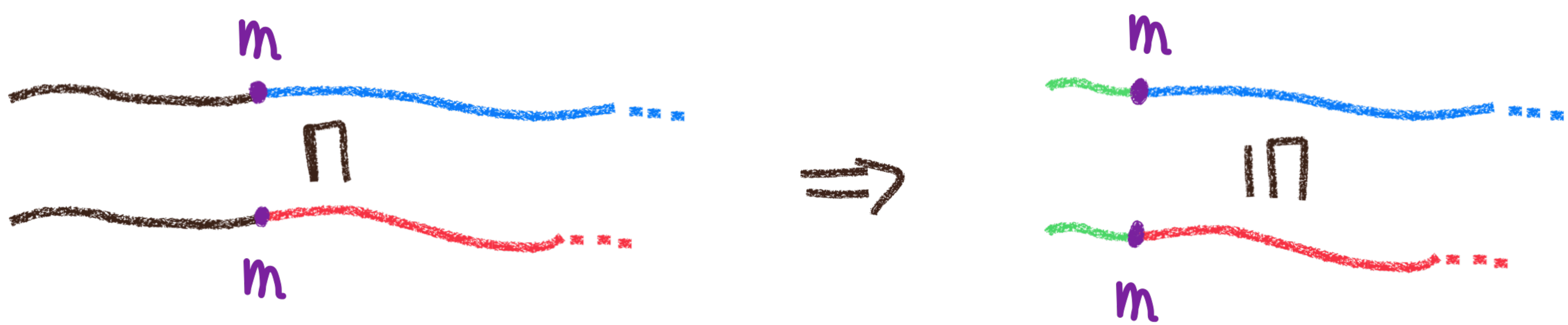

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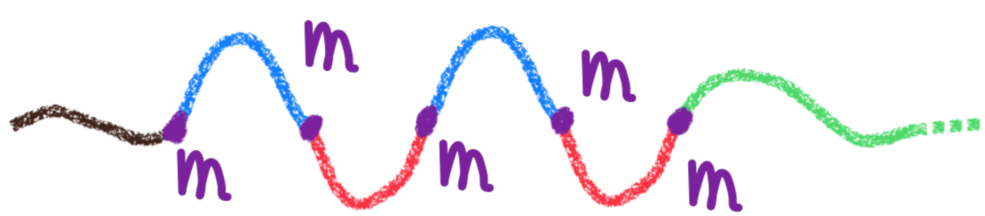

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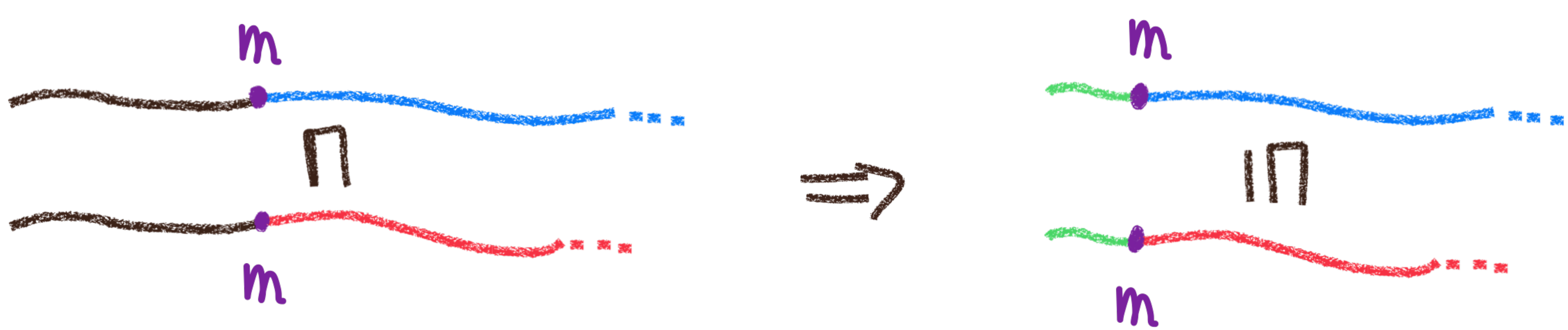

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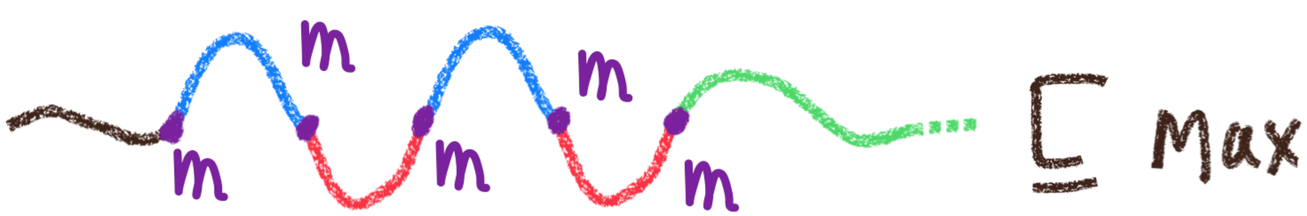

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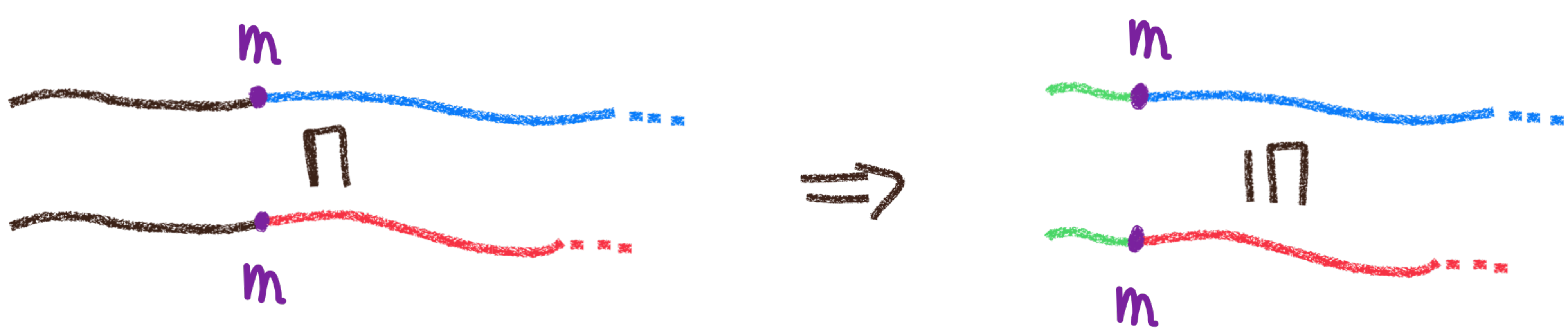

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→ We recover [GZ05] with $\mathcal{M} = \mathcal{M}_{\text{triv}}$

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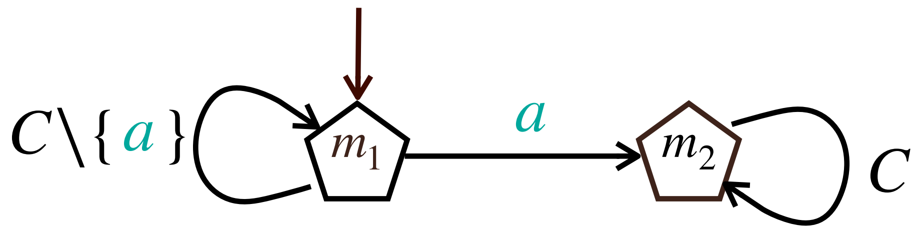

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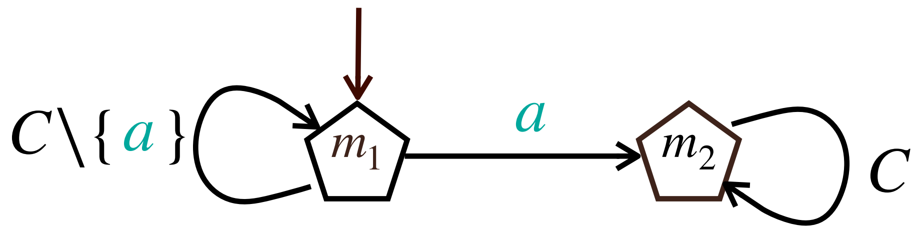

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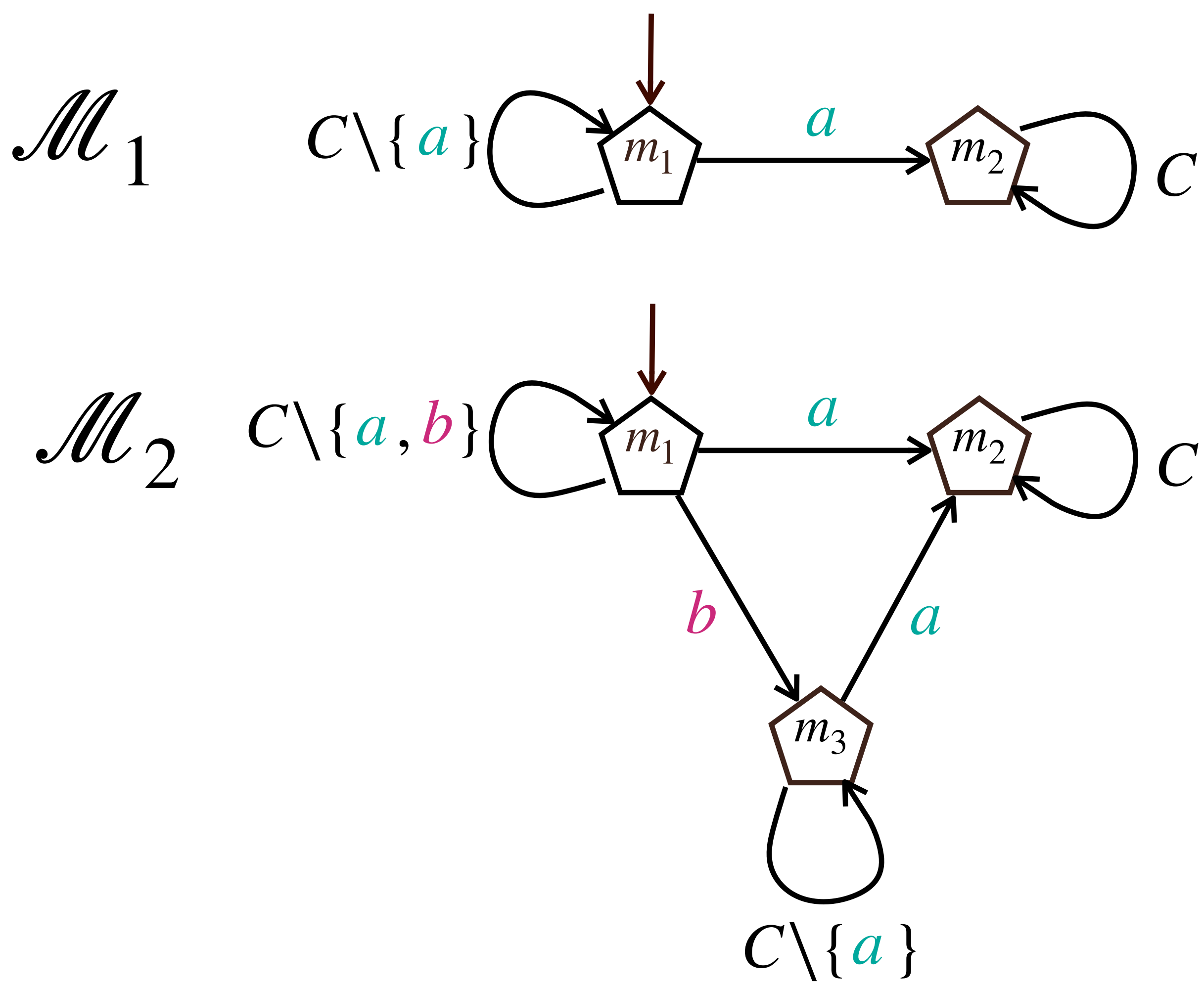

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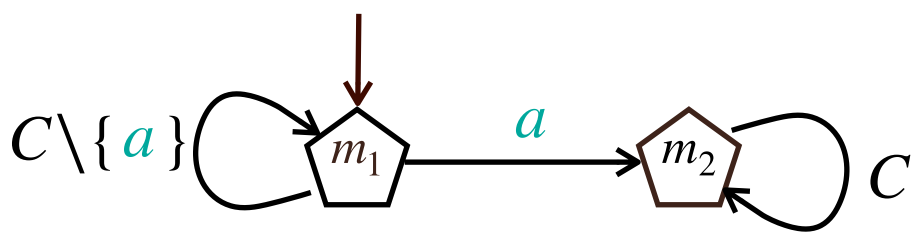

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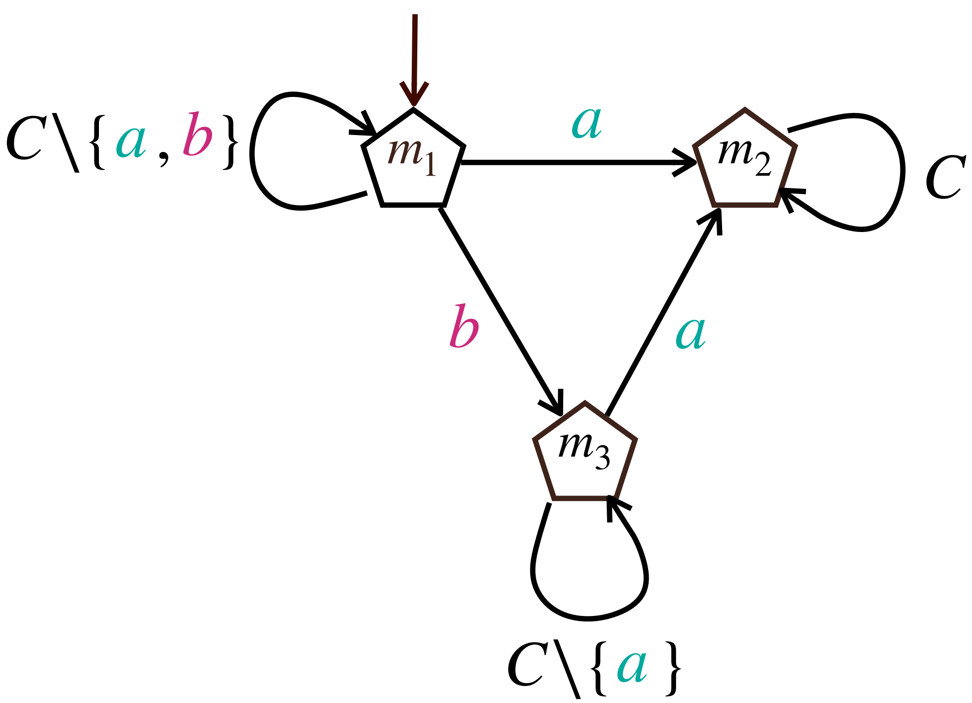

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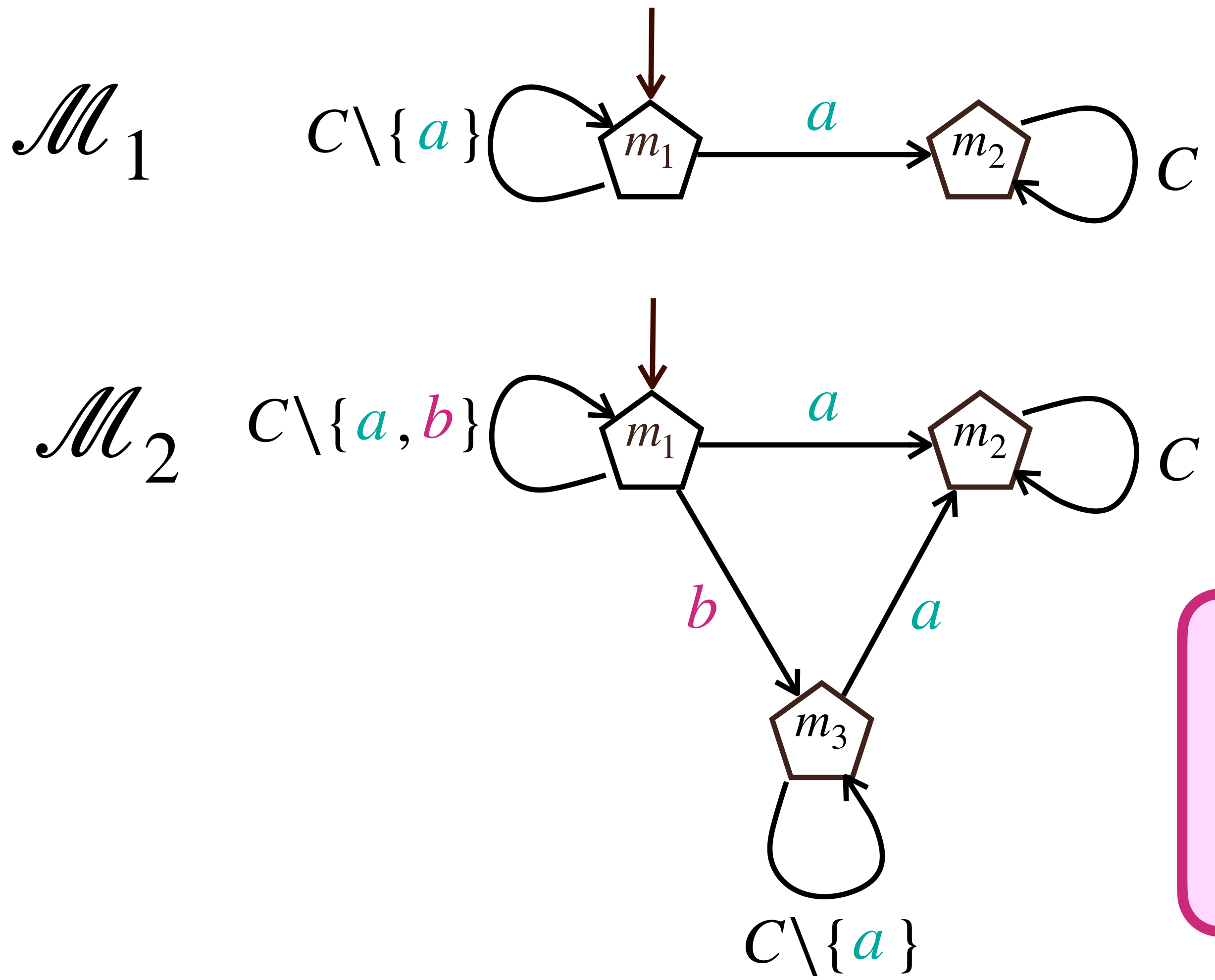

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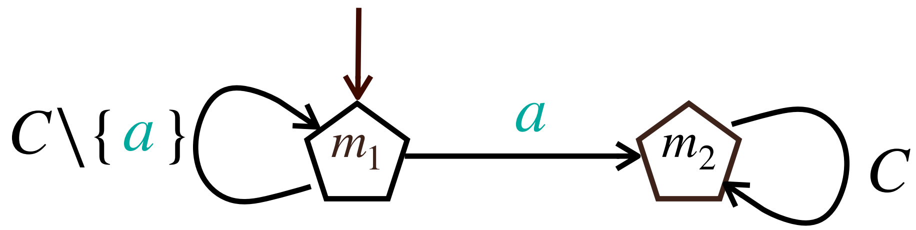

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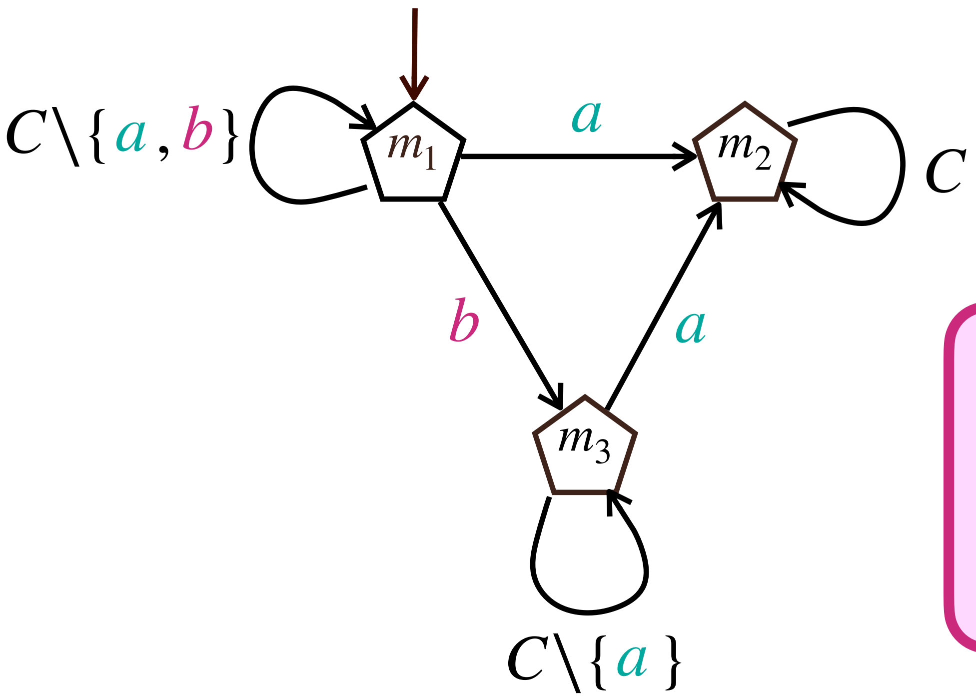

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→ Memory $\mathcal{M}_2$ is sufficient for both players in all finite games

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Finite games

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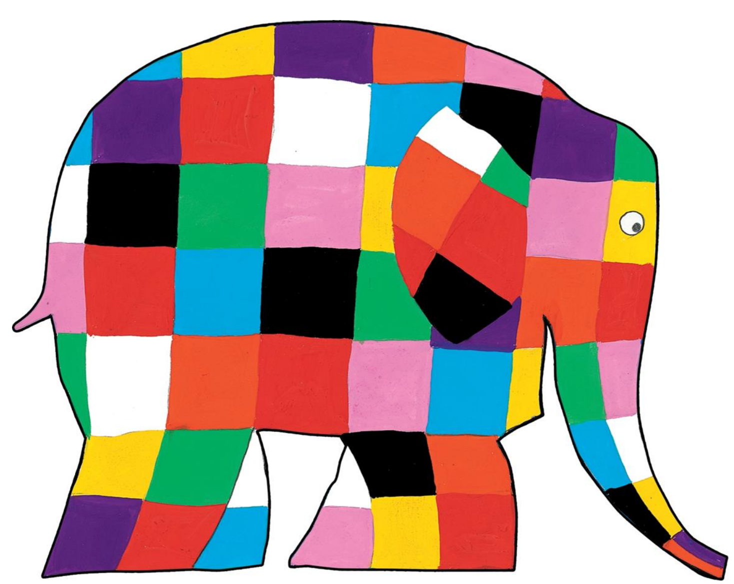
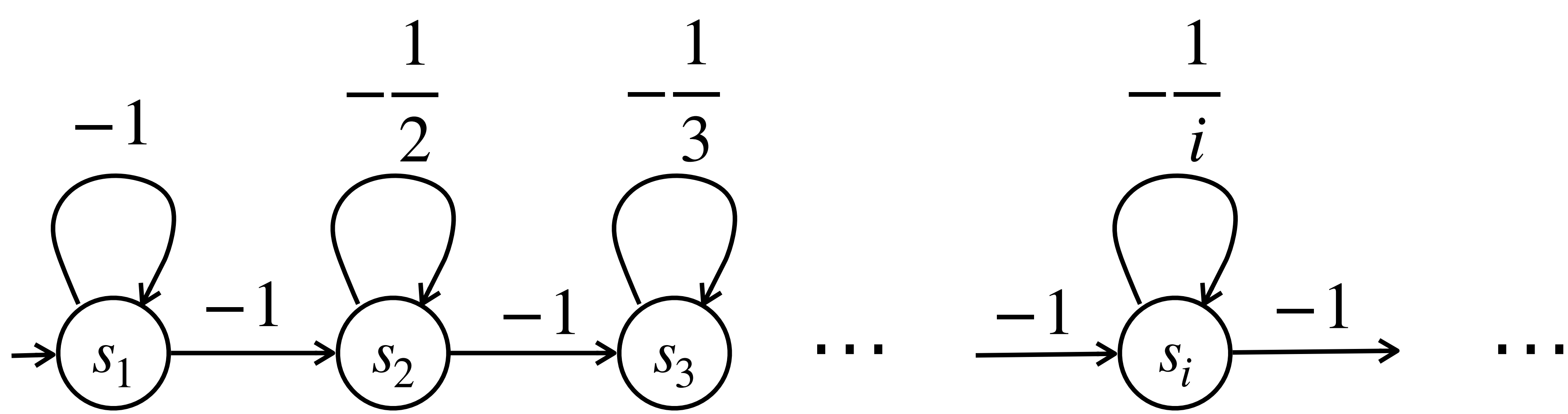
Characterizing positional and chromatic finite-memory determinacy in infinite games



The case of mean-payoff

- Objective for P_1: get non-negative (limsup) mean-payoff
- In finite games: **positional** strategies are sufficient to win
- In infinite games: **infinite memory** is required to win



A first insight [CN06]

- Let W be a prefix-independent objective.

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Characterization - Two-player games

The two following assertions are equivalent:

1. Positional optimal strategies are sufficient for W in all (infinite) games for both players;
2. W is a parity condition
   That is, there are $n \in \mathbb{N}$ and $\gamma : C \to \{0,1,\ldots,n\}$ such that
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Some language theory (1)

- Let $L \subseteq C*$ be a language of finite words

Right congruence

- Given $x, y \in C*$,

$$x \sim_L y \Leftrightarrow \forall z \in C*, \left(x \cdot z \in L \Leftrightarrow y \cdot z \in L\right)$$

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Myhill-Nerode Theorem

- L is regular if and only if $\sim_L$ has finite index;
  - There is an automaton whose states are classes of $\sim_L$, which recognizes L.

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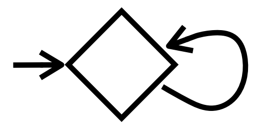
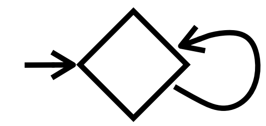
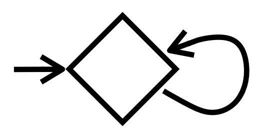
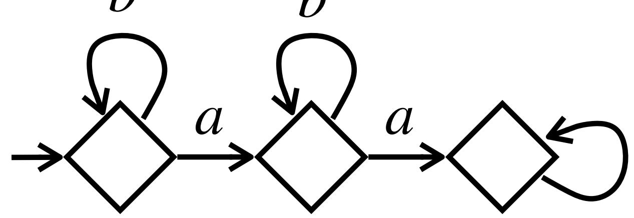
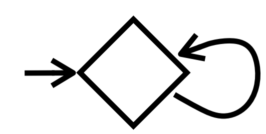
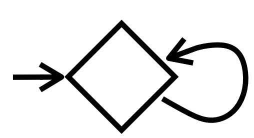
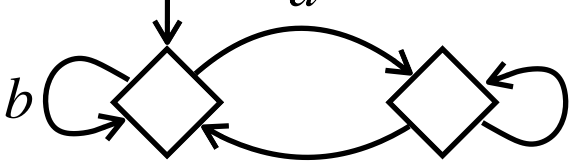
Link with ω-regularity?

- If L is ω-regular, then $\sim_L$ has finite index;
  - The automaton based on $\sim_L$ is a so-called prefix-classifier;
- The converse does not hold (e.g. all prefix-independent languages are such that $\sim_L$ has only one element).

Four examples

Objective	Prefix classifier $\mathscr{M}_{\sim}$	Suffcient memory
Parity objective		
Mean-payoff ≥ 0		No finite automaton
$C = \{a, b\}$ $W = b^*ab^*aC^\omega$		
$C = \{a, b\}$ $W = C^*(ab)^\omega$		

Characterization

- Let $W \subseteq C^\omega$ be a winning objective.

Characterization - Two-player games

If a finite memory structure $\mathcal{M}$ suffices to play optimally in one-player infinite arenas for both players, then the prefix-classifier $\mathcal{M}_\sim$ is finite and W is recognized by a parity automaton $(\mathcal{M}_\sim \otimes \mathcal{M}, \gamma)$, with $\gamma : M \times C \to \{0,1,\ldots,n\}$.

→ Generalizes [CN06] where both $\mathcal{M}$ and $\mathcal{M}_\sim$ are trivial

[CN06] Colcombet, Niwiński. On the positional determinacy of edge-labeled games (Theor. Comp. Science).
[BRV22] Bouyer, Randour, Vandenhove. Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs (STACS'22).

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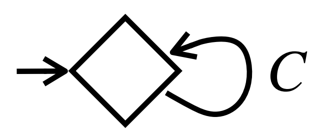
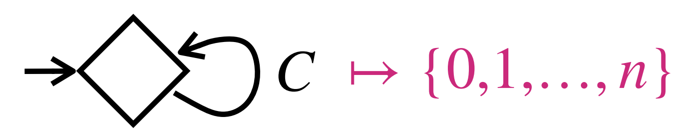
<tr><th>Objective</th><th>Prefix classifier $\mathscr{M}_\sim$</th><th>One-player memory</th></tr>
<tr><td>Parity objective</td><td>
</td><td>
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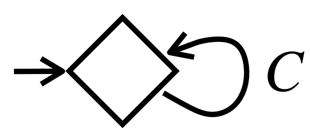
<tr><td>Mean-payoff ≥ 0</td><td>
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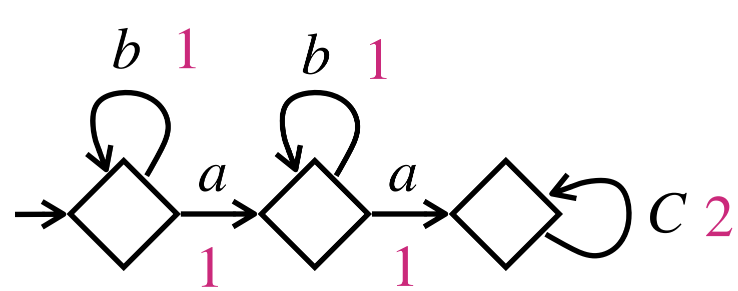
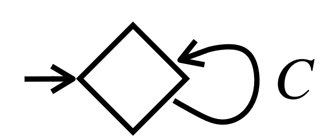
$W = b^*ab^*aC^\omega$</td><td>
</td><td>
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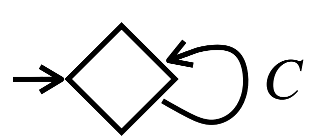
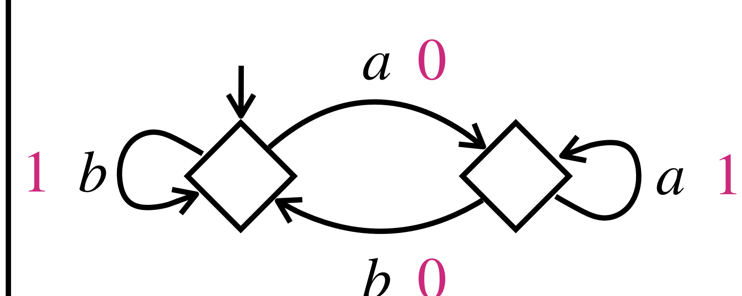
$W = C^*(ab)^\omega$</td><td>
</td><td>
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Lifting theorem

If W and W^c are finite-memory-determined in one-player infinite games, then W and W^c are finite-memory-determined in two-player infinite games.

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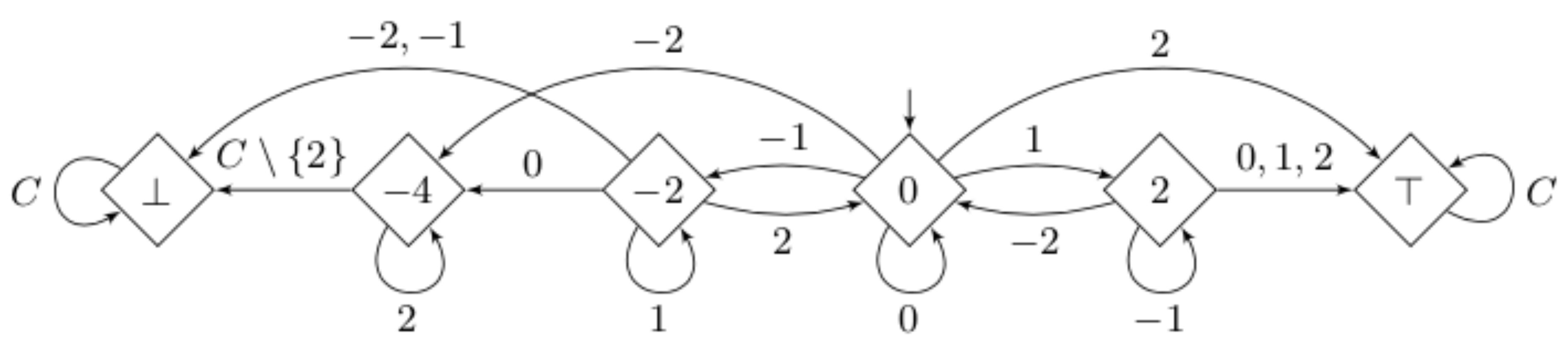
Characterization

W is finite-memory-determined in (two-player) infinite games if and only if W is ω-regular.

Some consequences

- Mean-payoff ≥ 0 is not ω-regular (even though it is positionally determined in finite games)

- Some discounted objectives are ω-regular:
e.g. condition $\mathsf{DS}_{\lambda}^{\geq 0}$ (with $\lambda \in (0,1) \cap \mathbb{Q}$, $C = [-k, k] \cap \mathbb{Z}$) is ω-regular if and only if $k < \frac{1}{\lambda} - 1$ or $\lambda = \frac{1}{n}$ for some $n \in \mathbb{N}_{>0}$



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  - Can be reduce/optimize the memory?
    E.g. is $\mathscr{M}_{\sim}$ necessary in the memory for two players?
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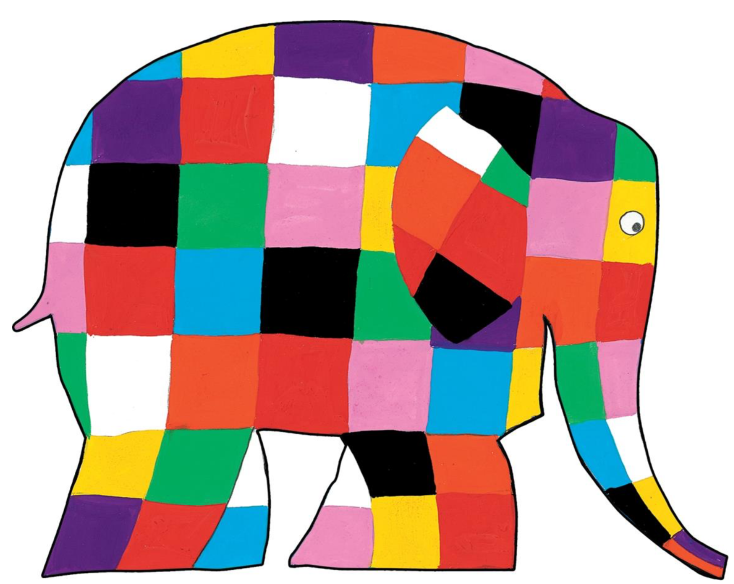
  - What about finite branching?

Conclusion



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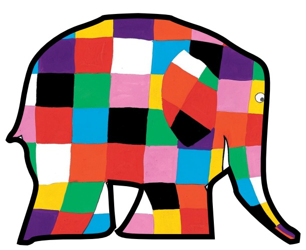

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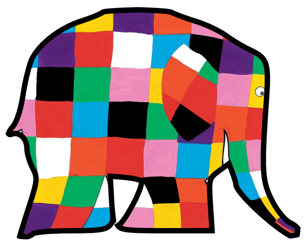

- Going further:
  - Games under **partial observation**, e.g. players with their own knowledge (of the game, of the other's choices, ...)
  - Half-positionality or half-finite-memory of objectives (preliminary result [BCRV22])

[BCRV22] Bouyer, Casares, Randour, Vandenhove. Half-Positional Objectives Recognized by Deterministic Büchi Automata (submitted).